DEGREE THREE INVARIANTS FOR SEMISIMPLE GROUPS OF TYPES $B$, $C$, AND $D$

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Abstract. We determine the group of reductive cohomological degree 3 invariants of all split semisimple groups of types $B$, $C$, and $D$. We also present a complete description of the cohomological invariants. As an application, we show that the group of degree 3 unramified cohomology of the classifying space $BG$ is trivial for all split semisimple groups $G$ of types $B$, $C$, and $D$.

1. Introduction

A degree $d$ cohomological invariant of an algebraic group $G$ defined over a field $F$ is a natural transformation of functors

$$G\text{-torsors} \to H^d$$

on the category of field extensions over $F$, where the functor $G\text{-torsors}$ takes a field $K/F$ to the set $G\text{-torsors}(K)$ of isomorphism classes of $G$-torsors over $K$ and the functor $H^d$ takes $K$ to the Galois cohomology $H^d(K) = H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$. All degree $d$ invariants of $G$ form a group $\text{Inv}^d(G)$. This notion was introduced by Serre, and since then it has been intensively studied by Merkurjev and Rost for $d = 3$ [10, 19].

In this paper, we study degree 3 cohomological invariants of split semisimple groups of Dynkin types $B$, $C$, and $D$. Thus from now on we shall focus on degree 3 invariants. Let $G$ be a split reductive group over a field $F$. An invariant in $\text{Inv}^3(G)$ is called normalized if it vanishes on trivial $G$-torsors. Such invariants form a subgroup $\text{Inv}^3(G)_{\text{norm}}$ of $\text{Inv}^3(G)$, thus $\text{Inv}^3(G) = \text{Inv}^3(G)_{\text{norm}} \oplus H^3(F)$. A normalized invariant in $\text{Inv}^3(G)_{\text{norm}}$ is called decomposable if it is given by a cup product of a degree 2 invariant with a constant invariant of degree 1. The subgroup of decomposable invariants of degree 3 is denoted by $\text{Inv}^3(G)_{\text{dec}}$. The quotient group $\text{Inv}^3(G)_{\text{norm}}/\text{Inv}^3(G)_{\text{dec}}$ is called the group of indecomposable invariants and is denoted by $\text{Inv}^3(G)_{\text{ind}}$. This group has been completely determined for all split simple groups in [10], [19], [4] and for some semisimple groups in [17], [1], [2], and [15].

Let $G$ be a split semisimple group over $F$. A strict reductive envelope of $G$ is a split reductive group $G_{\text{red}}$ over $F$ such that the derived subgroup of $G_{\text{red}}$ is $G$ and the center of $G_{\text{red}}$ is a torus. Then, by [18, §10] the restriction map

$$\text{Inv}^3(G_{\text{red}})_{\text{ind}} \to \text{Inv}^3(G)_{\text{ind}}$$

is injective and its image is independent of the choice of a strict reductive envelope $G_{\text{red}}$. This image is called the subgroup of reductive indecomposable invariants of $G$. 


and is denoted by $\text{Inv}^3(G)_{\text{red}}$. Recently, this subgroup has been completely computed for all split simple groups in [13] and for all split semisimple groups of type $A$ in [17].

In the present paper, we determine the group of reductive indecomposable invariants of all split semisimple groups of types $B$, $C$, and $D$, which completes the cohomological invariants of classical groups. In particular, if each component of the corresponding root system of type $B$ (respectively, type $C$) has rank at least 2 (respectively, even rank), then the group of indecomposable invariants is also determined as follows (see Theorem 5.1, Theorem 5.5, Theorem 5.6, and Corollary 5.2):

**Theorem 1.1.** Let $G$ be an arbitrary split semisimple group of one of the following types: $B$, $C$, and $D$, i.e., $G = \prod_{i=1}^{m} \text{Spin}_{2n_i+1}/\mu$ $(n_i \geq 1)$, $(\prod_{i=1}^{m} \text{Sp}_{2n_i})/\mu$ $(n_i \geq 1)$, and $(\prod_{i=1}^{m} \text{Spin}_{2n_i})/\mu$ $(n_i \geq 3)$ respectively for some central subgroup $\mu$ and $m \geq 1$. Let $R$ be the subgroup of $Z$ whose quotient is the character group $\mu^\ast$, where

$$
Z := \bigoplus_{i=1}^{m} Z_i, \quad Z_i = \begin{cases} (\mathbb{Z}/2\mathbb{Z})e_i & \text{if } G \text{ is of type } B \text{ or } C, \\
(\mathbb{Z}/4\mathbb{Z})e_i & \text{if } G \text{ is of type } D, n_i \text{ odd,} \\
(\mathbb{Z}/2\mathbb{Z})e_{i1} \bigoplus (\mathbb{Z}/2\mathbb{Z})e_{i2} & \text{if } G \text{ is of type } D, n_i \text{ even,}
\end{cases}
$$

denotes the character group of the center of the corresponding simply connected semisimple group.

1. **Assume that $G$ is of type $B$.** Let $l = \dim R$. Then,

$$
\text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{l_1-l_2},
$$

where $l_1 = \dim\langle e_i \in R \mid n_i \leq 2 \rangle$, $l_2 = \dim\langle e_i + e_j \in R \mid e_i, e_j \not\in R, n_i = n_j = 1 \rangle$. In particular, if $n_i \geq 2$ for all $1 \leq i \leq m$, then

$$
\text{Inv}^3(G)_{\text{ind}} = \text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{l_1-l_2}.
$$

2. **Assume that $G$ is of type $C$.** Let $s$ denote the number of ranks $n_i$ divisible by 4 and $l = \dim \langle R \cap (\bigoplus_{4n_i} Z_i) \rangle$. Then,

$$
\text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{s+l_1-l_2},
$$

where $l_1 = \dim\langle e_i \in R \rangle$ and $l_2 = \dim\langle e_i + e_j \in R \mid e_i, e_j \not\in R, n_i \equiv n_j \equiv 1 \mod 2 \rangle$. In particular, if $n_i \equiv 0 \mod 2$ for all $i$, then

$$
\text{Inv}^3(G)_{\text{ind}} = \text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{s+l_1-l_2}.
$$

3. **Assume that $G$ is of type $D$.** Let

$$
\bar{R} = \{(\bar{r}_1, \ldots, \bar{r}_m) \in \bigoplus_{i=1}^{m} (\mathbb{Z}/2\mathbb{Z})\bar{e}_i \mid \sum_{i=1}^{m} r_i \in R \}, \quad \text{where } r_i = \begin{cases} 2\bar{r}_i e_i & \text{if } n_i \text{ odd,} \\
\bar{r}_i e_{i1} + \bar{r}_i e_{i2} & \text{if } n_i \text{ even,}
\end{cases}
$$

$R_{i1} = R \cap Z_i$ for odd $n_i$, and $R_{i1} = R \cap Z_i$ for even $n_i$. Set

$$
R' = \bar{R} \cap \bigoplus_{\text{odd } n_i, R_{i1} \neq \mathbb{Z}_i} (\mathbb{Z}/2\mathbb{Z})\bar{e}_i \text{ with } l = \dim R', \quad I_1 = \{i \mid Z_i = R_{i1} \text{ or } R'_{i1}, n_i \neq 3 \},
$$

$I_2 = \{i \mid R'_{i1} = 0, 4 \mid n_i \} \cup \{i \mid R'_{i1} = (\mathbb{Z}/2\mathbb{Z})e_{i1} \text{ or } (\mathbb{Z}/2\mathbb{Z})e_{i2}, n_i \geq 6, 4 \mid n_i \}$ with $s_i = |I_i|$. }
Then, we have
\[ \text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{e_1 + s_2 + i - l_1 - l_2}, \]
where
\[ l_1 = |\{i \mid 4 \nmid n_i, R_{i_1} = 2Z_i \text{ or } R_{i_1} = (\mathbb{Z}/2\mathbb{Z})(e_{i_1} + e_{i_2})\}|, \quad l_2 = \dim(\bar{e}_i + \bar{e}_j \mid R_{i_1} = R_{i_j} = 0, 2e_i + 2e_j \in R). \]

For each type of \(B, C, \) and \(D\), our main theorem can be restated as follows (see Propositions \[6.3, 6.7, 6.13\]): Assume that \(F\) is an algebraically closed field. For type \(B\), let \(G_{\text{red}} = (\prod_{i=1}^{m} \Gamma_{2n_i+1})/\mu\), where \(\Gamma_{2n_i+1}\) is the split even Clifford group \([12, \S 23]\) and let
\[ R \to \text{Inv}^3(G_{\text{red}})_{\text{norm}} \]
be the homomorphism given by \(r \mapsto \text{e}_3(\phi[r])\), where \(\phi[r]\) is the quadratic form defined in Remark \[6.2\] and \(\text{e}_3\) denotes the Arason invariant. Then, this morphism is surjective and its kernel is the subspace
\[ \langle e_i, e_j + e_k \in R \mid e_j, e_k \notin R, n_i \leq 2, n_j = n_k = 1 \rangle. \]

