1 Introduction

Preliminary

- Course home page: http://math.kaist.ac.kr/~schoi/GT2010.html
  Old: http://mathsci.kaist.ac.kr/~schoi/dgorb.htm

Some advanced references for the course

- W. Thurston, Lecture notes...: A chapter on orbifolds, 1977. (This is the principal source)
- W. Thurston, Three-dimensional geometry and topology, PUP, 1997
- R.W. Sharp, Differential geometry: Cartan’s generalization of Klein’s Erlangen program.
- T. Ivey and J.M. Landsberg, Cartan For Beginners: Differential geometry via moving frames and exterior differential systems, GSM, AMS
- M. Berger, Geometry I, Springer

Outline of the orbifolds part

- See the book introduction also.
- Manifolds and differentiable structures: Background materials..
- Lie groups and geometry: Geometry and discrete groups
- Topology of orbifolds: topology and covering spaces.
- The topology of 2-orbifolds: cutting and pasting, classification (not complete yet)
- The geometry of orbifolds
- The deformation space of hyperbolic structures on 2-orbifolds.
- Note that the notes are incomplete... I will try to correct as we go along. The orders may change...
Helpful preliminary knowledge for this chapter:

- Hatcher’s "Algebraic topology" Chapters 0,1. (better with Chapter 2...) [http://www.math.cornell.edu/~hatcher/AT/ATpage.html](http://www.math.cornell.edu/~hatcher/AT/ATpage.html)
- "Introduction to differentiable manifolds" by Munkres
- "Riemannian manifolds" by Do Carmo.
- R. Bishop and R. Crittendon, Geometry of manifolds.
- W. Thurston, Three-dimensional geometry and topology, Princeton Univ. press.
- W. Thurston, Geometry and Topology of 3-manifolds [http://www.msri.org/publications/books/gt3m](http://www.msri.org/publications/books/gt3m)

Section 1: Manifolds and differentiable structures (Intuitive account)

- Manifolds
- Simplicial manifolds
- Lie groups.
- Pseudo-groups and $G$-structures.
- Differential geometry and $G$-structures.
- Principal bundles and connections, flat connections

Section 2: Lie groups and geometry

- Projective geometry and conformally flat geometry
- Euclidean geometry
- Spherical geometry
- Hyperbolic geometry and three models
- Discrete groups: examples

2 Manifolds and differentiable structures (Intuitive account)

2.1 Aim

- The following theories for manifolds will be transferred to the orbifolds. We will briefly mention them here as a "review" and will develop them for orbifolds later (mostly for 2-dimensional orbifolds).
We follow coordinate-free approach to differential geometry. We do not need to actually compute curvatures and so on.

- \( \mathcal{G} \)-structures
- Covering spaces
- Riemannian manifolds and constant curvature manifolds
- Lie groups and group actions
- Principal bundles and connections, flat connections

2.2 Manifolds

Topological spaces.

- Quotient topology
- We will mostly use cell-complexes: Hatcher’s AT P. 5-7 (Consider finite ones for now.)
- Operations: products, quotients, suspension, joins; AT P.8-10

Manifolds.

- A topological \( n \)-dimensional manifold (\( n \)-manifold) is a Hausdorff space with countable basis and charts to Euclidean spaces \( E^n \); e.g curves, surfaces, 3-manifolds.
- The charts could also go to a positive half-space \( H^n \). Then the set of points mapping to \( \mathbb{R}^{n-1} \) under charts is well-defined is said to be the boundary of the manifold. (By the invariance of domain theorem)
- \( \mathbb{R}^n, H^n \) themselves or open subsets of \( \mathbb{R}^n \) or \( H^n \).
- \( S^n \) the unit sphere in \( \mathbb{R}^{n+1} \). (use http://en.wikipedia.org/wiki/Stereographic_projection
- \( \mathbb{R}P^n \) the real projective space. (use affine patches)

Manifolds.

- An \( n \)-ball is a manifold with boundary. The boundary is the unit sphere \( S^{n-1} \).
- Given two manifolds \( M_1 \) and \( M_2 \) of dimensions \( m \) and \( n \) respectively. The product space \( M_1 \times M_2 \) is a manifold of dimension \( m + n \).
- An annulus is a disk removed with the interior of a smaller disk. It is also homeomorphic to a circle times a closed interval.
- The \( n \)-dimensional torus \( T^n \) is homeomorphic to the product of \( n \) circles \( S^1 \).
More examples

- Let $T_n$ be a group of translations generated by $T_i : x \mapsto x + e_i$ for each $i = 1, 2, \ldots, n$. Then $\mathbb{R}^n / T_n$ is homeomorphic to $T^n$.

