Classifying the ends of convex real projective $n$-orbifolds

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Abstract

- We consider n-orbifolds modeled on real projective geometry. These include hyperbolic manifolds and orbifolds. There are nontrivial deformations of hyperbolic orbifolds to real projective ones.

- Among open real projective orbifolds that are topologically tame, we consider ones with radial ends and totally geodesic ends. We will present our work to classify these ends with some natural conditions.
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   - Orbifolds and $\mathbb{RP}^n$-structures

2 Convex $\mathbb{RP}^n$-orbifolds with radial or totally geodesic ends
   - Examples: Global and Local
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   - Types of ends and the classification goal
   - Convexity and convex domains
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4 PC-ends

5 The properties of lens-shaped ends
   - The equivalence of lens condition with umec.

6 Non-properly convex (NPCC) ends
Orbifolds

By an \( n \)-dimensional orbifold, we mean a Hausdorff 2nd countable topological space with

- a fine open cover \( \{ U_i, i \in I \} \)
- with models \( (\tilde{U}_i, G_i) \) where \( G_i \) is a finite group acting on the \( \tilde{U}_i \subset \mathbb{R}^n \), and
- a map \( p_i : \tilde{U}_i \to U_i \) inducing \( \tilde{U}_i / G_i \cong U_i \) where

  (compatibility) for each \( i, j, x \in U_i \cap U_j \), there exists \( U_k \) with \( x \in U_k \subset U_i \cap U_j \) and the inclusion

  \( U_k \to U_i \) induces \( \tilde{U}_k \to \tilde{U}_i \) with respect to \( G_k \to G_i \).

Good orbifolds

Our orbifolds are of form \( M/\Gamma \) for a simply connected manifold \( M \) and a discrete group \( \Gamma \) acting on \( M \) properly discontinuously.
Topography of our orbifolds

- Let $O$ denote an $n$-dimensional orbifold with finitely many ends with end neighborhoods, closed $(n-1)$-dimensional orbifold times an open interval. (strongly tame).
- Equivalently, $O$ has a compact suborbifold $K$ so that $O - K$ is a disjoint union $\Omega_i \times [0, 1)$ for closed $n-1$-orbifolds $\Omega_i$. 
Real projective and affine geometry

Real projective geometry

- Recall that the **real projective space**

  \[ \mathbb{RP}^n := P(\mathbb{R}^{n+1}) := \mathbb{R}^{n+1} - \{O\} / \sim \text{ under} \]

  \[ \vec{v} \sim \vec{w} \text{ iff } \vec{v} = s\vec{w} \text{ for } s \in \mathbb{R} - \{O\}. \]

- \( \text{GL}(n + 1, \mathbb{R}) \) acts on \( \mathbb{R}^{n+1} \) and \( \text{PGL}(n + 1, \mathbb{R}) \) acts faithfully on \( \mathbb{RP}^n \).

Projective sphere geometry

- Recall that the **real projective sphere**

  \[ S^n := S(\mathbb{R}^{n+1}) := \mathbb{R}^{n+1} - \{O\} / \sim \text{ under} \]

  \[ \vec{v} \sim \vec{w} \text{ iff } \vec{v} = s\vec{w} \text{ for } s > 0. \]

- \( \text{GL}(n + 1, \mathbb{R}) \) acts on \( \mathbb{R}^{n+1} \) and \( \text{SL}_\pm(n + 1, \mathbb{R}) \) acts faithfully on \( S^n \).
**Properly convex domain**

- An **affine subspace** $\mathbb{R}^n$ can be identified with $\mathbb{RP}^n - V$ where $V$ a hyperspace. Geodesics agree.

- $\text{Aff}(\mathbb{R}^n) = \text{Aut}(\mathbb{RP}^n - V)$.

- A **convex subset** of $\mathbb{RP}^n$ is a convex subset of an affine subspace.

- A **properly convex subset** of $\mathbb{RP}^n$ is a precompact convex subset of an affine subspace.

- A convex domain $\Omega$ is properly convex iff $\Omega$ does not contain a complete real line.

**Sphere version**

- An open hemisphere is the affine subspace in $\mathbb{S}^n$ with boundary a hypersphere $V$. Now, the geometry is exactly the same as the above.

- $\text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \text{GL}(n, \mathbb{R})$. 
Real projective structures on orbifolds

A discrete group $\Gamma$ acts on a simply connected manifold $M$ properly discontinuously.

A $\mathbb{RP}^n$-structure on $M/\Gamma$ is given by

- an immersion $D : M \rightarrow \mathbb{RP}^n$
- equivariant with respect to a homomorphism $h : \Gamma \rightarrow \text{PGL}(n+1, \mathbb{R})$.

$\Gamma$ is the fundamental group of $M/\Gamma$.

- $M$ an interior of a conic in $\mathbb{RP}^n$ of sign $(-, + \cdots, +)$. Discrete $\Gamma \subset \text{PO}(n,1)$ and $M/\Gamma$ is a hyperbolic orbifold and a convex $\mathbb{RP}^n$-orbifold.

- A $\mathbb{RP}^n$-structure on $M/\Gamma$ is \textit{convex} if $D$ is a diffeomorphism to a convex domain
  $$D(M) \subset A^n \subset \mathbb{RP}^n$$

- Identify $M = D(M)$ and $\Gamma$ with its image under $h$.

- A $\mathbb{RP}^n$-structure on $M/\Gamma$ is \textit{properly convex} if so is $D(M)$.
Figure: The developing images of convex $\mathbb{RP}^n$-structures on 2-orbifolds deformed from hyperbolic ones: $S^2(3, 3, 5)$ and $D^2(2, 7)$

**Lift to real projective spheres**

Obtain the lift $D : M \to S^n$ and consider $h : \Gamma \to \text{Aut}(S^n)$ as an identification. Usually, $D$ is an imbedding and we identify $M$ with $D(M)$. If $M$ is convex or properly convex, then $D(M)$ is in an open hemisphere identifiable to $A^n$. 
Dual real projective orbifolds

Dual domains

- An open convex cone $C$ in $\mathbb{R}^{n+1}$ is dual to $C^*$ in $\mathbb{R}^{n+1,*}$ if $C^*$ is the set of linear forms taking positive values on $\text{Cl}(C) - \{O\}$.
- A convex open domain $\Omega$ in $rpn$ is dual to $\Omega^*$ in $P(\mathbb{R}^{n+1,*}) = \mathbb{RP}^{n*}$ if $\Omega$ corresponds to an open convex cone $C$ and $\Omega^*$ to its dual $C^*$.

