1 Introduction

About this lecture

- Sets (HTP Sections 1.3, 1.4)
- Quantifiers and sets (HTP 2.1)
- Equivalences involving quantifiers (HTP 2.2)
- More operations on sets (HTP 2.3)
- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://moodle.kaist.ac.kr`
- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.
- A mathematical introduction to logic, H. Enderton, Academic Press.
- `http://plato.stanford.edu/contents.html` has much resource.
- Introduction to set theory, Hrbacek and Jech, CRC Press.

Some helpful references

- `http://en.wikipedia.org/wiki/Truth_table`,
- `http://logik.phl.univie.ac.at/~chris/gateway/formular-uk-zentral.html`, complete (i.e. has all the steps)
- `http://svn.oriontransfer.org/TruthTable/index.rhtml`, has xor, complete.

2 Sets

Sets

- A set is a collection....This naive notion is fairly good.
- The set theory is compatible with logic.
- Symbols $\in, \{\}$. (belong, included)
- $\{\{\}, \{\{\}\}, \{\{\{\}\}\}\}$
- $\{a\}$. We hold that $a \in \{a, b, c, ...\}$.

- The main thrust of the set theory is the theory of description by Russell.
- $P(x)$: x is a variable. $P(x)$ is the statement that x is a prime number
- $y \in \{x|P(x)\}$ is equivalent to $P(y)$. That is the truth set of P.
- Sets $\leftrightarrow$ Properties
- $D(p, q)$: p is divisible by q.
- A set $B = \{x|x$ is a prime number $\}$.
- $x \in B$. What does this mean?

Axioms of the set theory (Naive version)

- There exists a set which has no elements. (Existence)
- Two sets are equal if and only if they have the same elements. (Extensionality)
- There exists a set $B = \{x \in A|P(x)\}$ if A is a set. (Comprehension)
- For any two sets, there exists a set that they both belong to. That is, if A and B are sets, there is $\{A, B\}$. (Pairing)
- For any collection of sets, there exists a unique set that contains all the elements that belong to at least one set in the collection. (Union)

Axioms of the set theory (Naive version)

- Given each set, there exists a collection of sets that contains among its elements all the subset of the given set. (Power set)
- An inductive set exists (Infinity)
- Let $P(x, y)$ be a property that for every x, there exists unique y so that $P(x, y)$ holds. Then for every set A, there is a set B such that for every $x \in A$, there is $y \in B$ so that $P(x, y)$ holds. (Substitution)
- Zermelo-Fraenkel theory has more axioms...The axiom of foundation, the axiom of choice.(ZFC)

Example

- $\{x|x^2 > 9\}$.
- $\mathbb{R} = \{x|x$ is a real number. $\}$.
- $\mathbb{Q} = \{x|x$ is a rational number.$\}$
- $\mathbb{Z} = \{x|x$ is an integer.$\}$.

- $\mathbb{N} = \{x|x$ is a natual number.$\}$.
- $y \in \{x \in A|P(x)\}$ is equivalent to $y \in A \wedge P(y)$.
- $\emptyset$ is the empty set.

Operations on sets

- $A \subset B$ if and only if $\forall x(x \in A \rightarrow x \in B)$.
- $A \cap B = \{x|x \in A \wedge x \in B\}$.
- $A \cup B = \{x|x \in A \vee x \in B\}$.
- $A \cap B \subset A \cup B$.
- $A - B = \{x|x \in A \wedge x \notin B\}$.
- $A = \emptyset$ if and only if $\neg \exists x(x \in A)$.

Set theoretic problem

- When is the set empty?
- How can one verify two sets are disjoint, same, smaller, bigger, or none of the above?
- Answer: We use logic and the model theory.
- $A \subset B$ means $x \in A \rightarrow x \in B$.
- Equality of A and B means $x \in A$ if and only if $x \in B$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?
- $x \in A \cup (B \cap C)$
- $x \in A \vee (x \in B \wedge x \in C)$.
- $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$. DM.
- Thus, $x \in A \cup (B \cap C) \leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$.
- One can use Venn diagrams....

More set theoretic problem

- Compare $(A - B) - C$, $(A - B) \cap (A - C)$, $(A - B) \cup (A - C)$.
- $x \in (A - B) \wedge x \notin C$. $(x \in A \wedge x \notin B) \wedge \notin C$.
- $(x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)$.
- $(A - B) \cap (A - C)$.
- We can show $(A - B) - C \subset (A - B) \cup (A - C)$.
- Is $(A - B) \cup (A - C) \subset (A - B) - C$?
-
- Use logic to find examples.

More set theoretic problem

- Comparing $(A - B) - C$ and $(A - B) \cup (A - C)$.
- $x \in (A - B) \wedge x \notin C$ and $(x \in A \wedge x \notin B) \wedge \notin C$.
- $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$.
- $\forall x((x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)) \rightarrow (x \in A \wedge x \notin B) \wedge x \notin C)$ is invalid.
- Find the counter-example...(Using what?)