For type \(C\), let \(G_{\text{red}} = (\prod_{i=1}^{m} \text{GSp}_{2n_i})/\mu\), where \(\text{GSp}_{2n_i}\) is the group of symplectic similitudes \([12, \S 12]\) and let
\[ \bigoplus_{4 \mid n_i} (\mathbb{Z}/2\mathbb{Z})e_i \bigoplus (R \cap (\bigoplus_{4 \mid n_i} (\mathbb{Z}/2\mathbb{Z})e_i)) \to \text{Inv}^3(G_{\text{red}})_{\text{norm}} \]
be the homomorphism given by \(e_i \mapsto \Delta_i\) for \(i\) such that \(4 \mid n_i\) and \(r \mapsto \text{e}_3(\phi[r])\) for \(r \in R \cap (\bigoplus_{4 \mid n_i} (\mathbb{Z}/2\mathbb{Z})e_i)\), where \(\phi[r]\) is the quadratic form defined in \([52]\) and \(\Delta_i\) is the invariant in \([53]\) induced by the Garibaldi-Parimala-Tignol invariant \([11]\). Then, this morphism is surjective and its kernel is given by
\[ \langle e_i, e_j + e_k \in R \mid e_j, e_k \notin R, n_j \equiv n_k \equiv 1 \mod 2 \rangle. \]

For type \(D\), let \(G_{\text{red}} = (\prod_{i=1}^{m} \Omega_{2n_i})/\mu\), where \(\Omega_{2n_i}\) is the extended Clifford group \([12, \S 13]\) and let
\[ \bigoplus_{i \in I_1 \cup I_2} (\mathbb{Z}/2\mathbb{Z})\bar{e}_i \bigoplus R' \to \text{Inv}^3(G_{\text{red}})_{\text{norm}} \]
be the homomorphism given by \(\bar{e}_i \mapsto \text{e}_{3,i}\) for \(i \in I_1\), \(\bar{e}_i \mapsto \Delta'_i\) for \(i \in I_2\), and \(r \mapsto \text{e}_3(\phi[r])\) for \(r \in R'\), where \(\text{e}_{3,i}\) denotes the invariant in \([66]\) induced by the Arason invariant, \(\Delta'_i\) denotes the invariant in \([67]\) given by the invariant of \(\text{PGO}_{2n_i}^+\) (see \([19\), Theorem 4.7]), and \(\phi[r]\) is the quadratic form defined in \([52]\). Then, the morphism is surjective, and its kernel is given by
\[ \langle \bar{e}_i, \bar{e}_j + \bar{e}_k \in R' \mid \bar{e}_j, \bar{e}_k \notin R', n_j \equiv n_k \equiv 1 \mod 2 \rangle. \]

Therefore, our main result (Theorem \[11.1\]) tells us that for all split semisimple groups of types \(B, C, D\) there are essentially two types of degree three reductive invariants given by the Arason invariant \(\text{e}_3\) and the Garibaldi-Parimala-Tignol invariant \(\Delta_i\) (and its analogue \(\Delta'_i\)) and no other invariants exist.
An invariant $\alpha \in \text{Inv}^3(G)$ is said to be \textit{unramified} if for any field extension $K/F$ and any element $\eta \in G$-torsors$(K)$, its value $\alpha(\eta)$ is contained in $H^3_{\text{nr}}(K)$, where $H^3_{\text{nr}}(K)$ denotes the subgroup in $H^3(K)$ of all unramified elements defined by

$$H^3_{\text{nr}}(K) = \bigcap_v \ker\left( \partial_v : H^3(K) \to H^2(F(v)) \right)$$

for all discrete valuations $v$ on $K/F$ and their residue homomorphisms $\partial_v$. The subgroup of all unramified invariant in $\text{Inv}^3(G)$ will be denoted by $\text{Inv}^3_{\text{nr}}(G)$. By a theorem of Rost, we have an isomorphism

$$\text{Inv}^3_{\text{nr}}(G) \simeq H^3_{\text{nr}}(F(BG)), \tag{1}$$

where $BG$ is the classifying space of $G$ (see [18], [25]).

A generalized version of Noether’s problem asks whether the classifying space $BG$ of an algebraic group $G$ is stably rational or retract rational (see [6], [16]). A way of detecting non-retract rationality is to use unramified cohomology as the following statement: the classifying space $BG$ is not retract rational if there exists a non-constant unramified invariant of degree $d$ for some $d$ [16]. In fact, Saltman gave the first counter example over an algebraically closed field to the original Noether’s question by providing certain finite groups which have a non-constant unramified invariant of degree 2 [21]. However, the generalized Noether’s problem is still open for a connected algebraic group over an algebraically closed field.

In [5], Bogomolov showed that connected groups have no nontrivial degree 2 unramified invariants, i.e., $\text{Inv}^2_{\text{nr}}(G) = 0$ for a connected group $G$. In [22] and [23], Saltman showed that the group $\text{Inv}^3_{\text{nr}}(\text{PGL}_n)$ is trivial. Recently, Merkurjev has shown that the group $\text{Inv}^3_{\text{nr}}(G)$ is trivial if $G$ is a split simple group [18] or a split semisimple group of type $A$ [14] over an algebraically field $F$ of characteristic 0.

Using the main theorem above we determine the group of unramified invariants of a split semisimple groups of types $B$, $C$, and $D$ (see Theorems 6.5, 6.10, 6.15).

**Theorem 1.2.** Let $G = (\prod_{i=1}^m \text{Spin}_{2n_i+1})/\mu$ ($n_i \geq 1$) or $(\prod_{i=1}^m \text{Sp}_{2n_i})/\mu$ ($n_i \geq 1$) or $(\prod_{i=1}^m \text{Spin}_{2n_i})/\mu$ ($n_i \geq 3$) defined over an algebraically closed field $F$ of characteristic 0, $m \geq 1$, where $\mu$ is an arbitrary central subgroup. Then, there are no nontrivial unramified degree 3 invariants for $G$, i.e., $\text{Inv}^3_{\text{nr}}(G) = H^3_{\text{nr}}(F(BG)) = 0$.

This paper is organized as follows. In Section 2 we recall some basic definitions and facts used in the rest of the paper. Sections 3-5 are devoted to the computation of the group of degree 3 invariants of a split semisimple group $G$ of types $B$, $C$, and $D$. In the last section, we present a description of the degree 3 invariants of $G$ and a proof of the second main result.

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2. Cohomological invariants of degree 3

In this section we recall some basic notions concerning degree 3 invariants following [10, 19]. We shall frequently use these in the following sections.

2.1. Invariant quadratic forms. Let $\tilde{G}$ be a split semisimple simply connected group of Dynkin type $D$, i.e., $\tilde{G} = G_1 \times \cdots \times G_m$ for some integer $m \geq 1$, where each $G_i$ is a split simple simply connected group of type $D$. Consider the natural action of the Weyl group $W = W_1 \times \cdots \times W_m$ of $\tilde{G}$ on the weight lattice $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_m$, where $W_i$ (resp. $\Lambda_i$) is the Weyl group (resp. the weight lattice) of $G_i$. Then, the group of $W$-invariant quadratic forms $S^2(\Lambda)^W$ on $\Lambda$, denoted by $Q(\tilde{G})$, is a sum of cyclic groups

$$Q(\tilde{G}) = \mathbb{Z}q_1 \oplus \cdots \oplus \mathbb{Z}q_m,$$

where $q_i$ is the normalized Killing form of $G_i$, for $1 \leq i \leq m$.

Consider an arbitrary split semisimple group $G$ of Dynkin type $D$, i.e., $G = \tilde{G}/\mu$, where $\mu$ is a central subgroup. Let $T$ be a split maximal torus of $G$ and let $T^*$ be the group of characters of $T$. Then, the subgroup $Q(G)$ of $W$-invariant quadratic forms on $T^*$ is given by

$$Q(G) = S^2(T^*) \cap Q(\tilde{G}).$$

2.2. Degree 3 invariants. Consider the Chern class map $c_2 : \mathbb{Z}[T^*] \to S^2(T^*)$ defined by $c_2(\sum \lambda_i^\alpha) = \sum_{i<j} \lambda_i \lambda_j$ [19, §3c], where $\mathbb{Z}[T^*]$ is the group ring of the maximal torus $T$ in Section 2.1 and $\lambda_i \in T^*$. Since $(T^*)^W = 0$, the restriction of $c_2$ induces a group homomorphism

$$c_2 : \mathbb{Z}[T^*]^W \to Q(G)$$

We shall write $\text{Dec}(G)$ for the image of $c_2$ in (3). For $\lambda \in T^*$, we denote by $\rho(\lambda) = \sum_{\chi \in W(\Lambda)} e^\chi$, where $W(\Lambda)$ is the $W$-orbit of $\lambda$. Then, the subgroup $\text{Dec}(G)$ is generated by $c_2(\rho(\lambda)) = -\frac{1}{2} \sum_{\chi \in W(\Lambda)} \chi^2$. By [19, Theorem 3.9], the indecomposable invariants of $G$ is determined by the following exact sequence

$$0 \to \text{Inv}^3(G)_{\text{dec}} \to \text{Inv}^3(G)_{\text{norm}} \to Q(G)/\text{Dec}(G) \to 0.$$ 

In particular, if $F$ is algebraically closed, then we have $\text{Inv}^3(G)_{\text{norm}} = Q(G)/\text{Dec}(G)$.

