1 Introduction

Outline

- The deformation space of (X, G)-structures on an orbifold.
  - Definition
  - The local homeomorphism theorem
    - The isotopy lemma
    - Proof.

Some helpful references

- S. Choi, Geometric structures on orbifolds and holonomy representations, Geometriae Dedicata 104: 161 – 199, 2004.
- W. Thurston, Lecture notes...: A chapter on orbifolds, 1977.
- R. Canary, D. Epstein, and P. Green, Notes on notes of Thurston, In: London Math. Soc. Lecture Note Ser. 111, Cambridge University Press, Cambridge, 1987, pp. 3 – 92.
- W. Lok, Deformations of locally homorgeneous spaces and kleinian groups, PhD Thesis, Columbia University, 1984.

2 Definition

Definition of the deformation space of (X, G)-structures on orbifolds

- Consider the set $\mathcal{M}(M)$ of all (X, G)-structures on an orbifold M.
- We introduce an equivalence relation $\sim$: two (X, G)-structures μ_1 and μ_2 are equivalent if there is an isotopy $\phi : M \to M$ so that $\phi^*(\mu_1) = \mu_2$.
- The deformation space of (X, G)-structures on M is $\mathcal{M}/\sim$.
- We reinterpret the space as
  - The set of diffeomorphisms $f : M \to M'$ for M an orbifold and M' an (X, G)-orbifold.
  - The equivalence relation $f : M \to M'$ and $g : M \to M$" if exists an (X, G)-diffeomorphism $h : M' \to M$" so that $h \circ f$ is isotopic to g.
  - The quotient space is same as above.

Another interpretations

- Identify $\pi_1(M)$ with $\pi_1(M \times I)$.
- Consider the set of diffeomorphisms $f : \tilde{M} \to \tilde{M}'$ equivariant with respect to isomorphism $f_* : \pi_(M) \to \pi_1(M')$ for an (X, G)-orbifold M'.
- We introduce an equivalence relation: Given $f : \tilde{M} \to \tilde{M}'$ and $g : \tilde{M} \to \tilde{M}$", we say that they are equivalent if there exists an (X, G)-map $\phi : \tilde{M}' \to \tilde{M}$" so that $\phi \circ f$ is isotopic to g by an isotopy $\tilde{M} \times I \to \tilde{M}''$ equivariant with respect to both $\phi_* \circ f_*$ and g_* which are equal.
- Denote this set by $\mathcal{D}_I(M)$.
- This set is again one-to-one relation with the above space since we can always lift diffeomorphisms and isotopies.

Isotopy-equivalence space.

- $\mathcal{S}(M)$ is defined as follows.
- Consider the set of $(D, \tilde{f} : \tilde{M} \to \tilde{M}')$ where $f : M \to M'$ is a diffeomorphism for orbifolds M and M' and $D : \tilde{M}' \to X$ is a diffeomorphism equivariant with respect to a homomorphism $h : \pi_1(M') \to G$.
- Two $(D, \tilde{f})$ and $(D', \tilde{f}' : \tilde{M} \to \tilde{M}$") are equivalent if there is a diffeomorphism $\phi : M' \to M''$ so that $D' \circ \tilde{\phi} = D$ and an isotopy $H : M \times I \to M''$ equivariant with respect to $\tilde{f}'_* : \pi_1(M) \to \pi_1(M'')$ so that $\phi \circ f = H_0$ and $f' = H_1$.
- We can finally give topology on this space by C_1 topology using $D \circ \tilde{f}$.

The topology of the deformation space

- There is a natural action of G on $\mathcal{S}(M)$ given by $g(D, \tilde{f}) = (g \circ D, \tilde{f}), g \in G$.
- The quotient space $\mathcal{D}(M)$ is the deformation space.
- Proof:
  - We show $\mathcal{D}_I(M)$ is one-to-one equivalent to $\mathcal{S}(M)/G$.
  - Given an element $\tilde{f} : \tilde{M} \to \tilde{M}'$, there is a developing map $D : \tilde{M}' \to X$ equivariant with respect to $h : \pi_1(M') \to G$
  - If $\tilde{f} : \tilde{M} \to \tilde{M}'$ and $\tilde{f}' : \tilde{M} \to \tilde{M}$" are equivalent, then there is an (X, G)-diffeomorphism $M' \to M$" and hence two global charts D' and D" differ only by an element of G.
  - Conversely, given $(D, \tilde{f})$, we obviously obtain an (X, G)-structure on M'.
  - If $(D, \tilde{f})$ and $(D', \tilde{f}')$ are equivalent, then there is a diffeomorphism $\phi : M' \to M''$ so that $D' \circ \tilde{\phi} = g \circ D$. This means $\phi' : M' \to M''$ is an (X, G)-diffeomorphism.