- A connected sum of two $n$-manifolds $M_1$ and $M_2$. Remove the interiors of two closed balls from $M_i$ for each $i$. Then each $M_i$ has a boundary component homeomorphic to $S^{n-1}$. We identify the spheres.

- Take many 2-dimensional tori or projective plane and do connected sums. Also remove the interiors of some disks. We can obtain all compact surfaces in this way. [http://en.wikipedia.org/wiki/Surface](http://en.wikipedia.org/wiki/Surface)

2.3 Discrete group actions

Some homotopy theory (from Hatcher’s AT)

- $X, Y$ topological spaces. A homotopy is a $f : X \times I \rightarrow Y$.

- Maps $f$ and $g : X \rightarrow Y$ are homotopic if $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$ for all $x$. The homotopic property is an equivalence relation.

- Homotopy equivalences of two spaces $X$ and $Y$ is a map $f : X \rightarrow Y$ with a map $g : Y \rightarrow X$ so that $f \circ g$ and $g \circ f$ are homotopic to $I_X$ and $I_Y$ respectively.

Fundamental group (from Hatcher’s AT)

- A path is a map $f : I \rightarrow X$.

- A linear homotopy in $\mathbb{R}^n$ for any two paths.

- A homotopy class is an equivalence class of homotopic map relative to endpoints.

- The fundamental group $\pi (X, x_0)$ is the set of homotopy class of path with endpoints $x_0$.

- The product exists by joining. The product gives us a group.

- A change of base-points gives us an isomorphism (not canonical)

- The fundamental group of a circle is $\mathbb{Z}$. Brouwer fixed point theorem

- Induced homomorphisms. $f : X \rightarrow Y$ with $f(x_0) = y_0$ induces $f_* : \pi (X, x_0) \rightarrow \pi (Y, y_0)$. 
Van Kampen Theorem (AT page 43–50)

- Given a space $X$ covered by open subsets $A_i$ such that any two or three of them meet at a path-connected set, $\pi(X, \ast)$ is a quotient group of the free product $\ast \pi(A_i, \ast)$.
- The kernel is generated by $i_j^*(a)i_k^*(a)$ for any $a$ in $\pi(A_i \cap A_j, \ast)$.
- For cell-complexes, these are useful for computing the fundamental group.
- If a space $Y$ is obtained from $X$ by attaching the boundary of 2-cells. Then $\pi(Y, \ast) = \pi(X, \ast)/N$ where $N$ is the normal subgroup generated by "boundary curves" of the attaching maps.
- Bouquet of circles, surfaces,...

Covering spaces and discrete group actions

- Given a manifold $M$, a covering map $p : \tilde{M} \to M$ from another manifold $\tilde{M}$ is an onto map such that each point of $M$ has a neighborhood $O$ s.t. $p|p^{-1}(O) : p^{-1}(O) \to O$ is a homeomorphism for each component of $p^{-1}(O)$.
- The coverings of a circle.
- Consider a disk with interiors of disjoint smaller disks removed. Cut remove edges and consider...
- The join of two circles example: See Hatcher AT P.56–58
- Homotopy lifting: Given two homotopic maps to $M$, if one lifts to $\tilde{M}$ and so does the other.
- Given a map $f : Y \to M$ with $f(y_0) = x_0$, $f$ lifts to $\tilde{f} : Y \to \tilde{M}$ so that $\tilde{f}(y_0) = \tilde{x}_0$ if $f_* \pi(Y, y_0) \subset p_* (\pi_*(\tilde{M}, \tilde{x}_0))$.

Covering spaces and discrete group actions

- The automorphism group of a covering map $p : M' \to M$ is a group of homeomorphisms $f : M' \to M'$ so that $p \circ f = f$. (also called deck transformation group.)
- $\pi_1(M)$ acts on $\tilde{M}$ on the right by path-liftings.
- A covering is regular if the covering map $p : M' \to M$ is a quotient map under the action of a discrete group $\Gamma$ acting properly discontinuously and freely. Here $M$ is homeomorphic to $M'/\Gamma$.
- One can classify covering spaces of $M$ by the subgroups of $\pi(M, x_0)$. That is, two coverings of $M$ are equal iff the subgroups are the same.
- Covering spaces are ordered by subgroup inclusion relations.
- If the subgroup is normal, the corresponding covering is regular.
• A manifold has a universal covering, i.e., a covering whose space has a trivial fundamental group. A universal cover covers every other coverings of a given manifold.