- A projective geodesic is an arc developing into a straight line in $\mathbb{RP}^n$.

- Given a properly convex real projective $n$-orbifold $\Omega/\Gamma$, there exists a dual one $\Omega^*/\Gamma^*$ with dual group given by

$$\Gamma \ni g \leftrightarrow g^{-1}, T \in \Gamma^*.$$

- There exists a diffeomorphism $\Omega/\Gamma \leftrightarrow \Omega^*/\Gamma^*$. (Vinberg)
Real projective structures on the ends

<table>
<thead>
<tr>
<th>Radial end (R-ends):</th>
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<tbody>
<tr>
<td>Each end $E$ has an end neighborhood foliated by lines developing into lines ending at a common point. The space of leaves gives us the <em>end orbifold</em> $\Sigma_E$ with a transverse real projective structure.</td>
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<tr>
<th>Totally geodesic end (T-ends):</th>
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<tbody>
<tr>
<td>Each end has an end neighborhood completed by a closed totally geodesic orbifold of codim 1. The orbifold $S_E$ is called an <em>ideal boundary</em> of the end or the end neighborhood. Clearly, it has a real projective structure.</td>
</tr>
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</table>
Some definitions for radial ends

- A subdomain $K$ of an affine subspace $A^n$ in $\mathbb{R}P^n$ is said to be *horospherical* if it is strictly convex and the boundary $\partial K$ is diffeomorphic to $\mathbb{R}^{n-1}$ and $\text{bd} K - \partial K$ is a single point.

- $K$ is *lens-shaped* if it is a convex domain in $A^n$ and $\partial K$ is a disjoint union of two strictly convex $(n-1)$-cells $\partial_+ K$ and $\partial_- K$.

- A *cone* is a domain $D$ in $A^n$ that has a point $v \in \text{bd} D$ called a *cone-point* so that

$$D = v \ast K - \{v\} \text{ for some } K \subset \text{bd} D.$$  

- A *cone $D$ over* a lens-shaped domain $L$ is a convex submanifold that contains $L$ so that

$$D = v \ast \partial_+ L - \{v\}$$

for $v \in \text{bd} D$ for a boundary component $\partial_+ L$ of $\partial L$ and $\partial_+ L \subset \text{bd} D$.

- We can allow one component $\partial_+ L$ be not smooth. In this case, we call these *generalized lens* and *generalized lens-cone*. 
The universal covers of horospherical and lens shaped ends. The radial lines form cone-structures.
Definition on ends continued

- A **totally-geodesic subdomain** is a convex domain in a hyperspace. A **cone-over** a totally-geodesic domain $A$ is a cone over a point $x$ not in the hyperspace.

- In general, a **sum** of convex sets $C_1, \ldots, C_m$ in $\mathbb{R}^{n+1}$ in independent subspaces $V_i$, we define

$$C_1 + \cdots + C_m := \{v | v = c_1 + \cdots + c_m, c_i \in C_i\}.$$

- A **join** of convex sets $\Omega_i$ in $\mathbb{RP}^n$ is given as

$$\Omega_1 \ast \cdots \ast \Omega_m := \Pi(C_1 + \cdots C_m)$$

where each $C_i$ corresponds to $\Omega_i$ and these subspaces are independent. (We can relax this last condition)
Example

- There is a **census of small hyperbolic orbifolds** with graph-singularity. (See the paper by D. Heard, C. Hodgson, B. Martelli, and C. Petronio [33])
- S. Tillman constructed an example on $S^3$ with a handcuff graph singularity.
- Some examples are obtained by myself on the **double orbifold of the hyperbolic ideal regular tetrahedron** [13] and by Lee on **complete hyperbolic cubes** by numerical computations (unpublished)
- These have **lens type or horospherical ends** by our theory to be presented.
End orbifold

**Figure**: The handcuff graph

**Cusp ends**

Let $M$ be a complete hyperbolic manifolds with cusps. $M$ is a quotient space of the interior $\Omega$ of a conic in $\mathbb{RP}^n$ or $S^n$. Then the horoballs form the **horospherical ends**. Any end with a projective diffeomorphic end neighborhood is also called a **cusp**.
Examples of ends

**Totally geodesic R-end (lens type)**

Suppose that hyperbolic $M$ has a totally geodesic imbedded surface $S$ homotopic to the ends.

- Then $\pi_1(S)$ fixes a point outside the conic, and acts on a lens-shaped domain that is an $\epsilon$-neighborhood of $S$ in $\Omega/\pi_1(M)$.
- Add the cone over the lens-shaped domain to $M$ to obtain the examples of real projective manifolds with radial ends.
- *(hyperideal extension of the hyperbolic manifolds as real projective manifolds.)*

**Bending examples by Johnson-Millson**

One can obtain a lens-shaped end by doing a bending construction also; i.e., find a radial direction totally geodesic submanifold of codimension 1 and find an element of $\text{SL}(n + 1, \mathbb{R})$ commuting with the holonomy and use it to change the holonomy.
Examples of ends

**Proposition 2.1**

A topologically tame properly convex real projective orbifold $O$ with radial or totally geodesic ends with admissible end fundamental groups. \textbf{Each end fundamental group is generated by closed curves about singularities or has the holonomy fixing the end vertex with eigenvalues 1.} (e.g. $\pi_1(E)$ simple case.) If an end is totally geodesic and properly convex, then the end is of \textit{lens-type}.

**Proposition 2.2**

Let $O$ be a 3-orbifold with the end orbifolds each of which is homeomorphic to a sphere $S_{3,3,3}$ with three singularities of order 3. Then the orbifold has \textit{lens-type ends or horospherical ends}.