3. The group $Q(G)$ for semisimple groups $G$ of types $B$, $C$, and $D$

In the present section, we shall compute the group $Q(G)$ for types $B$, $C$, and $D$.

3.1. Type $B$. Let $G = (\prod_{i=1}^m \text{Spin}_{2n_i+1})/\mu$ be an (arbitrary) split semisimple group of type $B$, $m, n_i \geq 1$, where $\mu \simeq (\mu_2)^k$ is a central subgroup for some $k \geq 0$. Let $T$ be the split maximal torus of $G$ (i.e., $T = (\mathbb{G}_m)^n/\mu$) and let

$$R = \{r = (r_1, \ldots, r_m) \in (\mathbb{Z}/2\mathbb{Z})^m \mid f_p(r) = 0, 1 \leq p \leq k\}$$
be the subgroup of $(\mathbb{Z}/2\mathbb{Z})^m$ whose quotient is the character group $\mu^*$ for some linear polynomials $f_p \in \mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_m]$. Consider the following commutative diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & R & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^m & \longrightarrow & \mu^* & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & T^* & \longrightarrow & \prod_{i=1}^m \mathbb{Z}^{n_i} & \longrightarrow & \mu^* & \longrightarrow & 0 \\
\end{array}
\]

where $T^*$ is the corresponding character group and the middle map $\prod_{i=1}^m \mathbb{Z}^{n_i} \to (\mathbb{Z}/2\mathbb{Z})^m$ is given by

\[
\sum a_{ij}w_{ij} \mapsto (a_{1n_1}, \ldots, a_{mn_m})
\]

for $1 \leq i \leq m$ and $1 \leq j \leq n_i$, where $w_{ij}$ denote the fundamental weights for the $i$th component of the root system of $G$. For the rest of this subsection, we simply write $a_i$ and $w_i$ for $a_{in_i}$ and $w_{in_i}$, respectively. Then, it follows from (5) that

\[
T^* = \{ \sum a_{ij}w_{ij} | f_p(a_1, \ldots, a_m) \equiv 0 \mod 2 \}.
\]

Let $I = \{1, \ldots, m\}$ and let $I_1 = \{i \in I | f_p(e_i) = 0, 1 \leq p \leq k\}$, where $\{e_1, \ldots, e_m\}$ denotes the standard basis of $\mathbb{Z}^m$. We write the relations $f_p(a_1, \ldots, a_m) \equiv 0 \mod 2$ as

\[
(a_{i_1}, \ldots, a_{i_k})^T = B \cdot (a_{j_1}, \ldots, a_{j_l})^T + (2c_1, \ldots, 2c_k)^T
\]

for some distinct $i_1, \ldots, i_k, j_1, \ldots, j_l$ such that $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} = I \setminus I_1$ and some $k \times l$ binary matrix $B = (b_{ij})$ (i.e., $b_{ij} = 0$ or 1) with $c_p \in \mathbb{Z}$. Then, we have

\[
\sum a_{ij}w_{ij} = \sum_{1 \leq i \leq m, 1 \leq j \leq n_i-1} a_{ij}w_{ij} + \sum_{i \in I_1} a_iw_i + \sum_{p=1}^k 2c_pw_{ip} + \sum_{s=1}^l a_{js}(w_{js} + g_s)
\]

where $g_s = (w_{i_1}, \ldots, w_{i_k}) \cdot B_s$ and $B_s$ is the $s$-th column of $B$, thus we obtain the following $\mathbb{Z}$-basis of $T^*$:

\[
\{w_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n_i-1} \cup \{w_i\}_{i \in I_1} \cup \{2w_{ij}\}_{1 \leq p \leq k} \cup \{w_{js} + g_s\}_{1 \leq s \leq l}.
\]

Let $v_p = 2w_{ip}$ and $h_p(t_1, \ldots, t_l) = b_{p1}t_1 + \cdots + b_{pl}t_l \in \mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_l]$ for $1 \leq p \leq k$. Since the group $Q(\tilde{G})$ is generated by the normalized Killing forms $q_i = \begin{cases} 2w_i^2 - w_{in_i-1}w_i + \sum_{j=1}^{n_i-1} w_{ij}^2 & \text{if } n_i \geq 1, \\ w_i^2 & \text{if } n_i = 1 \end{cases}$ for all $1 \leq i \leq m$, any element of $Q(G)$ is of the form $q = \sum_{i=1}^m d_i q_i$ for some $d_i \in \mathbb{Z}$. Therefore, with respect to the basis (8) we have

\[
q = q' + \frac{1}{4} \sum_{p=1}^k v_p^2[\delta_{ip}d_{ip} + h_p(\delta_{j_1}d_{j_1}, \ldots, \delta_{j_l}d_{j_l})] + \frac{1}{2} \sum_{1 \leq i < j \leq k} v_i v_j h_i(\delta_{j_1}d_{j_1}, b_{j_1}, \ldots, \delta_{j_l}d_{j_l}, b_{j_l})
\]
for some quadratic form $q'$ with integer coefficients, where

$$\delta_i = \begin{cases} 
2 & \text{if } n_i \geq 2, \\
1 & \text{if } n_i = 1.
\end{cases}$$

Hence, by (2) we obtain $q = \sum_{i=1}^{m} d_i q_i \in Q(G)$ if and only if

$$\delta_i p d_i p + h_p (\delta_{j_1} d_{j_1}, \ldots, \delta_{j_i} d_{j_i}) \equiv 0 \mod 4$$

and

$$h_p (\delta_{j_1} d_{j_1} b_{j_1}, \ldots, \delta_{j_i} d_{j_i} b_{j_i}) \equiv 0 \mod 2$$

for all $1 \leq p \leq k$. In particular, since two systems of equations $\{f_p(t_1, \ldots, t_m)\}$ and $\{t_i p + h_p(t_{j_1}, \ldots, t_{j_i})\}$ are equivalent we replace the condition (9) by

$$f_p (\delta_1 d_1, \ldots, \delta_m d_m) \equiv 0 \mod 4.$$

Equivalently, we can compute $Q(G)$ with respect to a basis of $R$ as follows. Let

$$R_1 = \langle e_i \mid e_i \in R \rangle$$

and

$$R_2 = \langle e_i + e_j \mid e_i, e_j \in R, e_i, e_j \notin R_1 \rangle$$

be the subspaces of $R$. We first choose $\{w_i\}_{i \in I_1}$ as a part of basis of $T^*$. Then, for the remaining part of a basis of $T^*$ we write a given basis of $R$ as

$$(e_{j_1}, \ldots, e_{j_l})^T = C(e_{i_1}, \ldots, e_{i_l})^T$$

for some $i_1, \ldots, i_k, j_1, \ldots, j_l$ with $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} = I \setminus I_1$ and some $l \times k$ binary matrix $C$ such that all basis elements of the form $e_i + e_j$ in $R_2$ is a part of (13). Then, we have the same $\mathbb{Z}$-basis of $T^*$ as in (8) by replacing $g_s$ in (8) with $g_s = C_s \cdot (w_{i_1}, \ldots, w_{i_k})$, where $C_s$ is the $s$-th row of $C$. The rest of the computation is the same as in the previous one.

In particular, if either $R = R_1 \oplus R_2$ or $n_i \geq 2$ for all $1 \leq i \leq m$, then the condition (10) becomes trivial, thus

**Proposition 3.1.** Let $G = ((\prod_{i=1}^{m} \text{Spin}_{2n_i+1})/\mu, m, n_i \geq 1$, where $\mu \simeq (\mu_2)^k$ is a central subgroup for some $k \geq 0$. Let $R = \{r \in (\mathbb{Z}/2\mathbb{Z})^m \mid f_p(r) = 0, 1 \leq p \leq k\}$ be the subgroup of $(\mu_2^\ast)^\ast$ whose quotient is the character group $\mu^\ast$ for some linear polynomials $f_j \in \mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_m]$. Assume that either $n_i \geq 2$ for all $i$ or $R = R_1 \oplus R_2$, where $R_1$ and $R_2$ are the subgroups of $R$ defined in (12). Then, we have

$$Q(G) = \{\sum_{i=1}^{m} d_i q_i \mid f_p (\delta_1 d_1, \ldots, \delta_m d_m) \equiv 0 \mod 4\}, \text{ where } \delta_i = \begin{cases} 
2 & \text{if } n_i \geq 2, \\
1 & \text{if } n_i = 1.
\end{cases}$$

**3.2. Type C.** Let $G = ((\prod_{i=1}^{m} \text{Sp}_{2n_i})/\mu$ be a split semisimple group of type $C$, where $m, n_i \geq 1$ and $\mu \simeq (\mu_2)^k$ is a central subgroup for some $k \geq 0$. Let $T$ be the split maximal torus of $G$ and let $R$ be the subgroup of $(\mathbb{Z}/2\mathbb{Z})^m$ as in (4). Then, we have the same commutative diagram (5), replacing the middle vertical map (6) by

$$\sum a_{ij} e_{ij} \mapsto (\sum_{j=1}^{n_1} \tilde{a}_{1j}, \ldots, \sum_{j=1}^{n_m} \tilde{a}_{mj}),$$
where $e_{ij}$ denote the standard basis for the $i$th component of $\prod_{i=1}^{m} \mathbb{Z}^{n_i}$. Then, by (5) we have

$$T^* = \{ \sum a_{ij} e_{ij} \mid f_p(\sum_{j=1}^{n_1} a_{1j}, \ldots, \sum_{j=1}^{n_m} a_{mj}) \equiv 0 \mod 2 \}. \quad (14)$$

We simply write $e_i$ for $e_{ii}$. Let $e'_{ij} = e_{ij} - e_i$ for all $1 \leq i \leq m$ and $2 \leq j \leq n_i$ and let $a_i = \sum_{j=1}^{n_i} a_{ij}$. Then, we apply the same argument as in type $B$ so that we have the following $\mathbb{Z}$-basis of $T^*$

$$\{ e'_{ij} \}_{1 \leq i \leq m, 2 \leq j \leq n_i} \cup \{ e_i \}_{i \in I_1} \cup \{ 2e_{ip} \}_{1 \leq p \leq k} \cup \{ e_{is} + g_s \}_{1 \leq s \leq l}, \quad (15)$$

where $B$ is the binary matrix as in (7) and $g_s = (e_{i_1}, \ldots, e_{i_q}) \cdot B_s$.