• \( \tilde{M} \) has the covering automorphism group \( \Gamma \) isomorphic to \( \pi_1(M) \). A manifold \( M \) equals \( \tilde{M}/\Gamma \) for its universal cover \( \tilde{M} \). \( \Gamma \) is a subgroup of the deck transformation group.

  – Let \( \tilde{M} \) be \( \mathbb{R}^2 \) and \( T^2 \) be a torus. Then there is a map \( p : \mathbb{R}^2 \to T^2 \) sending \( (x, y) \) to \( ([x], [y]) \) where \( [x] = x \mod 2\pi \) and \( [y] = y \mod 2\pi \).

  – Let \( M \) be a surface of genus 2. \( \tilde{M} \) is homeomorphic to a disk. The deck transformation group can be realized as isometries of a hyperbolic plane.

2.4 Simplicial manifolds

Simplicial manifolds

• An \( n \)-simplex is a convex hull of \( n + 1 \)-points (affinely independent). An \( n \)-simplex is homeomorphic to \( B^n \).

• A simplicial complex is a locally finite collection \( S \) of simplices so that any face of a simplex is a simplex in \( S \) and the intersection of two elements of \( S \) is an element of \( S \). The union is a topological set, a polyhedron.

• We can define barycentric subdivisions and so on.

• A link of a simplex \( \sigma \) is the simplicial complex made up of simplices opposite \( \sigma \) in a simplex containing \( \sigma \).
• An \(n\)-manifold \(X\) can be constructed by gluing \(n\)-simplices by face-identifications. Suppose \(X\) is an \(n\)-dimensional triangulated space. If the link of every \(p\)-simplex is homeomorphic to a sphere of \((n - p - 1)\)-dimension, then \(X\) is an \(n\)-manifold.

• If \(X\) is a simplicial \(n\)-manifold, we say \(X\) is orientable if we can give an orientations on each simplex so that over the common faces they extend each other.

2.5 Surfaces

Surfaces

Canonical construction
Given a polygon with even number of sides, we assign identification by labeling by alphabets \(a_1, a_2, \ldots, a_1^{-1}, a_2^{-1}, \ldots\), so that \(a_i\) means an edge labelled by \(a_i\) oriented counter-clockwise and \(a_i^{-1}\) means an edge labelled by \(a_i\) oriented clockwise. If a pair \(a_i\) and \(a_i^{-1}\) occur, then we identify them respecting the orientations.

• A bigon: We divide the boundary into two edges and identify by labels \(a, a^{-1}\).

• A square: We identify the top segment with the bottom one and the right side with the left side. The result is a 2-torus.

• Any closed surface can be represented in this manner.

• A \(4n\)-gon. We label edges 
\[
a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_n, b_n, a_n^{-1}, b_n^{-1}.
\]
The result is a connected sum of \(n\) tori and is orientable. The genus of such a surface is \(n\).

• A \(2n\)-gon. We label edges \(a_1a_1a_2b_2\ldots a_nb_n\). The result is a connected sum of \(n\) projective planes and is not orientable. The genus of such a surface is \(n\).

• The results are topological surfaces and a 2-dimensional simplicial manifold.

• We can remove the interiors of disjoint closed balls from the surfaces. The results are surfaces with boundary.
The fundamental group of a surface can now be computed. A surface is a cell complex starting from a 1-complex which is a bouquet of circles and attached with a cell. (See AT P.51)

\[ \pi(S) = \{a_1, b_1, ..., a_g, b_g | [a_1, b_1][a_2, b_2]...[a_g, b_g] \} \]

for orientable \( S \) of genus \( g \).

An Euler characteristic of a simplicial complex is given by \( E - F + V \). This is a topological invariant. We can show that the Euler characteristic of an orientable compact surface of genus \( g \) with \( n \) boundary components is \( 2 - 2g - n \).

