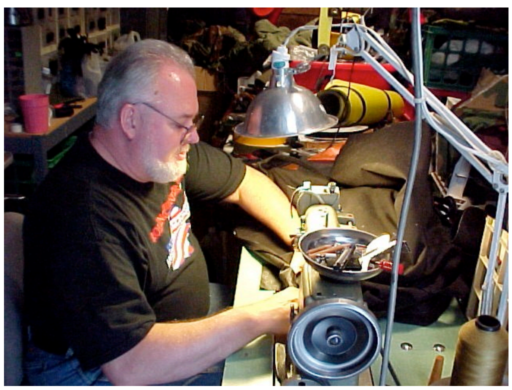
1 Introduction

Outline

- Topological operations on 2-orbifolds: constructions and decompositions
  - Classifications of 1-dimensional suborbifolds of 2-orbifolds
  - Definition of splitting and sewing of 2-orbifolds
  - Regular neighborhoods of 1-orbifolds
  - Reinterpretation of splitting and sewing.
  - Identification interpretations of splitting and sewing

Outline

- 

Some helpful references

- S. Choi and W. Goldman, The deformation spaces of convex RP 2 -structures on 2-orbifolds American Journal of Mathematics 127, 5, 1019–1102 (2005)
- Y. Matsumoto and J. Montesinos-Amilibia, A proof of Thurston's uniformization theorem of geometric orbifolds, Tokyo J. Mathematics 14, 181–196 (1991)

2 Main

2.1 Classifications of 1-dimensional suborbifolds of 2-orbifolds

2-orbifolds

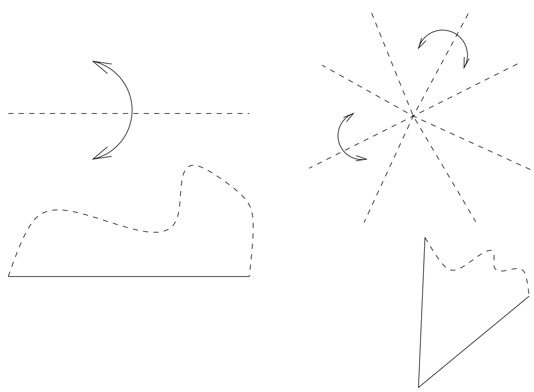
- Recall that 2-orbifold have three types of singularities: silvered points in open arcs, isolated cone-points, and isolated corner-reflector points. The singular points of a two-dimensional orbifold fall into three types:
  (i) The mirror point: $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflections on the y-axis.

(ii) The cone-points of order n: $\mathbb{R}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acting by rotations by angles $2\pi m/n$ for integers m.

(iii) The corner-reflector of order n: $\mathbb{R}^2/D_n$ where D_n is the dihedral group generated by reflections about two lines meeting at an angle π/n.

2-orbifolds

- The actions here are isometries on $\mathbb{R}^2$.



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The triangulations of 2-orbifolds and classification

- Theorem: Any 2-orbifold is obtained from a smooth surface with corner by silvering some arcs and putting cone-points and corner-reflectors.
- A 2-orbifold is classified by the underlying smooth topology of the surface with corner and the number and orders of cone-points, corner-reflectors, and the boundary pattern of silvered arcs.
- proof: basically, strata-preserving isotopies.

Classifications of 1-dimensional suborbifolds of 2-orbifolds

- A *suborbifold* Q' on a subspace $X_{Q'} \subset X_Q$ is the subspace so that each point of $X_{Q'}$ has a neighborhood in X_Q modeled on an open subset U of $\mathbb{R}^n$ with a finite group Γ preserving $U \cap \mathbb{R}^d$ where $\mathbb{R}^d \subset \mathbb{R}^n$ is a proper subspace, so that $(U \cap \mathbb{R}^d, \Gamma')$ is in the orbifold structure of Q'.
- Here Γ' denotes the restricted group of Γ to $U \cap \mathbb{R}^d$, which is in general a quotient group.

Classifications of 1-dimensional suborbifolds of 2-orbifolds

- A compact 1-orbifold is either a closed arc, a segment, a segment with one silvered endpoint, or a segment with two silvered end-point.
- Properly and nicely imbedded 1-orbifolds in a 2-orbifold with boundary. (nice means that only boundary goes to boundary.)
  - No silvered-point case: An imbedded closed arc avoiding boundary or a segment with two endpoints in the boundary
  - One silvered-point case: A segment with silvered endpoint at a cone-point of order two or a silvered arc and the other endpoint in the boundary.
  - Two silvered-point case: A segment with silvered endpoints at cone-points or order two or in silvered arcs.