2.1 The local homeomorphism theorem

The representation space

- Suppose that π is finitely-presented. In particular if M is a compact n-orbifold, this is true.
- Denote by $g_1, ..., g_n$ the set of generators and $R_1, ..., R_m$ be the set of relations.
- The set of homomorphisms $\pi_1(M) \to G$ can be identified with a subset of G^n by sending a homomorphism h to $(h(g_1), ..., h(g_n))$. This clearly injective map.
- This image can be described as an algebraic subset defined by relations $R_1, ..., R_m$.
- This follows since if the relation is satisfied, then we can obtain the representation conversely.
- Denote the space by $Hom(\pi, G)$.
- There is an action of G on $Hom(\pi, G)$ given by the action $(g \star h)(\cdot) = gh(\cdot)g^{-1}$
- We denote by $Rep(\pi, G)$ the quotient space $Hom(\pi, G)/G$.

The map hol

- We can define $PH : \mathcal{S}(M) \to Hom(\pi, G)$. The main purpose of this section is to show that PH is a local homeomorphism.
- We send $(D, \tilde{f})$ to the associated homomorphism $h : \pi \to G$.
- PH is continuous: If $D' \circ \tilde{f}'$ is sufficiently close to $D \circ f$ in a sufficiently large compact subset of $\tilde{M}$, then the holonomy $h'(g_i)$ of generators g_i is as close to the original $h(g_i)$ as possible.
- The local homeomorphism result was very important for the study of deformations of (X, G)-structures on manifolds, as first observed by Weil. The same can be said for orbifolds.
- For manifolds, Thurston gave a proof. Later J. Morgan gave a lecture of it, which is written up by in his Ph.D. thesis. Also, Canary and Epsten gave a proof of it also.

The stable representations

- There is a dense open subset, called the stable subset, of $Hom(\pi, G)$ where G acts properly. Denote this space by $Hom^s(\pi, G)$ and its quotient by $Rep^s(\pi, G)$.
- If we denote by $\mathcal{D}^s(M)$ the subset of $\mathcal{D}$ whose holonomies are in the stable region. Then there is a local homeomorphism $hol : \mathcal{D}^s(M) \to Rep^s(\pi, G)$ since the right action on developing map gives a conjugation action on holonomy homomorphisms.

2.2 The proof of the local homeomorphism

The isotopy lemma

- Each point of $Hom(G_x, G)$ has a neighborhood S which is a cone over a semi-algebraic set. Here G_x is a stabilizer group or any other finitely presented group.
- Let F be a compact subset of an open ball B. Let G_B be a finite group acting on F and B. Let S be a cone-neighborhood of an element of $Hom(G_B, G)$.
  - Let $H : F \times [0, \epsilon] \times S \to X$ be a map such that $H(h) : F \times [0, \epsilon'] \to X$ is a G_B equivariant isotopy for each $h' \in S$.
  - Then there exists an extention $H' : B \times [0, \epsilon''] \to X$ which is a G_B equivariant isotopy.

The outline of the proof

- Step I: we realize the orbifold M as a union of model open sets. We choose covers $\{U_1, ..., U_k\}$, $\{W_1, ..., W_k\}$, $\{V_1, ..., V_k\}$ such that $Cl(U_i) \subset W_i$, $Cl(W_i) \subset V_i$: One can choose nice covering which are model neighborhoods.
- Step II: We define a section $s : O \subset Hom(\pi_1(M_0), G) \to \mathcal{S}(M_0)$ so that $PH \circ s$ is the identity. This is accomplished by deforming model neighborhoods and patching conjugating diffeomorphisms.
- Step III: We show that PH is locally injective. From this it follows that PH is a local homeomorphism.

Step II

- We choose O as a cone-neighborhood of a point in the representation space.
- We identify V_i with open subsets of X and find the conjugating diffeomorphisms.
- We identified the deformed V_i by V_i'. If we choose V_is well and take sufficiently small deformations, then the result is an orbifold, call it M'.
- We will show that M' is diffeomorphic to M.
  - We can use U_is to obtain M' as well.
  - Now defined maps from the highest intersection set of U_i. The number will be $n + 1$ if M has dimension n.
  - Define inductively from $n + 1$-intersections and n-intersections and so on.
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The step III

- In this step, we show PH is locally injective.
  - We control the size of O so that for sufficiently small O, $s(O)$ is contained in any given small open neighborhood of original (X, G)-isotopy class.
  - Next, if the image under PH is the same two points for a small open neighborhood of $\mathcal{S}(M)$, then we can find an orbifold-diffeomorphism between the two that is very close to identity in the C_1-sense.
  - Now use the exponential map on $\tilde{M}$ to obtain an isotopy of this map to identity.
  - This proves the injectivity.
- The local homeomorphism property of PH follows by seeing that the section s can be defined on open subsets which maps into a given neighborhood. Thus, an image of an open set is open.