Let $v_p = 2e_{ip}$ and let $h_p$ be the polynomial defined as in type $B$. Since the normalized Killing forms are given by

$$q_i = e_{i1}^2 + \cdots + e_{im}^2,$$

for any $q \in Q(G)$ there exist $d_i \in \mathbb{Z}$ such that $q = \sum_{i=1}^{m} d_i q_i$, thus with respect to the basis (15) we have

$$q = q' + \frac{1}{4} \sum_{p=1}^{k} v_p^2 [n_p d_{ip} + h_p(n_{q_1} d_{j_1}, \ldots, n_{q_k} d_{j_k})] + \frac{1}{2} \sum_{1 \leq i < j \leq k} v_i v_j h_i(n_{j_1} d_{j_1}, \ldots, n_{j_k} d_{j_k})$$

for some quadratic form $q'$ with integer coefficients. Therefore, by the same argument as in type $B$ we have $q = \sum_{i=1}^{m} d_i q_i \in Q(G)$ if and only if

$$h_p(n_{j_1} d_{j_1}, \ldots, n_{j_k} d_{j_k}) \equiv 0 \mod 2 \quad (16)$$

and $f_p(n_{1d_1}, \ldots, n_{md_m}) \equiv 0 \mod 4$ for all $1 \leq p \leq k$.

Similar to the case of type $B$, if $R = R_1 \oplus R_2$ or $n_i$ is even for all $1 \leq i \leq m$, then the first condition in (16) becomes obvious, thus

**Proposition 3.2.** Let $G = (\prod_{i=1}^{m} \text{Sp}_{2n_i})/\mu$, $m, n_i \geq 1$, where $\mu \simeq (\mu_2)^k$ is a central subgroup for some $k \geq 0$. Let $R$, $R_1$, and $R_2$ be the groups as in (4) and (12). Assume that either $n_i$ is even for all $i$ or $R = R_1 \oplus R_2$. Then, we have

$$Q(G) = \{ \sum_{i=1}^{m} d_i q_i \mid f_p(n_{1d_1}, \ldots, n_{md_m}) \equiv 0 \mod 4 \}. \quad (17)$$

### 3.3. Type D

Let $G = (\prod_{i=1}^{m} \text{Spin}_{2n_i})/\mu$ be a split semisimple group of type $D$, $m \geq 1$, $n_i \geq 3$, where $\mu \simeq (\mu_2)^{k_1} \times (\mu_4)^{k_2}$ is a subgroup of the center $Z(\prod_{i=1}^{m} \text{Spin}_{2n_i})$ for some $k_1, k_2 \geq 0$. We shall denote the character group $Z(\prod_{i=1}^{m} \text{Spin}_{2n_i})^*$ by

$$Z := \bigoplus_{i=1}^{m} Z_i, \quad (17)$$

where $Z_i = \begin{cases} (\mathbb{Z}/4\mathbb{Z}) e_i & \text{if } n_i \text{ odd}, \\ (\mathbb{Z}/2\mathbb{Z}) e_i \oplus (\mathbb{Z}/2\mathbb{Z}) e_i & \text{if } n_i \text{ even}. \end{cases}$

Let $T$ be the split maximal torus of $G$ and let

$$R = \{ r \in Z \mid f_p(r) = 0, 1 \leq p \leq k \}$$
be the subgroup of $Z$ such that $\mu^* \simeq Z/R$ for some linear polynomials $f_1, \ldots, f_k \in \mathbb{Z}/2\mathbb{Z}[T_1, \ldots, T_m]$ with $k = k_1 + k_2$, where $T_i$ denotes a 2-tuple $(t_i, t_{i2})$ of variables (resp. a variable $t_i$) if $n_i$ is even (resp. odd) and the coefficients of $t_{i1}$ and $t_{i2}$ in $f_p$ are either 0 or 2. Then, we have the same diagram (5), replacing the middle vertical map (6) by $\prod_{i=1}^m \mathbb{Z}^{n_i} \to Z$,

$$\sum_{j=1}^{n_i} a_{i,j} w_{i,j} \mapsto A_i := \begin{cases} \left( \frac{(a_{i,n_i-1} - a_{i,n_i} + 2S_i)}{e_i} \right) e_i & \text{if } n_i \text{ odd}, \\ \left( \frac{(a_{i,n_i-1} + S_i)}{e_{i1}} \right) e_{i1} + \left( \frac{(a_{i,n_i} + S_i)}{e_{i2}} \right) e_{i2} & \text{if } n_i \text{ even}, \end{cases}$$

where $S_i = \sum_{j=1}^{[(n_i-1)/2]} a_{i,2j-1}$ and $w_{i,j}$ denote the fundamental weights for the $i$th component of the root system of $G$. Therefore, by (5) we have

$$T^* = \{ \sum_{i=1}^m a_{i,j} w_{i,j} | f_p(\sum_{i=1}^m A_i) = 0, 1 \leq p \leq k \}.$$  

(18)

Let $I_1 = \{ i \in I | f_p(e_i) = 0 \text{ or } f_p(e_{i1}) = f_p(e_{i2}) = 0 \text{ for all } 1 \leq p \leq k \}$ and $I' = I \setminus I_1$. In view of the argument in the case of type $B$ we may assume that each relation $f_p(\sum_{i=1}^m A_i) = 0$ can be written as

$$\delta_p a_p = b_p + 4c_p,$$

where $b_p = \begin{cases} \delta_p a_p + f_p(\sum_{i=1}^m A_i) & \text{if } a_p = a_{i,n_i} \text{ with odd } n_i, \\ \delta_p a_p - f_p(\sum_{i=1}^m A_i) & \text{otherwise}, \end{cases}$

for some distinct $a_p \in \{ a_{i,n_i-1}, a_{i,n_i} | i \in I' \}$ with $\delta_p \in \{1, 2\}$ and $c_p \in \mathbb{Z}$ such that the terms $a_1, \ldots, a_k$ do not appear in $b_1, \ldots, b_k$ and each coefficient of $a_{i,l}$ in $b_p$ is divisible by $\delta_p$.

Let $W_1 = \{ w_{i,2j-1} | i \in I', 1 \leq j \leq [(n_i - 1)/2] \} \cup \{ w_{i,n_i-1}, w_{i,n_i} | i \in I' \}$. We simply write $w_p \in W_1$ for $w_{i,n_i-1}$ (resp. $w_{i,n_i}$) if $a_p = a_{i,n_i-1}$ (resp. $a_p = a_{i,n_i}$). Set

$$g_{i,l} = s_1(i, l) w_1 + \cdots + s_k(i, l) w_k \text{ and } W' = W_1 \setminus \{ w_1, \ldots, w_k \},$$

where $s_p(i, l)$ denotes the coefficient of $a_{i,l}$ in $b_p/\delta_p$. Then, we obtain the following $\mathbb{Z}$-basis of $T^*$:

$$\{ w_{i,j} | i \in I, \forall j \cup \{ w_{i,2j} | i \in I', 1 \leq j \leq [(n_i - 2)/2] \} \cup \{ \frac{4}{\delta_p} w_p \}_{1 \leq p \leq k} \cup \{ w_{i,l} + g_{i,l} \}_{w_i,l \in W'}.$$  

(19)

Let $v_p = \frac{4}{\delta_p} w_p$ and $v_{i,l} = w_{i,l} + g_{i,l}$. Assume that for each $p$, $w_p$ is a fundamental weight for the $i_p$-th component of the root system of $G$. As the normalized Killing forms are given by

$$q_i = \left( \sum_{j=1}^{n_i} w_{i,j}^2 \right) - \left( w_{i,n_i-2} w_{i,n_i} + \sum_{j=1}^{n_i-2} w_{i,j} w_{i,j+1} \right),$$
for any \( q \in Q(G) \) there exist \( d_i \in \mathbb{Z} \) such that \( q = \sum_{i=1}^{m} d_i q_i \). Hence, with respect to the basis \((19)\) we obtain

\[
q = q' + \frac{1}{16} \sum_{p=1}^{k} v_p^2 \delta_p^2 [d_i + \sum_{w_{i,l} \in W'} d_i s_p(i, l)^2] + \frac{1}{8} \sum_{1 \leq u \leq k} v_p v_u \delta_p \delta_u [\sum_{w_{i,l} \in W'} d_i s_p(i, l) s_u(i, l)]
\]

\[
- \frac{1}{2} \sum_{p=1}^{k} v_p \delta_p \left[ \sum_{w_{i,l} \in W'} v_i d_i s_p(i, l) \right]
\]

for some quadratic form \( q' \) with integer coefficients. Hence, \( q = \sum_{i=1}^{m} d_i q_i \in Q(G) \) if and only if

\[
(20) \quad \delta_p^2 [d_i + \sum_{w_{i,l} \in W'} d_i s_p(i, l)^2] \equiv 0 \mod 16, \quad \sum_{w_{i,l} \in W'} d_i \delta_p \delta_u s_p(i, l) s_u(i, l) \equiv 0 \mod 8,
\]

\[
(21) \quad \text{and } d_i \delta_p s_p(i, l) \equiv 0 \mod 2
\]

for all \( 1 \leq p \leq k, 1 \leq p < u \leq k \), and all \((i, l)\) such that \( w_{i,l} \in W' \).