Orbifold Euler-characteristic for 2-orbifolds due to Satake

- We define the Euler characteristic to be

$$\chi(X) = \sum_{c_i} (-1)^{\dim(c_i)} (1/|\Gamma(c_i)|),$$

  where c_i ranges over the open cells and $|\Gamma(c_i)|$ is the order of the group Γ_i associated with c_i.
- If X is finitely covered by another orbifold X', then $\chi(X') = r\chi(X)$ where r is the number of sheets for regular points. This follows since the sum of the order of local groups in the inverse image of the elementary neighborhood is always r.
- The Euler-characteristic of 1-orbifold: a circle O, a segment 1, a segment with one silvered-point $1/2$, a full 1-orbifold O.

Orbifold Euler-characteristic for 2-orbifolds due to Satake

- For 2-orbifolds Σ_1, Σ_2 meeting in a compact 1-orbifold Y in the interior forming a 2-orbifold Σ as a union, we have the following additivity formula:

$$\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2) - \chi(Y), \tag{1}$$

- To be verified by counting cells with weights since the orders of singular points in the boundary orbifold equal the ambient orders.

Orbifold Euler-characteristic for 2-orbifolds due to Satake

- Suppose that a 2-orbifold Σ with or without boundary has the underlying space X_Σ and m cone-points of order q_i and n corner-reflectors of order r_j and n_Σ boundary full 1-orbifolds.

- Then the following generalized Riemann-Hurwitz formula is very useful:

$$\chi(\Sigma) = \chi(X_\Sigma) - \sum_{i=1}^{m}\left(1 - \frac{1}{q_i}\right) - \frac{1}{2}\sum_{j=1}^{n}\left(1 - \frac{1}{r_j}\right) - \frac{1}{2}n_\Sigma, \qquad (2)$$

  which is proved by a doubling argument and cutting and pasting.

2.2 Definition of Splitting and sewing 2-orbifolds

Definition of Splitting and sewing 2-orbifolds

- Let S be a very good orbifold so that its underlying space X_S is a pre-compact open surface with a path-metric admitting a compactification to a surface with boundary.
- Let $\hat{S}$ be a very good cover, that is, a finite regular cover, of S, so that S is orbifold-diffeomorphic to $\hat{S}/F$ where F is a finite group acting on $\hat{S}$.
- Since $X_{\hat{S}} = \hat{S}$ is also pre-compact and has a path-metric, complete it to obtain a compact surface $X'_{\hat{S}}$.
- $X'_{\hat{S}}/F$ with the quotient orbifold structure is said to be the *orbifold-completion* of S.
- Let S be a 2-orbifold with an embedded circle or a full 1-orbifold l in the interior of S. The completion S' of $S - l$ is said to be obtained from *splitting* S along l. Since $S - l$ has an embedded copy in S', we see that there exists a map $S' \to S$ sending the copy to $S - l$. Let l' denote the boundary component of S corresponding to l under the map.
- Conversely, S is said to be obtained from *sewing* S' *along* l'.
- If the interior of the underlying space of l lies in the interior of the underlying space of S, then the components of S' are said to be *decomposed components of* S *along* l, and we also say that S *decomposes* into S' along l.
- Of course, if l is a union of disjoint embedded circles or full 1-orbifolds, the same definition holds.

Silvering and clarifying

- There are two distinguished classes of splitting and sewing operations:
- A simple closed curve boundary component can be made into a set of mirror points and conversely in a unique manner.
- a boundary point has a neighborhood which is realized as a quotient of an open ball by a $\mathbb{Z}_2$-action generated by a reflection about a line.

- A boundary full 1-orbifold can be made into a 1-orbifold of mirror points and two corner-reflectors of order two and conversely in a unique manner: (a boundary point has a neighborhood which is a quotient space of a dihedral group of order four acting on the open ball generated by two reflections.)
- The forward process is called *silvering* and the reverse process *clarifying*.

2.3 Regular neighborhoods of 1-orbifold

The classification of Euler-characteristic zero orbifold

- Let A be a compact annulus with boundary. The quotient orbifold of an annulus has Euler characteristic zero.
- From Riemann-Hurwitz equation, all of the Euler characteristic zero 2-orbifolds with nonempty boundary:

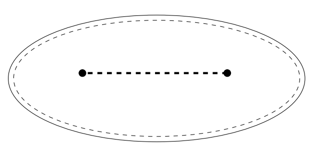
  (1) an annulus, (2) a Möbius band, (3) an annulus with one boundary component silvered (a *silvered annulus*),

  (4) a disk with two cone-points of order two with no mirror points (a $(;2,2)$-*disk*),

  (5) a disk with two boundary 1-orbifolds, two edges (*a silvered strip*),

  (6) a disk with one cone-point and one boundary full 1-orbifold (a *bigon with a cone-point of order two*), that is, it has only one edge, and

  (7) a disk with two corner-reflectors of order two and one boundary full 1-orbifold (a *half-square*). (It has three edges.)