The hyperbolic structure examples of Tillman on the handcuff orbifold, my doubled ideal tetrahedral orbifolds, and Lee’s cubes realize concrete examples of these.
Back to Theory: p-ends, p-end neighborhood, p-end fundamental group

<table>
<thead>
<tr>
<th>End fundamental group</th>
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<tbody>
<tr>
<td>Given an end $E$ of $\mathcal{O}$, a system of connected end neighborhoods $U_1 \supset U_2 \supset \cdots$ of $\mathcal{O}$ gives such a system $U'_1 \supset U'_2 \subset \cdots$ in $\tilde{\mathcal{O}}$.</td>
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<tr>
<td>On each the end group $\Gamma_{\tilde{E}}$ acts. That is $U'<em>i / \Gamma</em>{\tilde{E}} \rightarrow U_i$, a homeomorphism.</td>
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<tr>
<td>There are called these proper pseudo-end neighborhood in $\tilde{\mathcal{O}}$ and defines a pseudo-end $\tilde{\mathcal{E}}$.</td>
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<tr>
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<tr>
<td>${ \tilde{E}</td>
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<tr>
<td>If $\tilde{E}$ is a p-end of radial type, we obtain a unique vertex $\mathbf{v}<em>{\tilde{E}}$ for each p-end $\tilde{E}$. $\Gamma</em>{\tilde{E}}$ acts on it.</td>
</tr>
<tr>
<td>If $\tilde{E}$ is a p-end of totally geodesic type, we obtain a properly convex domain $\tilde{S}<em>{\tilde{E}}$ in the boundary of $\mathcal{O}$ for each p-end $\tilde{E}$. $\Gamma</em>{\tilde{E}}$ acts on it.</td>
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R-ends

A point $v_{\tilde{E}} \in \mathbb{RP}^n$, the set of directions of lines from $v_{\tilde{E}}$ form a sphere $S_{v_{\tilde{E}}}^{n-1}$. $\Gamma_{\tilde{E}}$ acts on this.

- Given a radial p-end $\tilde{E}: v_{\tilde{E}}$, we obtain $R_{v_{\tilde{E}}} \subset S_{v_{\tilde{E}}}^{n-1}$ the space $\tilde{\Sigma}_{\tilde{E}}$ of directions of line segments from $v_{\tilde{E}}$ ending in $\tilde{O}$.

- For radial end, $\Sigma_{\tilde{E}} := \tilde{\Sigma}_{\tilde{E}} / \Gamma_{\tilde{E}}$ is the end orbifold with the transverse real projective structure associated with $E$. (or $\Sigma_E$)

T-ends

For a T-end, $S_{\tilde{E}} := \tilde{S}_{\tilde{E}} / \Gamma_{\tilde{E}}$ is an ideal boundary component corresponding to $E$. (or $S_E$.)
Real projective $n-1$-orbifolds associated with ends

**Usual assumption**

Let $O$ a properly convex and strongly tame real projective orbifolds with radial or totally geodesic ends. The holonomy homomorphism is strongly irreducible. (The end fundamental group is of infinite index.)

**R-ends:**

An R-end $E$ has an end orbifold $\Sigma_E$ admitting a real projective structure of dim $= n - 1$. The structure is convex real projective one. The structure can be

- **PC:** properly convex,
- **CA:** complete affine, or
- **NPCC:** convex, not properly convex, not complete.

**T-end**

A totally geodesic end $E$ has the ideal boundary orbifold $S_E$ admitting a real projective structure of dim $= n - 1$. Here the structure is properly convex.
The main classification aim

R-types

CA-ends Horospherical type end $\leftrightarrow$ complete end

PC-ends generalized Lens type – structurally stable (Anosov-type) condition on eigenvalues. (also quasi-lens type)

- In the reducible case, they are **totally geodesic** also, i.e., a cone over such a domain.

NPCC-ends Mixed types —The others are “joins” of horospherical ones or lens type ones.
(They are classifiable under the conditions.)

T-types

- Lens type – structurally stable (Anosov-type) condition on eigenvalues
  - In the reducible case, they are totally geodesic also. Can be made to radial end.

- In the reducible case, the totally geodesics and radial type can be converted to each other by some operations.
Classification aim

In each case, we will find the natural condition called the uniform middle eigenvalue condition and show that the ends are of lens type or horospherical type (quasi-lens, quasi-join). This is a classification since we can construct all these ends.

Nonradial types (we do not study these)

- Convex ends – Geometrically finite, or infinite. (Even topologically wild?)
- Ends that can be completed by a lower dimensional strata– sometimes correspond to "geometrical Dehn surgeries".
- Recent example by Cooper $S_{3\times3}^+/\text{SL}(3,\mathbb{Z})$ for the space of positive definite matrices.
Benoist’s "maximally complete" results on closed real projective orbifolds

In his papers "Convexes divisibles I-IV":

**Proposition 2.3 (Benoist)**

*Suppose that a discrete subgroup $\Gamma$ of $\text{PGL}(n + 1, \mathbb{R})$ acts on a properly convex $n$-dimensional open domain $\Omega$ so that $\Omega/\Gamma$ is compact. Then the following statements are equivalent.*

- Every finite index subgroup of $\Gamma$ has a finite center.
- Every FI subgroup of $\Gamma$ has a trivial center.
- Every FI subgroup of $\Gamma$ is irreducible in $\text{PGL}(n + 1, \mathbb{R})$. (or strongly irreducible).
- The Zariski closure of $\Gamma$ is semisimple.
- $\Gamma$ does not contain a normal infinite nilpotent subgroup.
- $\Gamma$ does not contain a normal infinite abelian subgroup.
Benoist’s result continued

- The group with the above property is said to be the group with *trivial virtual center*.

**Theorem 2.4 (Benoist)**

*Let $\Gamma$ be a discrete subgroup of $\text{PGL}(n+1, \mathbb{R})$ with a trivial virtual center. Suppose that a discrete subgroup $\Gamma$ of $\text{PGL}(n+1, \mathbb{R})$ acts on a properly convex $n$-dimensional open domain $\Omega$ so that $\Omega/\Gamma$ is compact. Then every representation of a component of $\text{Hom}(\Gamma, \text{PGL}(n+1, \mathbb{R}))$ containing the inclusion representation also acts on a properly convex $n$-dimensional open domain cocompactly.*

- We call the group such as above theorem a *vcf-group*. By above Proposition 2.3, we see that every representation of the group acts irreducibly.
Decompositions of reducible cases

Proposition 2.5 (Benoist-Koszul-Vey decomposition)

Let $\mathcal{O}$ be an $(n - 1)$-dimensional closed properly convex projective orbifold. Then