In fact, a closed orientable surface contains \( 3g - 3 \) disjoint simple closed curves so that the complement of its union is a disjoint union of pairs of pants, i.e., spheres with three holes. Thus, a pair of pants is an "elementary" surface.
2.6 Lie groups

Section 1: Lie groups

- A Lie group is a space of symmetries of some space. More formally, a Lie group is a manifold with a group operation \( G \times G \to G \) that satisfies
  - \( \circ \) is smooth.
  - the inverse \( \iota : G \to G \) is smooth also.

- Examples:
  - The permutation group of a finite set form a 0-dimensional manifold, which is a finite set.
  - \( \mathbb{R}, \mathbb{C} \) with + as an operation. (\( \mathbb{R}^+ \) with + is merely a Lie semigroup.)
  - \( \mathbb{R} - \{0\}, \mathbb{C} - \{0\} \) with * as an operation.
  - \( T^n = \mathbb{R}^n / \Gamma \) with + as an operation and \( O \) as the equivalence class of \((0, 0, ..., 0)\). (The three are abelian ones.)

- Products of Lie groups are Lie groups.

- A covering space of a connected Lie group form a Lie group.

- A Lie subgroup of a Lie group is a subgroup that is a Lie group with the induced operation and is a submanifold.
  - \( O(n, \mathbb{R}) \subset SL(n, \mathbb{R}) \subset GL(n, \mathbb{R}) \).
  - \( O(n - 1) \subset Isom(\mathbb{R}^n) \).

- A homomorphism \( f : G \to H \) of two Lie groups \( G, H \) is a smooth map that is a group homomorphism. The above inclusion maps are homomorphisms.

- The kernel of a homomorphism is a closed normal subgroup. Hence it is a Lie subgroup also.

- If \( G, H \) are simply connected, \( f \) induces a unique homomorphism of Lie algebra of \( G \) to that of \( H \) which is \( Df \) and conversely.
2.7 Lie algebras

### Lie algebras

- A Lie algebra is a vector space $V$ with an operation $[,] : V \times V \to V$ that satisfies:
  - $[x, x] = 0$ for $x \in L$. (Thus, $[x, y] = -[y, x]$.)
  - Jacobi identity $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.

- **Examples:**
  - Sending $V \times V$ to $O$ is a Lie algebra (abelian ones.)
  - Direct sums of Lie algebras is a Lie algebra.
  - A subalgebra is a subspace closed under $[,]$.
  - An ideal $K$ of $L$ is a subalgebra such that $[x, y] \in K$ for $x \in K$ and $y \in L$.
  - A homomorphism of a Lie algebra is a linear map preserving $[,]$.
  - The kernel of a homomorphism is an ideal.

### Lie groups and Lie algebras

- Let $G$ be a Lie group. A left translation $L_g : G \to G$ is given by $x \mapsto g(x)$.

- A left-invariant vector field of $G$ is a vector field so that the left translation leaves it invariant, i.e., $dL_g(X(h)) = X(gh)$ for $g, h \in G$.

- The set of left-invariant vector fields form a vector space under addition and scalar multiplication and is vector-space isomorphic to the tangent space at $I$. Moreover, $[,]$ is defined as vector-fields brackets. This forms a Lie algebra.

- The Lie algebra of $G$ is the the Lie algebra of the left-invariant vector fields on $G$.

- **Example:** The Lie algebra of $GL(n, \mathbb{R})$ is isomorphic to $gl(n, \mathbb{R})$:
  - For $X$ in the Lie algebra of $GL(n, \mathbb{R})$, we can define a flow generated by $X$ and a path $X(t)$ along it where $X(0) = I$.
  - We obtain an element of $gl(n, \mathbb{R})$ by taking the derivative of $X(t)$ at $0$ seen as a matrix.
  - The brackets are preserved.
  - A Lie algebra of an abelian Lie group is abelian.
Lie group actions

- A Lie group $G$-action on a smooth manifold $X$ is given by a smooth map $G \times X \rightarrow X$ so that $(g h)(x) = (g(h(x)))$ and $I(x) = x$. (left action)
- A right action satisfies $(x)(gh) = ((x)g)h$.
- Each Lie algebra element correspond to a vector field on $X$ by using a vector field.
- The action is faithful if $g(x) = x$ for all $x$, then $g$ is the identity element of $G$.
- The action is transitive if given two points $x, y \in X$, there is $g \in G$ such that $g(x) = y$.
- Example:
  - $GL(n, \mathbb{R})$ acting on $\mathbb{R}^n$.
  - $PGL(n + 1, \mathbb{R})$ acting on $\mathbb{R}P^n$.