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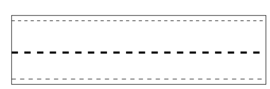

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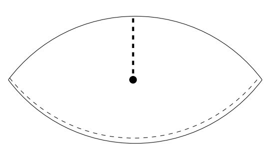
- 

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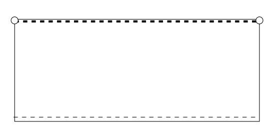

(7)

Proof

- To prove this, notice that the underlying space must have a nonnegative Euler characteristic and Riemann-Hurwitz formula.
- When the Euler characteristic of the space is zero, there are no cone-points, corner-reflectors, (1)(2)(3).

- Now the underlying space is a disk.
- No singular points in the boundary. Then (4) as there has to be exactly two cone-points of order two.
- If two boundary full 1-orbifolds, then no singular points in the interior and no corner-reflector can exist; thus, (5)
- Exactly one boundary full 1-orbifold.
- If a cone-point, then it has to be a unique one and of order two. (6) If there are no cone-points, but corner-reflectors, then exactly two corner-reflectors of order two and no more. (7)

Regular neighborhoods of 1-orbifold

- A circle or a 1-orbifold l in the interior of a 2-orbifold S, not homotopic to a point.
- l has a neighborhood of zero Euler characteristic considering its good cover.
- Since the inverse image of l consists of closed curves which represent generators.
  - For (1) and (2), l is the closed curve representing the generator of the fundamental group;
  - For (3), l is the mirror set that is a boundary component;
  - For (4), l is the arc connecting the two cone-points unique up to homotopy;
  - For (5), l is an arc connecting two interior points of two edges respectively;
  - For (6), l is an arc connecting an interior point of an edge and the cone-point of order two;
  - For (7), the edge in the topological boundary connecting the two corner-reflectors of order two.

Regular neighborhoods of 1-orbifold

- Given a 1-orbifold l and a neighborhood N of it in some ambient 2-orbifold, N is said to be a *regular neighborhood* if the pair (N, l) is diffeomorphic to one of the above.
- A 1-orbifold in a good 2-orbifold has a regular neighborhood which is unique up to isotopy.

Regular neighborhoods of 1-orbifold

- proof:
- The existence is proved above. The uniqueness up to isotopy is proved as follows:
- Each regular neighborhood fibers over a 1-orbifold with fibers connected 1-orbifolds in the orbifold sense.
- A regular neighborhood can be isotoped into any other regular neighborhood by contracting in the fiber directions.
- To see this, we can modify the proof of Theorem 5.3 in Chapter 4 of Hirsch to be adopted to an annulus with a finite group acting on it and an imbedded circle.

2.4 Splitting and sewing on 2-orbifolds reinterpreted

Splitting and sewing on 2-orbifolds reinterpreted

- Let l be a 1-orbifold embedded in the interior of an orbifold S.
- If one removes l from the interior of a regular neighborhood, we obtain either a union of one or two open annuli, or a union of one or two open silvered strip.
- In (2)-(4), an open annulus results. For (1), a union of two open annuli results. For (6)-(7), an open silvered strip results. For (5), we obtain a union of two open silvered strips.

Splitting and sewing on 2-orbifolds reinterpreted

- These can be easily completed to be a union of one or two compact annuli or a union of one or two silvered strips respectively.
- We can complete $S - l$ in this manner: We take a closed regular neighborhood N of l in S.
- We remove $N - l$ to obtain the above types and complete it and re-identify with $S - l$ to obtain a compactified orbifold. This process is the splitting of S along l.

Splitting and sewing on 2-orbifolds reinterpreted

- Conversely, we can describe sewing: Take an open annular 2-orbifold N which is a regular neighborhood of a 1-orbifold l.
- Suppose that l is a circle. We obtain $U = N - l$ which is a union of one or two annuli.
- Take an orbifold S' with a union l' of one (resp. two) boundary components which are circles.

- Take an open regular neighborhood of l' and remove l' to obtain V.
- U and V are the same orbifold. We identify $S' - l'$ and $N - l$ along U and V.
- This gives us an orbifold S, and it is easy to see that S is obtained from S' by sewing along l'.