Let \( c_{i1}(p), c_{i2}(p), c_i(p) \) denote the coefficients of \( t_{i1}, t_{i2}, t_i \) in \( f_p \), respectively. Note that \( c_{i1}(p) \) and \( c_{i2}(p) \) are either 0 or 2. Since

\[
\delta_p^2 + \sum_{l} \delta_p s_p(i_p, l)^2 = \sum_{l} \delta_p s_p(i, l)^2 = \begin{cases} 
8 & \text{if } c_i(p) = 2 \text{ or } c_i(p) + c_{i2}(p) = 4, \\
2n_i & \text{if } c_i(p) = \pm 1 \text{ or } c_i(p) + c_{i2}(p) = 2
\end{cases}
\]

for all \( p \) and \( i \neq i_p \), where the sums range over all \( l \) such that \( w_{i,l} \in W' \), the first equation in \((20)\) is equivalent to the following equation

\[
(22) \quad f_p(T_1, \ldots, T_m) \equiv 0 \mod 8, \text{ where } t_i = \begin{cases} 
\pm n_i d_i & \text{if } c_i(p) = \pm 1, \\
2d_i & \text{if } c_i(p) = 2,
\end{cases}
\]

\[
t_{i1} = \begin{cases} 
\frac{n_i d_i}{2} & \text{if } c_i(p) = 2, c_{i2}(p) = 0, \\
d_i & \text{if } c_i(p) + c_{i2}(p) = 4
\end{cases} \quad \text{and } t_{i2} = \begin{cases} 
\frac{n_i d_i}{2} & \text{if } c_i(p) = 0, c_{i2}(p) = 2, \\
d_i & \text{if } c_i(p) + c_{i2}(p) = 4.
\end{cases}
\]

Since we have

\[
\sum_{l} s_p(i, l) s_u(i, l) \equiv \begin{cases} 
\pm 2n_i & \text{mod } 8 \quad \text{if } c_i(p) c_i(u) \equiv \pm 1 \mod 4, \\
4 & \text{mod } 8 \quad \text{if } c_i(p) c_i(u) \equiv 2 \mod 4, \\
0 & \text{mod } 8 \quad \text{otherwise}
\end{cases}
\]

for all \( 1 \leq p < u \leq k \) such that \( \delta_p = \delta_u = 1 \), where the sum ranges over all \( l \) such that \( w_{i,l} \in W' \), the second equation in \((20)\) is equivalent to

\[
(23) \quad \sum_{\{i \in I'| c_i(p) c_i(u) \equiv 1 \mod 4\}} 2d_i + \sum_{\{i \in I'| c_i(p) c_i(u) \equiv 2 \mod 4\}} 4d_i \equiv 0 \mod 8
\]
if $\delta_p = \delta_u = 1$ and
\[
4 \sum_{i \in I''} d_i \equiv 0 \mod 8
\]
for some subset $I''$ of $I'$ otherwise.

4. The subgroup $\text{Dec}(G)$ for semisimple groups $G$ of types $B$, $C$, $D$

In this section we will compute the subgroup $\text{Dec}(G)$ of decomposable elements of $G$ for types $B$, $C$, and $D$.

4.1. Type $B$. Consider a split semisimple group $G = (\prod_{i=1}^m \text{Spin}_{2n_i+1})/\mu$ of type $B$, where $m, n_i \geq 1$ and $\mu \simeq (\mu_2)^k$ is a central subgroup for some $k \geq 0$. Let $I = \{1, \ldots, m\}$.

We first consider the case where $G$ is simply connected (i.e. $G = \tilde{G}$), equivalently $k = 0$. Since
\[
\text{Dec}(G_1 \times G_2) = \text{Dec}(G_1) \times \text{Dec}(G_2)
\]
for any two semisimple groups $G_1$ and $G_2$, it suffices to compute $\text{Dec}(\text{Spin}_{2n+1})$. Observe that $\text{Dec}(\text{Spin}_q) = \Z q$ as $c_2(\rho(w_1)) = -q$ and $\text{Dec}(\text{Spin}_q) = \Z q$ as $c_2(\rho(w_2)) = -q$. Similarly, $c_2(\rho(w_1)) = -2q \in \text{Dec}(\text{Spin}_{2n+1})$ for any $n \geq 2$. As the Weyl group of $\text{Spin}_{2n+1}$ contains a normal subgroup $(\Z/2\Z)^n$ generated by sign switching, we see that $2 \mid c_2(\rho(\lambda))$ for any $\lambda \in \Lambda$ (c.f. [10, Part II, §13]), thus $\text{Dec}(\text{Spin}_{2n+1}) = 2\Z q$. Therefore,
\[
\text{Dec}(G) = \delta'_1 \Z q_1 \oplus \cdots \oplus \delta'_m \Z q_m, \quad \text{where} \quad \delta'_i = \begin{cases} 2 & \text{if } n_i \geq 3, \\ 1 & \text{if } n_i = 1, 2. \end{cases}
\]

Now we assume that $G$ is adjoint (i.e. $G = \tilde{G}$), equivalently, $k = m$. Then, $\text{Dec}(\text{O}_3^+q) = 4\Z q$ as $c_2(\rho(2w_1)) = -4q$. Similarly, by the same argument as in the simply connected case, we see that $\text{Dec}(\text{O}_2^+q) = 2\Z q$ for $n \geq 2$ (see [19, Theorem 4.5]). Hence,
\[
\text{Dec}(G) = \delta_1 \Z q_1 \oplus \cdots \oplus \delta_m \Z q_m, \quad \text{where} \quad \delta_i = \begin{cases} 2 & \text{if } n_i \geq 2, \\ 4 & \text{if } n_i = 1. \end{cases}
\]

In general, we show that the subgroup $\text{Dec}(G)$ is determined by certain subgroups of $R$ introduced in Section 3.

**Proposition 4.1.** Let $G = (\prod_{i=1}^m \text{Spin}_{2n_i+1})/\mu$, $m, n_i \geq 1$, where $\mu$ is a central subgroup. Let $R$ be the subgroup of $(\mu_2)^* = (\Z/2\Z)^m$ such that $\mu^* = (\mu_2^*)^*/R$. Let $R'_1 = \langle e_i \mid e_i \in R, n_i \leq 2 \rangle$ and $R'_2 = \langle e_i + e_j \mid e_i + e_j \in R, e_i, e_j \notin R, n_i = n_j = 1 \rangle$ be two subspaces of $R$ with dim $R'_1 = l_1$ and dim $R'_2 = l_2$. Then,
\[
\text{Dec}(G) = \left( \bigoplus_{e_i \in R'_1} \Z q_i \right) \oplus \left( \bigoplus_{n_i \geq 2, e_i \notin R'_1} \Z q_i \right) \oplus \left( \bigoplus_{r=1}^{l_2} 2\Z q_r \right) \oplus \left( \bigoplus_{s=1}^{l_3} 4\Z q''_s \right),
\]
where \( l_3 = m - l_1 - l_2 - \{i \in I \mid n_i \geq 2, e_i \not\in R'_1 \} \) and \( q_{i} \) (resp. \( q''_{i} \)) is of the form \( q_{i} + q_{j} \) (resp. \( q_{i} \)) for some \( i, j \) such that \( \langle q_{r}, q''_{s} \rangle \mid 1 \leq r \leq l_2, 1 \leq s \leq l_3 \rangle = \langle q_{i} \mid n_i = 1, e_i \not\in R'_1 \rangle \) over \( \mathbb{Z} \).

**Proof.** Observe that by (25) and (26) we obtain

\[
\delta_1 \mathbb{Z} q_1 \oplus \cdots \oplus \delta_m \mathbb{Z} q_m \subseteq \text{Dec}(G) \subseteq \delta_1' \mathbb{Z} q_1 \oplus \cdots \oplus \delta_m' \mathbb{Z} q_m.
\]

Since we have

\[
-c_2(\rho(\chi)) = \begin{cases} q_i & \text{if } \chi = w_{i_1}, n_i = 1 \text{ or } \chi = w_{i_2}, n_i = 2, \\ 2(q_i + q_{i_2}) & \text{if } \chi = w_{i_1} + w_{i_2}, n_i = n_j = 1,
\end{cases}
\]

we see from (8) and (28) that the right hand side of (27) is contained in \( \text{Dec}(G) \).