- $\tilde{\mathcal{O}}$ is a join $K_1 \ast \cdots \ast K_r$ where $K_i$ is a properly convex compact domain of dimension $n_i \geq 0$ corresponding to a convex cone $C_i \subset \mathbb{R}^{n_i + 1}$.
- $\tilde{\mathcal{O}}$ is the image of $C_1 \oplus \cdots \oplus C_r$.
- $\pi_1(E)$ is virtually $\mathbb{Z}^{r-1} \times \Gamma_1 \times \cdots \Gamma_r$ for $r - 1 + \sum_{i=1}^r n_i = n - 1$.
- Each $\Gamma_j$ acts on $K_j^o$ cocompactly and the identity component of the Zariski closure is an irreducible Lie group and acts trivially on $K_l$ for $l \neq j$. ($\Gamma_j$ could be trivial.)
- The center $\mathbb{Z}^{r-1}$ acts trivially on each $K_j$

If we require $\Gamma_i$ to be hyperbolic, then $\Gamma_i$ has a Zariski closure a copy of

$$\text{SL}(n_j + 1, \mathbb{R}) \text{ or } \text{SO}(1, n_j).$$
Duality of ends

Proposition 2.6

Let $\mathcal{O}$ have the usual property. Then the dual real projective orbifold $\mathcal{O}^*$ is also strongly tame and has the same number of ends so that

- the set of ends of $\mathcal{O}$ $\leftrightarrow$ the set of ends of $\mathcal{O}^*$.
- the set of horospherical ends of $\mathcal{O}$ $\leftrightarrow$ the set of horospherical ones of $\mathcal{O}^*$.
- the set of T-ends of $\mathcal{O}$ $\leftrightarrow$ the set of ends of properly convex R-ends of $\mathcal{O}^*$. $(\tilde{S}_\mathcal{E})^* \simeq \tilde{\Sigma}_{\mathcal{E}^*}$.
- the set of properly convex R-ends of $\mathcal{O}$ $\leftrightarrow$ the set of T-ends of $\mathcal{O}^*$. Also, $(\tilde{\Sigma}_\mathcal{E})^* \simeq \tilde{S}_{\mathcal{E}^*}$.

The lens-type property also get preserved provided the p-end is reducible or $\mathcal{O}$ satisfies the triangle condition.
Definitions for ends

**Admissible group**

An *admissible group* is a finite extension of a finite product of $\mathbb{Z}^l \times \Gamma_1 \times \cdots \times \Gamma_k$ for infinite hyperbolic groups $\Gamma_i$ where $l \geq k$ or $k = 1$ holds and $l + k \leq n$ holds.

(i) $l \geq k - 1$ follows from the results of Benoist.

(ii) $k = 1$ and $l = 0$ if and only if the end fundamental group is hyperbolic.

(iii) If $l \leq k - 1$, $k$, then $\Sigma_{\tilde{E}}$ is always properly convex.

**Theorem 1 (Classification for the CA-ends)**

Let $\mathcal{O}$ be a real projective orbifold with usual property.

Then $\tilde{E}$ is complete if and only if $\tilde{E}$ is a cusp.
Classification of the PC-ends

Definition: umec

Let $\tilde{E}$ be a properly convex end. The end fundamental group $\Gamma_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition (umec) if each irreducible factor $\Gamma_i$ or the center $\mathbb{Z}^l$ of the appropriate finite index subgroup of $\Gamma_{\tilde{E}}$ satisfies

$$K^{-1}\text{length}(g) \leq \log(\lambda_1(g)/\lambda_{v_{\tilde{E}}}(g)) \leq K\text{length}(g),$$

for the largest eigenvalue modulus $\lambda_1(g)$ of $g$ and the eigenvalue of $g$ at $v_{\tilde{E}}$ for $g$ in $\Gamma_E$.

Definition: wumec

In the above condition, if each element $g$ of the center satisfies only

$$\lambda_1(g) \geq \lambda_{v_{\tilde{E}}}(g),$$

then $\Gamma_{\tilde{E}}$ satisfies the weak uniform middle eigenvalue condition. (wumec)
Main results

Theorem 2 (Main result for PC R-ends)

Let $\mathcal{O}$ be a real projective orbifold with usual property.

- Suppose that the end holonomy group of a properly convex end $\tilde{E}$ satisfies the uniform middle eigenvalue condition.

Then $\tilde{E}$ is of generalized lens type. If $\tilde{E}$ is reducible or $\mathcal{O}$ satisfies the triangle condition, then $\tilde{E}$ is of lens type. If we assume only weak middle eigenvalue condition, then the end can also be quasi-joined type.

Theorem 3 (Main result for T-ends)

Let $\mathcal{O}$ be a real projective orbifold with usual property.

- Suppose that each end holonomy group satisfies the uniform middle eigenvalue condition.
  (i.e., structurally stable conditions on ends)

Then each end is of lens type.

The NPCC-ends will be studied later.
Proposition 3.1

Let $\mathcal{O}$ be a real projective $n$-orbifold with usual assumptions.

- For a horospherical end $E$, the space of rays from the end point form a complete affine space of dimension $n - 1$.

- The only eigenvalues of $g$ for an element of a horospherical end fundamental group $\pi_1(E)$ are 1 or complex numbers of absolute value 1.

- $\pi_1(E)$ is virtually abelian and a finite index subgroup is in a conjugate of a parabolic subgroup of $\text{SO}(1, n)$ in $\text{SL}_\pm(n + 1, \mathbb{R})$. (Crampon-Marquis)

- An end point of a horospherical end cannot be on a segment in $\partial \tilde{\mathcal{O}}$. 
Theorem 3.2

Let $\mathcal{O}$ be a real projective $n$-orbifold with usual assumptions. Suppose that $E$ is a complete radial end of $\mathcal{O}$. Let $v_E \in S^n$ be the fixed point of holonomy group $h(\pi_1(E))$ corresponding to $E$. Then

- The eigenvalues of elements of $h(\pi_1(E))$ have **unit norms** only.
- A finite index subgroup of $h(\pi_1(E))$ is contained a unipotent group fixing $v_E$.
- $E$ is **horospherical**.
Tubular actions for PC-ends

- If a group $\Gamma$ of projective automorphisms fixes a pair of fixed points $v_{\tilde{E}}$ and $v_{\tilde{E}-}$, then $\Gamma$ is said to be *tubular*.

- A projection $\Pi_{v_{\tilde{E}}}: \mathbb{S}^n - \{v_{\tilde{E}}, v_{\tilde{E}-}\} \rightarrow \mathbb{S}^{n-1}_{v_{\tilde{E}}}$ sends every segment of length $\pi$ to the sphere of directions at $v_{\tilde{E}}$.