Lie algebras

- Given $X$ in the Lie algebra $g$ of $G$, there is an integral curve $X(t)$ through $I$. We define the exponential map $\exp : g \rightarrow G$ by sending $X$ to $X(1)$.
- The exponential map is defined everywhere, smooth, and is a diffeomorphism near $O$.
- The matrix exponential defined by
  \[ A \mapsto e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \]
  is the exponential map $gl(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$.

3 Pseudo-group and $G$-structures

Pseudo-groups

- In this section, we introduce pseudo-groups.
- However, we are mainly interested in classical geometries, Clifford-Klein geometries. We will be concerned with Lie group $G$ acting on a manifold $M$.
- Most obvious ones are euclidean geometry where $G$ is the group of rigid motions acting on the euclidean space $\mathbb{R}^n$. The spherical geometry is one where $G$ is the group $O(n + 1)$ of orthogonal transformations acting on the unit sphere $S^n$. 
Pseudo-groups

- Topological manifolds form too large a category to handle.
- To restrict the local property more, we introduce pseudo-groups. A pseudo-group $\mathcal{G}$ on a topological space $X$ is the set of homeomorphisms between open sets of $X$ so that
  - The domains of $g \in \mathcal{G}$ cover $X$.
  - The restriction of $g \in \mathcal{G}$ to an open subset of its domain is also in $\mathcal{G}$.
  - The composition of two elements of $\mathcal{G}$ when defined is in $\mathcal{G}$.
  - The inverse of an element of $\mathcal{G}$ is in $\mathcal{G}$.
  - If $g : U \rightarrow V$ is a homeomorphism for $U, V$ open subsets of $X$. If $U$ is a union of open sets $U_\alpha$ for $\alpha \in I$ for some index set $I$ such that $g|U_\alpha$ is in $\mathcal{G}$ for each $\alpha$, then $g$ is in $\mathcal{G}$.

- The trivial pseudo-group is one where every element is a restriction of the identity $X \rightarrow X$.
- Any pseudo-group contains a trivial pseudo-group.
- The maximal pseudo-group of $\mathbb{R}^n$ is $\text{TOP}$, the set of all homeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group $C^r$, $r \geq 1$, of the set of $C^r$-diffeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group $\text{PL}$ of piecewise linear homeomorphisms between open subsets of $\mathbb{R}^n$.
- $(G, X)$-pseudo group. Let $G$ be a Lie group acting on a manifold $X$. Then we define the pseudo-group as the set of all pairs $(g|U, U)$ where $U$ is the set of all open subsets of $X$.
- The group $\text{isom}(\mathbb{R}^n)$ of rigid motions acting on $\mathbb{R}^n$ or orthogonal group $O(n + 1, \mathbb{R})$ acting on $\mathbb{S}^n$ give examples.

3.1 $\mathcal{G}$-manifold

$\mathcal{G}$-manifold is obtained as a manifold glued with special type of gluings only in $\mathcal{G}$.

- Let $\mathcal{G}$ be a pseudo-group on $\mathbb{R}^n$. A $\mathcal{G}$-manifold is a $n$-manifold $M$ with a maximal $\mathcal{G}$-atlas.
A $\mathcal{G}$-atlas is a collection of charts (imbeddings) $\phi : U \to \mathbb{R}^n$ where $U$ is an open subset of $M$ such that whose domains cover $M$ and any two charts are $\mathcal{G}$-compatible.

Two charts $(U, \phi), (V, \psi)$ are $\mathcal{G}$-compatible if the transition map
\[ \gamma = \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \in \mathcal{G}. \]

One can choose a locally finite $\mathcal{G}$-atlas from a given maximal one and conversely.

A $\mathcal{G}$-map $f : M \to N$ for two $\mathcal{G}$-manifolds is a local homeomorphism so that if $f$ sends a domain of a chart $\phi$ into a domain of a chart $\psi$, then
\[ \psi \circ f \circ \phi^{-1} \in \mathcal{G}. \]
That is, $f$ is an element of $\mathcal{G}$ locally up to charts.