Splitting and sewing on 2-orbifolds reinterpreted

- l corresponds to a 1-orbifold l'' in S in a one-to-one manner. We can obtain (1),(2),(3)-type neighborhoods of l'' in this way.
- The operation in case (1) is said to be *pasting*, in case (2) *cross-capping*, and in case (3) *silvering* along simple closed curves.

Splitting and sewing on 2-orbifolds reinterpreted

- Suppose that l is a full 1-orbifold. $U = N - l$ is either an open annulus or a union of one (resp. two) silvered strips.
- The former happens if N is of type (4) and the latter if N is of type (5)-(7).
- In case (4), take an orbifold S' with a boundary component l' a circle. Then we can identify U with a regular neighborhood of l' removed with l' to obtain an orbifold S. Then l corresponds a full 1-orbifold l'' in S in a one-to-one manner. l'' has a type-(4) regular neighborhood. The operation is said to be *folding* along a simple closed curve.

Splitting and sewing on 2-orbifolds reinterpreted

- In the remaining cases, take an orbifold S' with a union l' of one (resp. two) boundary full 1-orbifolds. Take a regular neighborhood N of l' and remove them to obtain V. Identify U with V for $S' - l'$ and $N - l$ to obtain S. Then S is obtained from S' by sewing along l'. Again l corresponds to a full 1-orbifold l'' in S in a one-to-one manner.
- We obtain (5),(6), and (7)-type neighborhoods of l'' in this way, where the operations are said to be *pasting*, *folding*, and *silvering* along full 1-orbifolds respectively.
- In other words, silvering is the operation of removing a regular neighborhood and replacing by a silvered annulus or a half square. Clarifying is an operation of removing the regular neighborhood and replacing an annulus or a silvered strip.

Splitting and sewing on 2-orbifolds reinterpreted

- The Euler characteristic of an orbifold before and after splitting or sewing remains unchanged.
- proof: Form regular neighborhoods of the involved boundary components of the split orbifold and those of the original orbifold. They have zero Euler characteristic. Since their boundary 1-orbifolds have zero Euler characteristic, the lemma follows by the additivity formula (1).

2.5 Identification interpretations of splitting and sewing

Identification interpretations of splitting and sewing

- In the following we describe the topological identification process of the underlying space involved in these six types of sewings. The orbifold structure on the sewed orbifold should be clear.
- Let an orbifold Σ have a boundary component b. (Σ is not necessarily connected.) b is either a simple closed curve or a full 1-orbifold. We find a 2-orbifold Σ'' constructed from Σ by sewing along b or another component of Σ.

- (A) Suppose that b is diffeomorphic to a circle; that is, b is a closed curve. Let Σ' be a component of the 2-orbifold Σ with boundary component b'. Suppose that there is a diffeomorphism $f : b \to b'$. Then we obtain a bigger orbifold Σ'' glued along b and b' topologically.

  (I) The construction so that Σ'' does not create any more singular point results in an orbifold Σ'' so that

  $$\Sigma'' - (\Sigma - b \cup b')$$

  is a circle with neighborhood either diffeomorphic to an annulus or a Möbius band.

  (1) In the first case, $b \neq b'$ (pasting).
  (2) In the second case, $b = b'$ and $\langle f \rangle$ is of order two without fixed points (cross-capping).

(II) When $b = b'$, the construction so that Σ'' does introduce more singular points to occur in an orbifold Σ'' so that

$$\Sigma'' - (\Sigma - b)$$

is a circle of mirror points or is a full 1-orbifold with endpoints in cone-points of order two depending on whether $f : b \to b$

(1) is the identity map (silvering), or

(2) is of order two and has exactly two fixed points (folding).

- (B) Consider when b is a full 1-orbifold with endpoints mirror points.

  (I) Let Σ' be a component orbifold (possibly the same as one containing b) with boundary full 1-orbifold b' with endpoints mirror points where $b \neq b'$. We obtain a bigger orbifold Σ'' by gluing b and b' by a diffeomorphism $f : b \to b'$. This does not create new singular points (pasting).

  (II) Suppose that $b = b'$. Let $f : b \to b$ be the attaching map. Then

    (1) if f is the identity, then b is silvered and the end points are changed into corner-reflectors of order two (silvering).

    (2) If f is of order two, then Σ'' has a new cone-point of order two and has one-boundary component orbifold removed away. b corresponds to a mixed type 1-orbifold in Σ' (folding).

  – It is obvious how to put the orbifold structure on Σ'' using the previous descriptions using regular neighborhoods above.