On the other hand, let \( \chi \in \Lambda \setminus \bigcup \mathcal{W}(\lambda) \), where \( \Lambda \) is the weight lattice of \( G \) and the union ranges over all \( \lambda \) in (29). Then, as the Weyl group of \( G \) contains normal subgroups \((\mathbb{Z}/2\mathbb{Z})^\infty\) generated by sign switching, we see that

\[
c_2(\rho(\lambda)) = 4\left( \sum_{i \in J} a_i q_i \right) + 2\left( \sum_{i \in K} b_i q_i \right)
\]

for some \( a_i, b_i \in \mathbb{Z} \) and some subsets \( J \subseteq \{ i \in I \mid n_i = 1 \} \) and \( K \subseteq \{ i \in I \mid n_i \geq 2 \} \). Hence, any element of \( \text{Dec}(G) \) is contained in the right hand side of (27).

\[ \square \]

### 4.2. Type C

Let \( G = (\prod_{i=1}^{m} \text{Sp}_{2n_i})/\mu \) be a split semisimple group of type C, \( m, n_i \geq 1 \), where \( \mu \) is a central subgroup. As \( c_2(\rho(e_1)) = -q \), we have \( \text{Dec}(\text{Sp}_{2n}) = \mathbb{Z} q \).

Similarly, as \( c_2(\rho(2e_1)) = -4q \) and \( c_2(\rho(e_1 + e_2)) = -2(n - 1)q \), we have \( \frac{4}{\gcd(2, n)} q \in \text{Dec}(\text{PGSp}_{2n}) \). Moreover, since the Weyl group of \( \text{Sp}_{2n} \) contains a normal subgroup \((\mathbb{Z}/2\mathbb{Z})^\infty\) generated by sign switching, we see that \( \frac{4}{\gcd(2, n)} | c_2(\rho(\lambda)) \) for any \( \lambda \) in the root lattice (c.f. [10] Part II, §44), thus \( \text{Dec}(\text{PGSp}_{2n}) = \frac{4}{\gcd(2, n)} \mathbb{Z} q \) (see [19] §4b)].

Therefore, by (24) we have

\[
\delta_1 \mathbb{Z} q_1 \oplus \cdots \oplus \delta_m \mathbb{Z} q_m \subseteq \text{Dec}(G) \subseteq \mathbb{Z} q_1 \oplus \cdots \oplus \mathbb{Z} q_m, \quad \text{where } \delta_i = \begin{cases} 4 & \text{if } n_i \text{ odd,} \\ 2 & \text{if } n_i \text{ even.}
\end{cases}
\]

Similar to the case of type B, we determine the subgroup \( \text{Dec}(G) \) for type C.

**Proposition 4.2.** Let \( G = (\prod_{i=1}^{m} \text{Sp}_{2n_i})/\mu \), \( m, n_i \geq 1 \), where \( \mu \simeq (\mu_2)^k \) is a central subgroup. Let \( R \) be the subgroup of \((\mu_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m\) such that \( \mu^* = (\mu_2^m)^*/R \). Let \( R_1 = \langle e_i \mid e_i \in R \rangle \) and \( R'_2 = \langle e_i + e_j \mid e_i + e_j \in R, e_i, e_j \not\in R, n_i \equiv n_j \equiv 1 \text{ mod } 2 \rangle \) be two subspaces of \( R \) with \( \dim R'_2 = l_2 \). Then,

\[
\text{Dec}(G) = \left( \bigoplus_{e_i \in R_1} \mathbb{Z} q_i \right) \oplus \left( \bigoplus_{n_i \equiv 0 \text{ mod } 2, e_i \not\in R_1} 2\mathbb{Z} q_i \right) \oplus \left( \bigoplus_{r=1}^{l_2} 2\mathbb{Z} q_{r} \right) \oplus \left( \bigoplus_{s=1}^{l_3} 4\mathbb{Z} q_{s} \right),
\]

where \( l_3 = \{ i \in I \mid n_i \equiv 1 \text{ mod } 2, e_i \not\in R_1 \} \) and \( q_r \) (resp. \( q''_s \)) is of the form \( q_{i} + q_{j} \) (resp. \( q_{i} \)) for some \( i, j \) such that \( \langle q_{r}, q''_{s} \rangle \mid 1 \leq r \leq l_2, 1 \leq s \leq l_3 \rangle = \langle q_{i} \mid n_i = 1 \text{ mod } 2, e_i \not\in R_1 \rangle \) over \( \mathbb{Z} \).
Proof. Let $T$ be the split maximal torus of $G$. Then, by (14) we have
\begin{equation}
T^* = \{ \sum_{i,j} a_{ij} e^i_j + \sum a_i e_i | f_p(a_1, \ldots, a_m) \equiv 0 \mod 2 \},
\end{equation}
where $e^i_j = e_i - e_j$ for all $1 \leq i \leq m$ and $2 \leq j \leq n_i$. It follows by (30) that the first two summands and the last summand in the right hand side of (31) are contained in $\text{Dec}(G)$. Let $e_i + e_j \in R^\prime_2$. Then, by (32) we have $e_i + e_j \in T^*$, thus
\[-c_2(\rho(e_i + e_j)) = 2n_i q_i + 2n_i q_j \in \text{Dec}(G).\]
As both $n_i$ and $n_j$ are odd, $2q_i + 2q_j \in \text{Dec}(G)$. Therefore, the right hand side of (31) is contained in $\text{Dec}(G)$.

Let $\lambda \in \Lambda \setminus \{ \chi \in W(e_i + e_j) | n_i \equiv n_j \equiv 1 \mod 2, i \neq j \} \cup \{ \chi \in W(e_i) | i \in I \}$, where $\Lambda$ is the weight lattice of $G$. Then, it follows from the action of the normal subgroups $(\mathbb{Z}/2\mathbb{Z})^n$ of the Weyl group of $G$ that
\[c_2(\rho(\lambda)) = 4(\sum_{i \in J} a_{i} q_{i}) + 2(\sum_{i \in K} b_{i} q_{i})\]
for some subsets $J \subseteq \{ i \in I | n_i \equiv 1 \mod 2 \}$ and $K \subseteq \{ i \in I | n_i \equiv 0 \mod 2 \}$ and some $a_i, b_i \in \mathbb{Z}$. Therefore, the group $\text{Dec}(G)$ is contained in the right hand side of (31).

4.3. Type D. Let $G = (\prod_{i=1}^{m} \text{Spin}_{2n})/\mu$ be a split semisimple group of type $D$, where $m \geq 1$, $n_i \geq 3$ and $\mu$ is a central subgroup. Consider the case when $G$ is simple (i.e., $m = 1$ and $n_1 = n$). First of all, as
\begin{equation}
c_2(\rho(\omega_i)) = -2q, \quad c_2(\rho(2\omega_1)) = -8q, \quad c_2(\rho(\omega_2)) = \begin{cases} -4(n - 1)q & \text{if } n \neq 3, \\ -q & \text{if } n = 3. \end{cases}
\end{equation}
we have $2\mathbb{Z}q \subseteq \text{Dec}(\text{Spin}_{2n})$ for $n \neq 3$, $\text{Dec}(\text{Spin}_6) = \mathbb{Z}q$, $\frac{8}{\gcd(2,n)} \mathbb{Z}q \subseteq \text{Dec}(\text{PGO}^+_{2n})$. On the other hand, as the Weyl group of $\text{Spin}_{2n}$ contains a normal subgroup $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ generated by sign switching of even number of coordinates, we see that $2 | c_2(\rho(\lambda))$ for any $\lambda \in \Lambda$ with $n \geq 4$ and $\frac{8}{\gcd(2,n)} | c_2(\rho(\lambda'))$ for all $\lambda'$ in the root lattice with $n \geq 3$ (c.f. [10, Part II, §15]), thus $\text{Dec}(\text{Spin}_{2n}) = 2\mathbb{Z}q$ for any $n \neq 3$ and $\text{Dec}(\text{PGO}^+_{2n}) = \frac{8}{\gcd(2,n)} \mathbb{Z}q$ for any $n \geq 4$ (see [19, §4b]). Hence, by (24) we obtain
\begin{equation}
\delta_i \mathbb{Z}q_1 + \cdots + \delta_m \mathbb{Z}q_m \subseteq \text{Dec}(G) \subseteq \delta'_i \mathbb{Z}q_1 + \cdots + \delta'_m \mathbb{Z}q_m,
\end{equation}
where
\[\delta_i = \begin{cases} 8 & \text{if } n_i \text{ odd}, \\ 4 & \text{if } n_i \text{ even}, \end{cases}
\text{ and } \delta'_i = \begin{cases} 2 & \text{if } n_i \neq 3, \\ 1 & \text{if } n_i = 3. \end{cases}
\]