### 3.2 Examples

Examples

- $\mathbb{R}^n$ is a $\mathcal{G}$-manifold if $\mathcal{G}$ is a pseudo-group on $\mathbb{R}^n$.
- $f : M \to N$ be a local homeomorphism. If $N$ has a $\mathcal{G}$-structure, then so does $M$ so that the map in a $\mathcal{G}$-map. (A class of examples such as $\theta$-annuli and $\pi$-annuli.)
- Let $\Gamma$ be a discrete group of $\mathcal{G}$-homeomorphisms of $M$ acting properly and freely. Then $M/\Gamma$ has a $\mathcal{G}$-structure. The charts will be the charts of the lifted open sets.
- $T^n$ has a $C^r$-structure and a PL-structure.
- Given $(G, X)$ as above, a $(G, X)$-manifold is a $\mathcal{G}$-manifold where $\mathcal{G}$ is the restricted pseudo-group.
- A euclidean manifold is a $(\text{isom}(\mathbb{R}^n), \mathbb{R}^n)$-manifold.
- A spherical manifold is a $(O(n + 1), S^n)$-manifold.

### 4 Differential geometry and $\mathcal{G}$-structures

#### Differential geometry and $\mathcal{G}$-structures

- We wish to understand geometric structures in terms of differential geometric setting; i.e., using bundles, connections, and so on.
- Such an understanding help us to see the issues in different ways.
- Actually, this is not central to the lectures. However, we should try to relate to the traditional fields where our subject can be studied in another way.
- We will say more details later on.
4.1 Riemannian manifolds

Riemannian manifolds and constant curvature manifolds.

- A differentiable manifold has a Riemannian metric, i.e., inner-product at each
tangent space smooth with respect coordinate charts. Such a manifold is said to
be a Riemannian manifold.

- An isometric immersion (imbedding) of a Riemannian manifold to another one
is a (one-to-one) map that preserve the Riemannian metric.

- We will be concerned with isometric imbedding of \( M \) into itself usually.

- A length of an arc is the value of an integral of the norm of tangent vectors to the
arc. This gives us a metric on a manifold. An isometric imbedding of \( M \) into
itself is an isometry always.

- A geodesic is an arc minimizing length locally.

- A constant curvature manifold is one where the sectional curvature is identical
to a constant in every planar direction at every point.

- Examples:
  - A euclidean space \( E^n \) with the standard norm metric has a constant curva-
ture = 0.
  - A sphere \( S^n \) with the standard induced metric from \( \mathbb{R}^{n+1} \) has a constant
curvature = 1.
  - Find a discrete torsion-free subgroup \( \Gamma \) of the isometry group of \( E^n \) (resp.
\( S^n \)). Then \( E^n / \Gamma \) (resp. \( S^n / \Gamma \)) has constant curvature = 0 (resp. 1).

4.2 Lie groups and group actions

Lie groups and group actions.

- A Lie group is a smooth manifold \( G \) with an associative smooth product map
\( G \times G \to G \) with identity and a smooth inverse map \( \iota : G \to G \). (A Lie group
is often the set of symmetries of certain types of mathematical objects.)

- For example, the set of isometries of \( S^n \) form a Lie group \( O(n + 1) \), which is a
classical group called an orthogonal group.

- The set of isometries of the euclidean space \( \mathbb{R}^n \) form a Lie group \( \mathbb{R}^n \cdot O(n) \), i.e.,
an extension of \( O(n) \) by a translation group in \( \mathbb{R}^n \).
• Simple Lie groups are classified. Examples $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n, \mathbb{R})$, $O(n, m)$, $GL(n, \mathbb{C})$, $U(n)$, $SU(n)$, $SP(2n, \mathbb{R})$, $Spin(n)$, ...

• An action of a Lie group $G$ on a space $X$ is a map $G \times X \to X$ so that $(gh)(x) = g(h(x))$.

• For each $g \in G$, $g$ gives us a map $g : X \to X$ where the identity element correspond to the identity map of $X$.

• Examples: $\mathbb{R}^n \cdot O(n)$ on $\mathbb{R}^n$ and $O(n)$ on $S^n$.

4.3 Principal bundles and connections, flat connections

Principal bundles and connections, flat connections

• Let $M$ be a manifold and $G$ a Lie group. A principal fiber bundle $P$ over $M$ with a group $G$:
  - $P$ is a manifold.
  - $G$ acts freely on $P$ on the right. $P \times G \to P$.
  - $M = P/G$. $\pi : P \to M$ is differentiable.
  - $P$ is locally trivial. $\phi : \pi^{-1}(U) \to U \times G$.