- A convex tube in $\mathbb{S}^n$ is the closure of the inverse image of a convex domain $\Omega$ in $\mathbb{S}^{n-1}_{v_{\tilde{E}}}$.

- If an end $\tilde{E}$ has the end domain $\Omega_{\tilde{E}}$ on which the end fundamental group $\pi_1(\tilde{E})$ is acting, it follows that $h(\pi_1(\tilde{E}))$ acts on the tube domain $T_{\tilde{E}}$ associated with $\Omega_{\tilde{E}}$.

- Letting $v_{\tilde{E}}$ be $[0, \ldots, 0, 1]$, $h(g)$ of $g \in \pi_1(\tilde{E})$ is of form

$$
\begin{pmatrix}
\frac{1}{\lambda_{v_{\tilde{E}}}(g)} & 0 \\
\hat{h}(g) & 1 \\
\vec{b}_g & \lambda_{v_{\tilde{E}}}(g)
\end{pmatrix}
$$

where $\vec{b}_g$ is an $n \times 1$-vector and $\hat{h}(g)$ is an $n \times n$-matrix of determinant $\pm 1$ and $\lambda_{v_{\tilde{E}}}(g) > 0$ is a constant.
Note that the representation $\hat{h} : \pi_1(E) \to SL_{\pm}(n, \mathbb{R})$ is given by sending $g \mapsto \hat{h}(g)$. Here we have $\lambda_{\nu_E}(g) > 0$.

If $\Omega_{\bar{E}}$ is properly convex, then the convex tubular domain is *properly tubular* and the action is *properly tubular*.

One can also deform the examples using bendings and changing by cohomology $Hom(\pi_1(E), \mathbb{R}^+) = H^1(\pi_1(E), \mathbb{R})$.

**Theorem 4.1**

The space of representations of the end fundamental group $\pi_1(\Sigma)$ to $SL_{\pm}(n + 1, \mathbb{R})_{\nu_{\bar{E}}}$

is the fiber space $B$ over $\text{Hom}(\pi_1(\Sigma), SL_{\pm}(n, \mathbb{R}))/SL_{\pm}(n, \mathbb{R}) \times H^1(\pi_1(\Sigma), \mathbb{R})$ with fiber $H^1(\pi_1\Sigma, \mathbb{R}^{n^*}_{\nu_{\bar{E}}, \lambda})$. 
Affine actions dual to tubular actions

- Consider the dual projective space $P(\mathbb{R}^{n+1*})$. We can identify it with $P(\mathbb{R}^{n+1})$.
- Correspondingly for $S(\mathbb{R}^{n+1})$ and $S(\mathbb{R}^{n+1*})$. We also note that $(g^*)^* = g$.
- The action preserving $S_{\infty}^{n-1}$ of $S(\mathbb{R}^{n+1})$ acts on an affine space $A^n$ that is a component of the complement. The subgroup of projective automorphisms preserving $S_{\infty}^{n-1}$ is denoted by $\text{Aff}(A^n)$.
- By duality, $S_{\infty}^{n-1}$ corresponds to a point $v_{S_{\infty}^{n-1}}$. Thus, given a group $\Gamma$ in $\text{Aff}(A^n)$, we obtain dual groups $\Gamma^*$ acting on $P(\mathbb{R}^{n+1,*})$ fixing $v_{S_{\infty}^{n-1}}$ and $(\Gamma^*)^* = \Gamma$.
- Suppose that affine group $\Gamma$ acts on a properly convex open domain $\Omega$ in $S_{\infty}^{n-1}$. Then we call $\Gamma$ an \textit{asymptotically properly convex affine} action.
- Then the dual groups $\Gamma^*$ and $\Gamma^*,'$ act on properly tubular domains with vertices $v_{S_{\infty}^{n-1}}$ and $v_{S_{\infty}^{n-1}}^*$. The tubular domain corresponds to the dual domain $\Omega^* \subset S_{\infty}^{n-1*}$ of $\Omega \subset S_{\infty}^{n-1}$.
A properly tubular action is said to be *distanted* if there exists a properly convex compact \( \Gamma \)-invariant subset in the tubular domain disjoint from the vertices.

A properly convex affine action of \( \Gamma \) is said to be *asymptotically nice* if there exists a properly convex \( \Gamma \)-invariant open domain \( U \) in \( A^n \) with boundary in \( \Omega \subset S^{n-1}_\infty \) and \( U \) is in the intersection of all open hemispheres \( H \) supporting \( U \) at \( \text{bd} \Omega \) and \( A^n \) where for each point \( x \in \text{bd} \Omega \) there exists a supporting hemisphere \( H_x \) with \( \text{bd} H_x \) are not in \( S^{n-1}_\infty \).

**Proposition 4.2**

Let \( \Gamma \) and \( \Gamma^* \) be dual groups where \( \Gamma \) is affine and \( \Gamma^* \) is tubular. \( \Gamma = (\Gamma^*)^* \) acts in a properly convex affine manner and is asymptotically nice if and only if \( \Gamma^* \) acts in a properly tubular manner and distanted.
Theorem 4.3

Let $\Gamma$ give an irreducible properly convex affine action on the affine space $A^n$ acting on a properly convex domain $U \subset A^n$ with boundary in the convex domain $\overline{\Omega}$ for a properly convex domain $\Omega$ in $L$. Suppose that $\Gamma$ satisfies the uniform middle eigenvalue condition. Then $\Gamma$ is asymptotically nice.

Theorem 4.4

Let $\Gamma$ be a nontrivial properly convex tubular action and acts irreducibly on a properly convex domain $U$ and satisfy the uniform middle eigenvalue conditions. Then $\Gamma$ is distanced inside the tube where $\Gamma$ acts on.
The proof of Theorem 4.3

- We form the product $U\Omega \times A^n$ that is an affine bundle over the unit tangent bundle $U\Omega$ of $\Omega$. We take the quotient $U\Omega \times A^n$ by the diagonal action $g(\vec{u}, x) = (Dg\vec{u}, g(x))$ where $Dg$ is the differential of the action of $g$ on $\Omega$. We denote the quotient by $\mathbb{A}$ fibering over the smooth orbifold $U\Omega/\Gamma$ with fiber $A^n$.