3 Quantifiers and sets

Quantifiers and sets

- $A \cap B \subset B - C$. Translate this to logic
- $\forall x((x \in A \wedge x \in B) \rightarrow (x \in B \wedge x \notin C))$.
- If $A \subset B$, then A and $C - B$ are disjoint.
- $\forall x(x \in A \rightarrow x \in B) \rightarrow \neg\exists x(x \in A \wedge x \in (C - B))$.
- $\forall x(x \in A \rightarrow x \in B) \rightarrow \neg\exists x(x \in A \wedge x \in C \wedge x \notin B)$.

Examples

- For every number a, the equation $ax^2 + 4x - 2 = 0$ has a solution if and only if $a \geq -2$.
- Use $\mathbb{R}$.
- $\forall a(a \geq -2 \rightarrow \exists x \in \mathbb{R}(ax^2 + 4x - 2 = 0))$.
- Is this true? How does one verify this...

4 Equivalences involving quantifiers

Equivalences involving quantifiers

- $\neg\forall x \quad P(x) \leftrightarrow \exists x\neg P(x)$.
- $\neg\exists x \quad P(x) \leftrightarrow \forall x\neg P(x)$.
- Negation of $A \subset B$.
- $\neg\forall x(x \in A \rightarrow x \in B)$.
- $\exists x\neg(x \in A \rightarrow x \in B)$.
- $\exists x\neg(x \notin A \vee x \in B)$. MI. (conditional law)
- $\exists x(x \in A \wedge x \notin B)$. DM.
- There exists an element of A not in B.

- $\exists x \in A \quad P(x)$ is defined as $\exists x(x \in A \wedge P(x))$.
- $\forall x \in A \quad P(x)$ is defined as $\forall x(x \in A \rightarrow P(x))$.
- $\neg\forall x \in A \quad P(x) \leftrightarrow \exists x \in A\neg P(x)$.
- proof: $\neg\forall x(x \in A \rightarrow P(x))$.
- $\exists x\neg(x \in A \rightarrow P(x))$.
- $\exists x\neg(x \notin A \vee P(x))$.
- $\exists x(x \in A \wedge \neg P(x))$.
- $\exists x \in A\neg P(x)$.
- These are all equivalent statements

- $\neg\exists x \in A \quad P(x) \leftrightarrow \forall x \in A\neg P(x)$.
- proof: $\neg\exists x(x \in A \wedge P(x))$.
- $\forall x\neg(x \in A \wedge P(x))$.
- $\forall x(x \notin A \vee \neg P(x))$.
- $\forall x(x \in A \rightarrow \neg P(x)$.
- $\forall x \in A\neg P(x)$.
- These are all equivalent statements

5 More operations on sets

Indexed sets

- Let I be the set of indices $i = 1, 2, 3, ...$
- $p_1 = 2$, $p_2 = 3$, $p_3 = 5$,...
- $\{p_1, p_2, ...\} = \{p_i | i \in I\}$ is another set, called, an indexed set. (Actually this is an axiom)
- In fact I could be any set.
- $\{n^2 | n \in \mathbb{N}\}$, $\{n^2 | n \in \mathbb{Z}\}$.
- $\{\sqrt{x} | x \in \mathbb{Q}\}$

Family of sets

- Sets whose elements are sets are said to be a *family of sets*.
- We can also write $\{A_i | i \in I\}$ for A_i a set and I an index set.
- $\mathcal{F} = \{\{\}, \{\{\}\}, \{\{\{\}\}\}\}$
- Given a set A, the power set is defined: $P(A) = \{x | x \subset A\}$.
- $x \in P(A)$ is equivalent to $x \subset A$ and to $\forall y (y \in x \to y \in A)$.

The power set

- $P(A) \subset P(B)$. Analysis
- $\forall x (x \in P(A) \to x \in P(B))$.
- $\forall x ((\forall y (y \in x \to y \in A)) \to (\forall y (y \in x \to y \in B)))$.
- If $A \subset B$, then is $P(A) \subset P(B)$?
- To check this what should we do? Use our inference rules....

- $A \subset B \vdash \forall x ((\forall y (y \in x \to y \in A)) \to (\forall y (y \in x \to y \in B)))$.
- 1. $\forall x (x \in A \to x \in B)$. A.
- 2: $\forall y (y \in a \to y \in A)$ H.
- 3: $b \in a \to b \in A$.
- 4: $b \in A \to b \in B$.

- 5: $b \in a \to b \in B$.
- 6.: $\forall y(y \in a \to y \in B)$.
- 7. $(\forall y(y \in a \to y \in A)) \to \forall y(y \in a \to y \in B)$. 2-6
- 8. $\forall x((\forall y(y \in x \to y \in A)) \to \forall y(y \in a \to y \in B))$.