• Example 1: $L(M)$ the set of frames of $T(M)$. $GL(n, \mathbb{R})$ acts freely on $L(M)$. $\pi : L(M) \to M$ is a principal bundle.

• $P$ a bundle space, $M$ the base space. $\pi^{-1}(x)$ a fiber.

• $\pi^{-1}(x) = \{ug | g \in G\}$.

• A bundle can be constructed by mappings

$$\{ \phi_{\beta,\alpha} : U_{\alpha} \cap U_{\beta} \to G|U_{\alpha},U_{\beta}\text{"trivial" open subsets of }M\}$$

so that

$$\phi_{\gamma,\alpha} = \phi_{\gamma,\beta} \circ \phi_{\beta,\alpha}$$

for any triple $U_{\alpha},U_{\beta},U_{\gamma}$.

• $G'$, $G$ Lie groups. $f : G' \to G$ a monomorphism. $P(G',M) \to P(G,M)$ inducing identity $M \to M$ is called a reduction of the structure group $G$ to $G'$. There maybe many reductions for given $G'$ and $G$.

• $P(G,M)$ is reducible to $P(G',M)$ if and only if $\phi_{\alpha,\beta}$ can be taken to be in $G'$. (See Kobayashi-Nomizu, Bishop-Crittendon for details.)
Associated bundles

- Associated bundle: Let $F$ be a manifold with a left-action of $G$.
- $G$ acts on $P \times F$ on the right by
  \[ g : (u, x) \to (ug, g^{-1}(x)), \quad g \in G, \ u \in M, \ x \in F. \]
- The quotient space $E = P \times_G F$.
- $\pi_E$ is induced and $\pi_E^{-1}(U) = U \times F$. The structure group is the same.
- Example: Tangent bundle $T(M)$. $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$. Let $F = \mathbb{R}^n$. Obtain $L(M) \times_{GL(n, \mathbb{R})} \mathbb{R}^n$.
- Example: Tensor bundles $T^r_s(M)$. $GL(n, \mathbb{R})$ acts on $T^r_s(\mathbb{R})$. Let $F = T^r_s(\mathbb{R})$.

Connections

- $P(M, G)$ a principal bundle.
- A connection decomposes each $T_u(P)$ for each $u \in P$ into
  - $T_u(P) = G_u \oplus Q_u$ where $G_u$ is a subspace tangent to the fiber. ($G_u$ the vertical space, $Q_u$ the horizontal space.)
  - $Q_{ug} = (R_g)_*Q_u$ for each $g \in G$ and $u \in P$.
  - $Q_u$ depend smoothly on $u$.
- A horizontal lift of a piecewise-smooth path $\tau$ on $M$ is a piecewise-smooth path $\tau'$ lifting $\tau$ so that the tangent vectors are all horizontal.
- A horizontal lift is determined once the initial point is given.

- Given a curve on $M$ with two endpoints, the lifts defines a parallel displacement between fibers above the two endpoints. (commuting with $G$-actions).
- Fixing a point $x_0$ on $M$, this defines a holonomy group.
- The curvature of a connection is a measure of how much a horizontal lift of small loop in $M$ is a loop in $P$.
- The flat connection: In this case, we can lift homotopically trivial loops in $M^n$ to loops in $P$. Thus, the flatness is equivalent to local lifting of coordinate chart of $M$ to horizontal sections in $P$.
- A flat connection on $P$ gives us a smooth foliation of dimension $n$ transversal to the fibers.
• The associated bundle \( E \) also inherits a connection and hence horizontal liftings.

• The flatness is also equivalent to the local lifting property.

• The flat connection on \( E \) gives us a smooth foliation of dimension \( n \) transversal to the fibers.

• Summary: A connection gives us a way to identify fibers given paths on \( X \)-bundles over \( M \). The flatness gives us locally consistent identifications.

The principal bundles and \( G \)-structures.

• Given a manifold \( M \) of dimension \( n \), a Lie group \( G \) acting on a manifold \( X \) of dimension \( n \).

• We form a principal bundle \( P \) and then the associated bundle \( E \) fibered by \( X \) with a flat connection.

• A section \( f : M \to E \) which is transverse everywhere to the foliation given by the flat connection.

• This gives us a \((G, X)\)-structure and conversely a \((G, X)\)-structure gives us \( P, E, f \) and the flat connection.