For the remaining simple groups $\text{O}^+_2$ and $\text{HSpin}_{2n}$ ($n$ even), we also have $2\mathbb{Z}q \subseteq \text{Dec}(\text{O}^+_2)$ and $4\mathbb{Z}q \subseteq \text{Dec}(\text{HSpin}_{2n})$ by (33). Moreover, if $n = 4$, then we have
\begin{equation}
c_2(\rho(\omega_3)) = c_2(\rho(\omega_4)) = -2q,
\end{equation}
thus $2\mathbb{Z}q \subseteq \text{Dec}(\text{HSpin}_4)$. Then, by the action of the Weyl group as above we obtain $\text{Dec}(\text{O}^+_8) = 2\mathbb{Z}q$, $\text{Dec}(\text{HSpin}_{2n}) = 4\mathbb{Z}q$ for even $n \geq 6$, and $\text{Dec}(\text{HSpin}_4) = 2\mathbb{Z}q$ ([11, Theorem 5.1]). In general, we determine the subgroup $\text{Dec}(G)$ for type $D$. 
Proposition 4.3. Let $G = (\prod_{i=1}^{m} \text{Spin}_{2n_i})/\mathbf{\mu}$, $m \geq 1$, $n_i \geq 3$, where $\mathbf{\mu}$ is a central subgroup. Let $R$ be the subgroup of $(\mathbb{Z}^2)$ such that $\mathbf{\mu}^* = \mathbb{Z}/R$, $R_{1i} = R \cap \mathbb{Z}$ for odd $n_i$, and $R_{2i} = R \cap \mathbb{Z}$ for even $n_i$. Set

$$I_1' = \{ i \mid R_{1i} \neq 0, n_i \neq 3 \} \cup \{ i \mid R_{1i} = 2\mathbb{Z}, n_i = 3 \} \cup \{ i \mid R_{1i}' \neq 0, n_i = 4 \} \cup \{ i \mid e_{i1} + e_{i2} \in R_{1i}', n_i \geq 6 \}.$$

Moreover, if $2 \equiv i \equiv 3 \mod 2$, then by (18) we have $w_{i1} \in T^*$, thus by (33) $2q_i \in \text{Dec}(G)$. Similarly, if $e_{i1} \in R_{1i}'$ (resp. $e_{i2} \in R_{1i}'$) with $n_i = 4$, then by (18) we have $w_{i,3} \in T^*$ (resp. $w_{i,4} \in T^*$), thus by (35) $2q_i \in \text{Dec}(G)$. Therefore, the second summand in the right hand side of (36) is contained in $\text{Dec}(G)$. Moreover, if $2e_i + 2e_j \in R$ for some $i \neq j$, then again by (18) we obtain $w_{i,1} + w_{j,1} \in T^*$, thus we get

$$-c_2(\rho(w_{i,1} + w_{j,1})) = 4n_jq_i + 4n_iq_j \in \text{Dec}(G).$$

As both $n_i$ and $n_j$ are odd, by (34) $4q_i + 4q_j \in \text{Dec}(G)$. Hence, by (34) the right hand side of (36) is contained in $\text{Dec}(G)$.

Let $\lambda \in \Lambda \backslash \{ \chi \in W(w_{i,1}) \mid i \in I \} \cup \{ \chi \in W(w_{i,2}) \cup W(w_{i,3}) \mid n_i = 3 \} \cup \{ \chi \in W(w_{i,3}) \cup W(w_{i,4}) \mid n_i = 4 \} \cup \{ \chi \in W(w_{i,1} + w_{j,1}) \mid n_i \equiv n_j \equiv 1 \mod 2, i \neq j \in I \}$, where $I = \{ 1, \ldots, m \}$. Then, by the action of the normal subgroups $(\mathbb{Z}/2\mathbb{Z})^{n_i-1}$ of the Weyl group of $G$ we obtain

$$c_2(\rho(\lambda)) = 8(\sum_{i \in J} a_iq_i) + 4(\sum_{i \in K} b_iq_i)$$

for some $a_i, b_i \in \mathbb{Z}$ and some subsets $J \subseteq \{ i \in I \mid n_i \equiv 1 \mod 2 \}$ and $K \subseteq \{ i \in I \mid n_i \equiv 0 \mod 2 \}$. Hence, the group $\text{Dec}(G)$ is contained in the right hand side of (36).

5. Degree 3 invariants for semisimple groups $G$ of types $B$, $C$, $D$

We now determine the group of reductive indecomposable invariants of a split semisimple group of types $B$, $C$, and $D$ by using the results of Section 3. Propositions 4.1, 4.2, and 4.3.
5.1. Type B.

**Theorem 5.1.** Let \( G = (\prod_{i=1}^{m} \text{Spin}_{2n_i+1})/\mu, m, n_i \geq 1 \), where \( \mu \simeq (\mu_2)^k \) is a central subgroup. Let \( R \) be the subgroup of \((\mu_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m\) whose quotient is the character group \( \mu^* \). Then,

\[
\text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{m-l_1-l_2}, \quad \text{where}
\]

\( l_1 = \dim(e_i \mid e_i \in R, n_i \leq 2) \) and \( l_2 = \dim(e_i + e_j \mid e_i, e_j \in R, e_i, e_j \notin R, n_i = n_j = 1) \).

**Proof.** Let \( R = \{ r = (r_1, \ldots, r_m) \in (\mathbb{Z}/2\mathbb{Z})^m \mid f_p(r) = 0, 1 \leq p \leq k \} \) be the subgroup of \((\mathbb{Z}/2\mathbb{Z})^m\) whose quotient is the character group \( \mu^* \) for some linear polynomials \( f_j \in \mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_m] \). Let \( R_1 = \langle e_i \mid e_i \in R \rangle \) and \( R_2 = \langle e_i + e_j \mid e_i, e_j \in R, e_i, e_j \notin R_1 \rangle \) be the subgroups of \( R \).

By [13, Proposition 7.1], an indecomposable invariant of \( G \) corresponding to \( q = \sum_{i=1}^{m} d_i q_i \in Q(G) \) is reductive indecomposable if and only if the order \(|\bar{w}_{ij}|\) in \( \Lambda/T^* \) divides \( \delta_{ij} d_i \) for all \( i \) and \( j \), where

\[
\delta_{ij} = \begin{cases} 
2 & \text{if } j = n_i \geq 2, \\
1 & \text{if } n_i \geq 2, 1 \leq j \leq n_i - 1; \text{ or } n_i = j = 1.
\end{cases}
\]

Since \(|\bar{w}_{11}| = 1\) for some \( i \) such that \( n_i = 1 \) is equivalent to \( w_{11} \in T^* \) and

\[
|\bar{w}_{ij}| \leq \begin{cases} 
2 & \text{if } j = n_i \geq 2, \\
1 & \text{if } n_i \geq 2, 1 \leq j \leq n_i - 1,
\end{cases}
\]

we may assume that \( 2 | d_i \) for all \( i \) such that \( n_i = 1 \) which appear in the conditions \([11] \) and \([10] \), i.e., any reductive indecomposable invariant of \( G \) corresponding to \( q = \sum_{i=1}^{m} d_i q_i \in Q(G) \) satisfies

\[
f_p\left(\frac{d_1}{\epsilon_1}, \ldots, \frac{d_m}{\epsilon_m}\right) \equiv 0 \mod 2, \quad \text{where } \epsilon_i = \begin{cases} 
2 & \text{if } n_i = 1, \\
1 & \text{if } n_i \geq 2.
\end{cases}
\]

for all \( p \). Therefore, we have the corresponding element

\[
(37) \quad \sum_{i=1}^{m} \epsilon_i r_i q_i \in Q(G)
\]

for any \( r = (r_1, \ldots, r_m) \in R \).

Let \( R'_1 = \langle e_i \mid e_i \in R, n_i \leq 2 \rangle \) and \( R'_2 = \langle e_i + e_j \mid e_i, e_j \in R, e_i, e_j \notin R, n_i = n_j = 1 \rangle \) be two subgroups of \( R \) with \( \dim R'_1 = l_1 \) and \( \dim R'_2 = l_2 \). We choose a basis \( B \) of \( R \) as follows. First, choose all \( e_i \in R_1 \) so that \( e_i \in B \). Then, choose all \( e_i + e_j \in R'_2 \) so that \( e_i + e_j \in B \). Finally, we complete \( B \) with an arbitrary basis of a complementary subspace. Then, using this basis \( B \) and \([37] \), the result for the group of indecomposable reductive invariants then follows by Proposition 4.11. □

In particular, under the assumption that the ranks of all components of the root system of \( G \) are at least 2 we have the following result.
Corollary 5.2. Let \( G = (\prod_{i=1}^m \text{Spin}_{2n_i+1})/\mu, \) \( m, n_i \geq 1, \) where \( \mu \simeq (\mu_2)^k \) is a central subgroup. Let \( R \) be the subgroup of \( (\mu_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m \) whose quotient is the character group \( \mu^* \). Assume that \( n_i \geq 2 \) for all \( 1 \leq i \leq m. \) Then,
\[
\text{Inv}^3(G)_{\text{ind}} = \text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{m-k-l},
\]
where \( l = \dim(e_i \mid e_i \in R, n_i = 2). \)

Proof. By Theorem 5.1, it suffices to show that \( \text{Inv}^3(G)_{\text{ind}} \subseteq \text{Inv}^3(G)_{\text{red}}. \) Since \( n_i \geq 2 \) for all \( 1 \leq i \leq m, \) the inclusion follows directly from the proof of Theorem 5.1. \( \square \)

Remark 5.3. One can directly compute \( \text{Inv}^3(G)_{\text{ind}} \) using Propositions 3.3 and 4.1.