- $\forall x((\forall y(y \in x \to y \in A)) \to (\forall y(y \in x \to y \in B))) \vdash A \subset B$.
- 1. $\forall x((\forall y(y \in x \to y \in A)) \to (\forall y(y \in x \to y \in B)))$ A.
- 2.: $a \in A$ H.
- 3.:: $a \in \{a\}$. H (used as a hypothesis)
- 4.:: $a \in A$.
- 5.: $a \in \{a\} \to a \in A$. 3-4
- 6.: $(\forall y(y \in \{a\} \to y \in A)) \to (\forall y(y \in \{a\} \to y \in B))$
- 7.: $(a \in \{a\} \to a \in A) \to (a \in \{a\} \to a \in B)$.
- 8.: $a \in \{a\} \to a \in B$.
- 9.: $a \in \{a\}$ (True statement)
- 9.: $a \in B$.
- 10. $a \in A \to a \in B$.

- $\mathcal{F} = \{C_s | s \in S\}$ a family of sets.
- Define $\bigcup \mathcal{F}$ as the set of elements in at least one element of $\mathcal{F}$.
- $\bigcup \mathcal{F} = \{x | \exists A(A \in \mathcal{F} \wedge x \in A)\} = \{x | \exists A \in \mathcal{F}(x \in A)\}$.
- Define $\bigcap \mathcal{F}$ as the set of common elements of elements of $\mathcal{F}$.
- $\bigcap \mathcal{F} = \{x | \forall A(A \in \mathcal{F} \to x \in A)\} = \{x | \forall A \in \mathcal{F}(x \in A)\}$.
- Alternate notations: $\mathcal{F} = \{A_i | i \in I\}$.
- $\bigcap \mathcal{F} = \bigcap_{i \in I} A_i = \{x | \forall i \in I(x \in A_i)\}$.
- $\bigcup \mathcal{F} = \bigcup_{i \in I} A_i = \{x | \exists i \in I(x \in A_i)\}$.

Example

- $x \in P(\bigcup \mathcal{F})$. Analysis:
- $x \subset \bigcup \mathcal{F}$.
- $\forall y(y \in x \rightarrow y \in \bigcup \mathcal{F})$.
- $\forall y(y \in x \rightarrow \exists A \in \mathcal{F}(y \in A))$.
- Prove that $x \in \mathcal{F} \vdash x \in P(\bigcup \mathcal{F})$.
- $x \in \mathcal{F} \vdash \forall y(y \in x \rightarrow \exists A \in \mathcal{F}(y \in A))$.
- 1. $x \in \mathcal{F}$. A.
- 2.: $a \in x$ H.
- 3.: $\exists A \in \mathcal{F}(a \in A)$.
- 4. $a \in x \rightarrow (\exists A \in \mathcal{F}(a \in A))$ 2-3.
- 5. $\forall y(y \in x \rightarrow (\exists A \in \mathcal{F}(y \in A)))$

Example

- $x \in P(\bigcup \mathcal{F}) \vdash x \in \mathcal{F}$. Is this valid?
- Try to use refutation tree test.
- $x \in P(\bigcup \mathcal{F})$. $x \notin \mathcal{F}$.
- 1. $\forall y(y \in x \rightarrow \exists A \in \mathcal{F}(y \in A))$. 2. $x \notin \mathcal{F}$.
- 1. $\forall y(y \in x \rightarrow \exists A \in \mathcal{F}(y \in A))$. 2. $x \notin \mathcal{F}$. 3. $a \in x \rightarrow \exists A \in \mathcal{F}(a \in A)$.
- 1. $\forall y(y \in x \rightarrow \exists A \in \mathcal{F}(y \in A))$. 2. $x \notin \mathcal{F}$. 3. check $a \in x \rightarrow \exists A \in \mathcal{F}(a \in A)$. 4 (i) $a \notin x$ 4(ii) $\exists A(a \in A \wedge A \in \mathcal{F})$.
- 1. $\forall y(y \in x \rightarrow \exists A \in \mathcal{F}(y \in A))$. 2. $x \notin \mathcal{F}$. 3. check $a \in x \rightarrow \exists A \in \mathcal{F}(a \in A)$. 4 (i) $a \notin x$ open 4(ii) check $\exists A(a \in A \wedge A \in \mathcal{F})$ 5 (ii) $a \in A_0$ 6 (ii) $A_0 \in \mathcal{F}$.
- How do one obtain a counter-example? $x \notin \mathcal{F}$ and $a \notin x$.
- $\mathcal{F} = \{\{1,2\},\{1,3\}\}$. $x = \{1,2,3\}$. $a = 4$.
- $\mathcal{F} = \{\{1,2\},\{1,3\}\}$. $x = \{1,2,3\}$. $a = 3$. $a \in \{1,3\}$. $\{1,3\} \in \mathcal{F}$.