- Let $V^n$ be the vector space associated with $A^n$. Then we can form $U\Omega \times V^n$ and take the quotient under the diagonal action. We denote by $\mathbb{V}$ the fiber bundle over $U\Omega/\Gamma$ with fiber $V^n$.

- We give a decomposition of $\mathbb{V}$ into three parts

$$\mathbb{V_+} \oplus \mathbb{V_0} \oplus \mathbb{V_-}.$$

If $g \in \Gamma$ acts on $l$, then $V_+$ and $V_-$ are eigenspaces of the largest norm eigenvalue and the smallest norm eigenvalue of the linear part of $g$ equal to

$$\frac{1}{\lambda_{\mathbb{V}_E}(g)(n+1)/n} \hat{h}(g).$$

Hence on $\mathbb{V}_+$, $g$ acts by expending by $\lambda_1(g)/\lambda_{\mathbb{V}_E}(g)$ and on $\mathbb{V}_-$, $g$ acts by contracting by $\lambda_n(g)/\lambda_{\mathbb{V}_E}(g)$. 

Anosov-type decomposition

As in Section 4.4 of [30], $\mathbb{V} = \mathbb{V}_+ \oplus \mathbb{V}_0 \oplus \mathbb{V}_-$. By the uniform middle eigenvalue condition, there exists a fiberwise Euclidean metric $g$ on $\mathbb{V}$ with the following properties:

- the flat linear connection $\nabla_\mathbb{V}$ is bounded with respect to $g$.
- hyperbolicity: There exists constants $C, k > 0$ so that

$$||\tilde{\Phi}_t(v)|| \leq C \exp(kt)||v|| \text{ as } t \to -\infty$$  \hspace{1cm} (3)

for $v \in \mathbb{V}_-$ and

$$||\tilde{\Phi}_t(v)|| \geq C \exp(kt)||v|| \text{ as } t \to \infty$$  \hspace{1cm} (4)

for $v \in \mathbb{V}_+$.  

Recall that $U\Sigma$ is a recurrent set under the geodesic flow. A section $s : U\Sigma \to \mathbb{A}$ is \textit{neutralized} if $\nabla_{\phi}s$ is in $\mathbb{V}_0$ always where $\phi$ is the geodesic flow vector field in $U\Sigma$.

**Lemma 4.5**

A neutralized section exists on $U\Sigma$. This lifts to a map $\tilde{s}_0 : U\Omega \to E$ so that $\tilde{s}_0 \circ \gamma = \gamma \circ \tilde{s}_0$.

**Proof.**

Let $s$ a continuous section $U\Sigma \to \mathbb{A}$. We decompose

$$\nabla_{\phi}(s) = \nabla_{\pm}^A(s) + \nabla_0^A(s) + \nabla_{-}^A(s)$$

where $\nabla_\pm^A(s) \in \mathbb{V}_\pm$. Again

$$s_0 = s + \int_0^\infty (D\Phi_t)_*(\nabla_{-}^A(s))dt - \int_0^\infty (D\Phi_{-t})_*(\nabla_{+}^A(s))dt$$

is a continuous section and $\nabla_{\phi}(s_0) = \nabla_0^A(s_0)$.

Since $U\Sigma$ is connected, there exists a fundamental domain $F$ so that we can lift $s_0$ to $\tilde{s}_0'$ defined on $F$ mapping to $\mathbb{A}$. We can extend $\tilde{s}_0'$ to $U\Omega \to \Omega \times A^n$. \qed
Proposition 4.6

There exists a continuous function \( \hat{s} : G(\Omega) \to N_2(A^n) \) equivariant with respect to \( \Gamma \)-actions.

Given \( g \in \Gamma \) and for the unique geodesic \( l_g \) in \( \Omega \) where \( g \) acts on, \( \hat{s}(l_g) = N_2(g) \).

This gives a continuous map

\[
\tau : \text{bd}\Omega \times \text{bd}\Omega - \Delta \to N_2(A^n)
\]

again equivariant with respect to the \( \Gamma \)-actions. There exists a continuous function

\[
\tau : \text{bd}\Omega \times \text{bd}\Omega - \Delta \to S(\text{bd}\Omega).
\]
Proof of Theorem 4.3.

For each point $x \in \partial \Omega$, we obtain an $(n - 1)$-dimensional hemisphere $h(x)$ passing $E$ with $\partial h(x) \subset L$ supporting $\Omega$ by Lemma 4.7. There exists a half-space $H(x) \subset E$ bounded by $h(x)$ and containing $\Omega$.

We form $\bigcap_{x \in \partial \Omega} H(x)$. This set is not empty: suppose that for each compact set $K$, there exists a sequence $x_i$ so that $K \cap H(x_i) = \emptyset$ for sufficiently large $i$. Then for the accumulation point $x_0$ of $x$, we have $h(x_0) \subset L$ by the continuity of $\tau$. This contradicts Lemma 4.7.

Lemma 4.7

Let $(x, y) \in L^*$. Then $\tau(x, y)$ does not depend on $y$, and $h(x, y)$ is never a hemisphere in $L$ for every $(x, y) \in \Lambda^*$. Hence, we obtain a continuous function $\tau : \partial \Omega \rightarrow S(\partial \Omega)$. 

The properties of lens-shaped ends

- Let us consider a lens-shaped R-end. A **concave end-neighborhood** is an imbedded end neighborhood contained in a radial end neighborhood in $\tilde{O}$ that is a component in the lens-cone of a complement of the lens.

- A **trivial one-dimensional cone** is an open half space in $\mathbb{R}^1$ given by $x > 0$ or $x < 0$.

- We define the set $S(\mathbf{v}_{\mathcal{E}})$ of maximal segments from $\mathbf{v}_{\mathcal{E}}$ in the closure of an end-neighborhood of $\mathbf{v}_{\mathcal{E}}$. 
Theorem 5.1

Let $\mathcal{O}$ be a real projective orbifold with usual property. Let $\tilde{E}$ be a generalized lens-shaped radial $p$-end of $\tilde{\mathcal{O}}$ associated with a $p$-end vertex $\mathbf{v}_{\tilde{E}}$. Assume that $\pi_1(E)$ is irreducible and hyperbolic.

(i) Let $D \ast \mathbf{v}_{\tilde{E}} - \{\mathbf{v}_{\tilde{E}}\}$ be a lens-cone neighborhood for $\tilde{E}$. $\partial \text{Cl}(D) - \partial D$ is independent of the choice of $D$. That is, $D$ is strictly lens-shaped.

(ii) The closure in $\mathbb{S}^n$ of a concave end-neighborhood of $\mathbf{v}_{\tilde{E}}$ contains every segment $l$ in $\partial \tilde{\mathcal{O}}$ meeting the closure of a concave end neighborhood of $\mathbf{v}_{\tilde{E}}$ in $\tilde{\mathcal{O}}$. $\text{S}(\mathbf{v}_{\tilde{E}})$ is independent of the end-neighborhood and so is the union of $\text{S}(\mathbf{v}_{\tilde{E}})$, and $\bigcup \text{S}(\mathbf{v}_{\tilde{E}})$ equals the closure of any end neighborhood of $\mathbf{v}_{\tilde{E}}$ intersected with $\partial \tilde{\mathcal{O}}$.

(iii) Any concave end neighborhood $U$ of $\mathbf{v}_{\tilde{E}}$ under the covering map $\tilde{\mathcal{O}} \rightarrow \mathcal{O}$, covers the end neighborhood of $E$ of form $U / \pi_1(E)$.

(iv) $\text{S}(g(\mathbf{v}_{\tilde{E}})) = g(\text{S}(\mathbf{v}_{\tilde{E}}))$ for $g \in \pi_1(E)$ and $\text{S}(\mathbf{v}_{\tilde{E}})^0 \cap \text{S}(w) = \emptyset$ or $\mathbf{v}_{\tilde{E}} = w$ for end vertices $\mathbf{v}_{\tilde{E}}$ and $w$ where $\text{S}(\mathbf{v}_{\tilde{E}})^0$ is the relative interior of $\text{S}(\mathbf{v}_{\tilde{E}})$ in $\partial \tilde{\mathcal{O}}$. 
Theorem 5.2

Let $O$ be a real projective orbifold with usual property. Let $\tilde{E}$ be a generalized lens-shaped $p$-end with the $p$-end vertex $v_{\tilde{E}}$. Then the following statements hold:

(i) For $S^{n-1}_{v_{\tilde{E}}}$, we obtain

(i-1) Under $\hat{h}(\pi_1(E))$, $\mathbb{R}^n$ splits into $V_1 \oplus \cdots \oplus V_l$ and $\tilde{E}$ is the quotient of the sum $C_1 + \cdots + C_l$ for properly convex or trivial one-dimensional cones $C_i \subset V_i$ for $i = 1, \ldots, l$.

(i-2) The Zariski closure of $\hat{h}(\pi_1(E))$ is the product $G = G_1 \times \cdots \times G_l$ where $G_i$ is a reductive subgroup of $GL(V_i)$.

(i-3) Let $D_i$ denote the image of $C_i$ in $S^{n-1}_{v_{\tilde{E}}}$. The number of hyperbolic group factors of $\pi_1(E)$ is $\leq l$ and

$$\text{Cl}(\tilde{\Sigma}_{\tilde{E}}) = \text{Cl}(D_1) \ast \cdots \ast \text{Cl}(D_l)$$

and each hyperbolic group factor of $\pi_1(E)$ acts on exactly one $D_i$ divisibly and other factors trivially.

(i-4) $\pi_1(E)$ has a rank $l - 1$ free abelian group center.
The properties of lens-shaped ends

Theorem (Continued)

(ii) The $p$-end is strictly lens-shaped.

(iii) A concave $p$-end neighborhood of $\tilde{E}$ is a proper $p$-end neighborhood.

Theorem 5.3

Suppose that $\mathcal{O}$ satisfies the usual conditions and do not contain essential annulus or $\mathcal{O}$ is not virtually reducible. Let $\text{Hom}_v(E, \text{SL}_\pm(n + 1, \mathbb{R}))$ be the space of representations of the admissible $p$-end fundamental group $\pi_1(\tilde{E})$ of a $p$-end $\tilde{E}$ of $\mathcal{O}$. Then for the end $\tilde{E}$, the lens-shapedness is equivalent to strict lens-shapedness and the subspace of lens-shaped representations is open.
**Theorem 5.4**

Let $O$ satisfy the usual condition. Let $\Gamma_{\tilde{E}}$ the holonomy group of a properly convex $p$-end $\tilde{E}$ of a properly convex real projective orbifold with radial or totally geodesic ends. Assume that $O$ satisfies the triangle condition or $\tilde{E}$ is reducible. Then the following statements are equivalent:

(i) $\Gamma_{\tilde{E}}$ is of lens-type.

(ii) $\Gamma_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition.

**Theorem 4**

Let $O$ be as usual. Let $\tilde{S}_{\tilde{E}}$ be a totally geodesic ideal boundary of a totally geodesic $p$-end $\tilde{E}$ of $\tilde{O}$. Then the following conditions are equivalent:

(i) $\tilde{E}$ satisfies the uniform middle-eigenvalue condition.

(ii) $\tilde{S}_{\tilde{E}}$ has a lens-neighborhood in an ambient open manifold containing $\tilde{O}$ and hence $\tilde{E}$ has a lens-type $p$-end neighborhood in $\tilde{O}$. (We can find one in any neighborhood of $\tilde{E}$.)
Let $\tilde{E}$ be a p-end of $O$ with $\tilde{\Sigma}_{\tilde{E}}$ is convex but not properly convex and not complete, and let $U$ the corresponding end neighborhood with the end vertex $v_{\tilde{E}}$. Let $\tilde{E}$ denote the universal cover of $E$. Then $\tilde{\Sigma}_{\tilde{E}}$ is foliated by affine spaces (or open hemispheres) of dimension $i$. 

- the space of $i$-dimensional hemispheres with boundary $S^{i-1}_{\infty}$ equals projective $S^{n-i-1}$.

- Now going $S^n$. Each hemisphere $H^i \subset S^{n-1}$ with $\partial H^i = S^{i-1}_{\infty}$ corresponds to $H^{i+1}$ in $S^n$ whose common boundary $S^i_{\infty}$ that contains $v_{\tilde{E}}$. Note $S^i_{\infty}$ is $h(\pi_1(E))$-invariant.

- We let $N$ be the subgroup of $h(\pi_1(E))$ of elements inducing trivial actions on $S^{n-i-1}$. 

Nonproperly convex (NPCC) ends
Proposition 6.1

Let $\tilde{E}$ be a NPCC $p$-end of a properly convex $n$-orbifold $\mathcal{O}$ with usual conditions. Let $
abla_{\tilde{E}} : \pi_1(\tilde{E}) \to \text{Aut}(\mathbb{S}^{n-1})$ be the associated holonomy homomorphism for the corresponding end vertex $v_{\tilde{E}}$. Then

- $\tilde{\Sigma}_{\tilde{E}}$ is foliated by complete affine subspaces of dimension $i$, $i > 0$.
- $h(\pi_1(E))$ acts on the great sphere $\mathbb{S}^{i-1}_{\infty}$ of dimension $i - 1$ in $\mathbb{S}^{n-1}_{v_{\tilde{E}}}$.
- There exists an exact sequence

$$1 \to N \to \pi_1(\tilde{E}) \to N_K \to 1$$

where $N$ acts trivially on quotient great sphere $\mathbb{S}^{n-i-1}$ and $N_K$ acts faithfully on a properly convex domain $K$ in $\mathbb{S}^{n-i-1}$ isometrically with respect to the Hilbert metric $d_K$.

Proof.

These are proved in [9].
Some examples of NPCC p-ends: the join and the quasi-join

**Lens part** \( \nu_1 \ast L \) where \( L \) is a properly open convex domain in a hyperspace \( S'_1 \) outside \( \nu_1 \).

Let \( \Gamma_1 \) acts on \( \nu_1 \) and \( L \). \( L/\Gamma \) is compact. (coming from some lens-type end)

Assume \( \nu_1 \ast L \subset S^{n-i_0-1}_1 \) for a subspace.

**Horosphere** Let \( H \) be horosphere with vertex \( \nu_2 \) in a subspace \( S^{i_0+1}_2 \) with \( \Gamma_2 \) act on it. \( \partial H/\Gamma_2 \) is a compact suborbifold. (coming from hyperbolic cusp.)

**Complementary** We embed these in an affine subspace of \( \mathbb{RP}^n \) where

\[ \nu_1 \ast L \subset S_1, H \subset S_2, \text{ so that } S_1 \cap S_2 = \{ \nu_1 = \nu_2 \}, S'_1 \cap S_2 = \emptyset. \]

**Join** Obtain a join \( (\nu_1 \ast L) \ast H \). Extend \( \Gamma_2 \) trivially on \( S_1 \). Extend \( \Gamma_1 \) to an action on \( S_2 \) normalizing \( \Gamma_2 \).
**Joined action**  Find an infinite cyclic group action by $g$ fixing every points $S'_1$ and $S_2$. They correspond to different eigenspaces of $g$ of eigenvalues $\lambda_1, \lambda_2, \lambda_1 < \lambda_2$. Then $\Gamma_1 \times \Gamma_2 \times \langle g \rangle$ acts on the join.

**Quasi-join**  We multiply $g$ a translation $T$ in $S_2$ towards $v_2$. Then $\Gamma_2 \times \Gamma_1 \times \langle g \rangle$ now acts on properly convex domain whose closure meets $S_1$ at $v_2$ only. For example, for $(\lambda < 1, k > 0)$,

\[
g := \begin{pmatrix} \lambda^3 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & k & 0 & \frac{1}{\lambda} \end{pmatrix}, \quad n := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \nu & 1 & 0 \\ 0 & \frac{\nu^2}{2} & \nu & 1 \end{pmatrix}
\]

(helped from Benoist.)
Definition 5

We define $\lambda_1(g)$ to equal to the largest norm of the eigenvalue of $g$ associated with a vector $\vec{v}$, $[\vec{v}] \in \mathbb{S}^n - \mathbb{S}_0^\infty$, and $\lambda_{v_E}(g)$ the eigenvalue of $g$ at $v_E$.

Definition 6 (The uniform middle-eigenvalue condition relative to $N$):

We also assume that

$$N_K \cong \mathbb{Z}^l \times \Gamma_1 \ldots \Gamma_k, \ l \geq k - 1$$

acts on

$$\text{Cl}(K) = p_1 * \ldots * p_{l-k-1} * K_1 * \ldots * K_k$$

where $\Gamma_i$ is a hyperbolic group acting on properly convex domain $K_i$ for each $i$, $i = 1, \ldots, k$, and each $p_j$ is a singleton for $j = 1, \ldots l - k - 1$ with following conditions:

- there exists a constant $C > 0$ independent of $g$ and $i$ such that for $\pi_K(g) \in \Gamma_i - \{I\}$ or $\pi_K(g)$ in the central set $\mathbb{Z}^{l_0-1} - \{I\}$

$$C^{-1}\text{length}_K(g) \leq \log \frac{\lambda_1(g)}{\lambda_{v_E}(g)} \leq C\text{length}_K(g) \quad (5)$$

For elements of the center $\mathbb{Z}^{l_0-1}$, we may just require $\lambda_1(g) \geq \lambda_{v_E}(g)$ for all $g \in \mathbb{Z}^{l_0-1}$. Then $\Gamma$ satisfies the weakly uniform middle-eigenvalue condition.
Theorem 7

Let $\Sigma_{\tilde{E}}$ be the end orbifold of a NPCC radial p-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $O$ satisfying usual conditions. Let $\Gamma_{\tilde{E}}$ be the end fundamental group, and it satisfies the weakly uniform middle-eigenvalue condition. Then there exists a finite cover $\Sigma_{E'}$ of $\Sigma_{\tilde{E}}$ so that $E'$ is a quasi-join of a totally geodesic ends and a cusp end.

Proof Idea

We show that the $d_K$-length of the action of $g$ dominates the log of the eigenvalues in the directions of $S^{i}_{\infty}$. In the first case, when $N$ is discrete: here, for a leaf $l$ of $\tilde{\Sigma}_{\tilde{E}}$, $l/N$ maps to a horosphere and $N$ is in the cusp group and acts on a modified p-end neighborhood $U$. $\Gamma_{\tilde{E}}$ normalize $N$. By computation, we can show that $\Gamma_{\tilde{E}}$ splits into two parts: a semi-simple part and horospherical part.

When $N$ is indiscrete, we show that the foliation of $\tilde{\Sigma}_{\tilde{E}}$ has polynomial growth leaves. Then the closure of the holonomy is solvable using the work of Carrière and Molino. We use the syndetic hull and show that there is a cusp group acting on a modified p-end neighborhood $U$. We work as above.
Non-properly convex (NPCC) ends

Figure: A developing of the boundary of quasi-joined end neighborhood.
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