We present below another particular case of Theorem 5.1 (and Theorem 5.5), which follows by the exceptional isomorphism \( A_1 = B_1 = C_1. \) This result in turn determine the reductive invariants of semisimple groups of type \( A \) (see \( [17, \text{Theorem 7.1}] \)).

Corollary 5.4. Let \( G = (\prod_{i=1}^m \text{SL}_2)/\mu, \) \( m \geq 1, \) where \( \mu \simeq (\mu_2)^k \) is a central subgroup. Let \( R \) be the subgroup of \( (\mu_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m \) whose quotient is the character group \( \mu^* \). Then,
\[
\text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{m-k-l_1-l_2},
\]
where \( l_1 = \dim(e_i \mid e_i \in R) \) and \( l_2 = \dim(e_i + e_j \mid e_i, e_j \in R, e_i, e_j \notin R) \).

5.2. Type C.

Theorem 5.5. Let \( G = (\prod_{i=1}^m \text{Sp}_{2n_i})/\mu, \) \( m, n_i \geq 1, \) where \( \mu \simeq (\mu_2)^k \) is a central subgroup. Let \( R \) be the subgroup of \( (\mu_2^m)^* = \bigoplus_{i=1}^m (\mathbb{Z}/2\mathbb{Z}) e_i \) whose quotient is the character group \( \mu^* \) and let \( s \) denote the number of ranks \( n_i \) which are divisible by 4. Then,
\[
\text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{s+l-l_1-l_2}, \quad \text{where}
\]
\[
l_1 = \dim(e_i \mid e_i \in R), \quad l_2 = \dim(e_i + e_j \mid e_i, e_j \in R, e_i, e_j \notin R, n_i \equiv n_j \equiv 1 \mod 2), \quad \text{and} \quad l = \dim(R \cap (\bigoplus_{i=1}^m (\mathbb{Z}/2\mathbb{Z}) e_i)).
\]
In particular, if \( n_i \equiv 0 \mod 2 \) for all \( 1 \leq i \leq m, \) then
\[
\text{Inv}^3(G)_{\text{ind}} = \text{Inv}^3(G)_{\text{red}} = (\mathbb{Z}/2\mathbb{Z})^{s+l-l_1}.
\]

Proof. We apply arguments similar to the proof of type \( B. \) Let \( \alpha_{ij} \) denote the simple roots of the \( i \)th component of the root system of \( G \) and let \( \delta_{ij} \) be the square of the length of the coroot of \( \alpha_{ij}. \) Then, we have
\[
\delta_{ij} = \begin{cases} 
1 & \text{if } j = n_i \geq 1, \\
2 & \text{otherwise.}
\end{cases}
\]

By \( [32], \) we see that for each \( i \in I \) the element \( e_m \) has order 2 in \( \Lambda/T^* \) if and only if the term \( n_i d_i \) appears in the equation \( f_p(n_1 d_1, \ldots, n_m d_m) \equiv 0 \mod 4 \) of \( [16] \) for some \( 1 \leq p \leq k. \) Therefore, it follows from \( [13, \text{Proposition 7.1}] \) that any reductive indecomposable invariant of \( G \) corresponding to \( q = \sum_{i=1}^m d_i q_i \in Q(G) \) satisfies
\[
f_p(n_1 d_1/2, \ldots, n_m d_m/2) \equiv 0 \mod 2
\]
Then, we have
denote the character group of the center of $\prod R$ of indecomposable invariants. Hence, the result immediately follows from Proposition 4.2. If $\leq (38)$ Inv

Theorem 5.6. Let $G = (\prod_{i=1}^{m} Spin_{2n_i})/\mu$, $m \geq 1$, $n_i \geq 3$, where $\mu$ is a central subgroup. Let $R$ be the subgroup of $Z$ such that $\mu = Z/R$, $R_{1i} = R \cap Z_i$ for odd $n_i$, $R'_{1i} = R \cap Z_i$ for even $n_i$, and let

$R = \{(r_1, \ldots, r_m) \in \bigoplus_{i=1}^{m} (Z/2Z) \bar{e}_i \mid \sum_{i=1}^{m} r_i \in R\}, r_i := \begin{cases} 2r_i e_i & \text{if } n_i \text{ odd,} \\ r_i e_{i1} + \bar{r}_i e_{i2} & \text{if } n_i \text{ even,} \end{cases}$

where $Z := \bigoplus_{i=1}^{m} Z_i$ with $Z_i = \begin{cases} (Z/4Z) e_i & \text{if } n_i \text{ odd,} \\ (Z/2Z) e_{i1} \bigoplus (Z/2Z) e_{i2} & \text{if } n_i \text{ even.} \end{cases}$

denote the character group of the center of $\prod_{i=1}^{m} Spin_{2n_i}$. Set

$R' = R \cap \left( \bigoplus_{4|n_i, R_{1i}, R_{1j}\neq Z_i} (Z/2Z) \bar{e}_i \right)$ with $l = \dim R'$, $I_1 = \{i \mid Z_i = R_{1i} \text{ or } R'_{1i}, n_i \neq 3\}$,

$I_2 = \{i \mid R'_{1i} = 0, 4|n_i\} \cup \{i \mid R'_{1i} = (Z/2Z) e_{i1} \text{ or } (Z/2Z) e_{i2}, n_i \geq 6, 4|n_i\}$ with $s_i = |I_i|$. Then, we have

$\text{Inv}^3(G_{\text{red}}) = (Z/2Z)^{s_1 + s_2 + l_1 - l_2}$, where

$l_1 = \{|i \mid 4 \nmid n_i, R_{1i} = 2Z_i \text{ or } R'_{1i} = (Z/2Z) e_{i1} \text{ or } (Z/2Z) e_{i2}\}$, and $l_2 = \dim \langle \bar{e}_i + \bar{e}_j \rangle 2e_i + 2e_j \in R, R_{1i} = R_{1j} = 0\).$

Proof. We shall use the description of $Q(G)$ in Section 3.3. Let $\mu$ be a central subgroup such that $\mu \simeq (\mu_2)^{k_1} \times (\mu_4)^{k_2}$ for some $k_1, k_2 \geq 0$ and let $R = \{r \in Z \mid f_p(r) = 0, 1 \leq p \leq k\}$ be the subgroup of $Z$ such that $\mu^* \simeq Z/R$ for some linear polynomials $f_p \in Z/4Z[T_1, \ldots, T_m]$ with $k = k_1 + k_2$.

Let $\delta_{ij}$ denote the square of the length of the $j$th coroot of the $i$th component of the root system of $G$. Then, $\delta_{ij} = 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n_i$. Note that the order of the fundamental weight $w_{ij}$ in $\Lambda/T^*$ is trivial for all $j$ if and only if

$Z_i = \begin{cases} R_{1i} & \text{if } n_i \text{ odd,} \\ R'_{1i} & \text{if } n_i \text{ even.} \end{cases}$
Moreover, if \( c_i(p) = \pm 1 \) for some \( 1 \leq p \leq k \), where \( c_i(p) \) denotes the coefficient of \( t_i \) in \( f_p \), then \( R_{1i} = 0 \), thus \( 2w_{i,n_i} \not\in T^* \), i.e., \( \bar{w}_{i,n_i} = 4 \). Hence, by \cite[Proposition 7.1]{13} any reductive indecomposable invariant of \( G \) corresponding to \( q = \sum_{i=1}^mq_i \in Q(G) \) satisfies (21) and (23). Therefore, it follows by (22) that

\[
\text{Inv}^3(G)_{\text{red}} = \left\{ \sum_{i=1}^md_iq_i \mid f_p(\epsilon_1d_1, \ldots, \epsilon_md_m) \equiv 0 \mod 2 \right\}
\]

where, \( f_p \in \mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_m] \) denotes the image of \( f_p \) under the following map

\[
\mathbb{Z}/4\mathbb{Z}[T] \to \mathbb{Z}/4\mathbb{Z}[t_1, \ldots, t_m] \to \mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_m]
\]

given by \( 2t_i \mapsto t_i ; t_i \mapsto t_i \) and \( \epsilon_i = \begin{cases} \frac{1}{2} & \text{if } c_i(p) = 2 \text{ or } c_i(p) + c_i(2) = 4, \\ \frac{n_i}{4} & \text{otherwise.} \end{cases} \)

Let \( \bar{R} = \{ \bar{r} = (\bar{r}_1, \ldots, \bar{r}_m) \in (\mathbb{Z}/2\mathbb{Z})^m \mid f_p(\bar{r}) \equiv 0 \mod 2 \} \), equivalently

\[
\bar{R} = \{ (\bar{r}_1, \ldots, \bar{r}_m) \in (\mathbb{Z}/2\mathbb{Z})^m \mid \sum_{i=1}^mr_i \in R \}, \text{ where } r_i := \begin{cases} 2\bar{r}_1\epsilon_i & \text{if } n_i \text{ odd,} \\ \bar{r}_i\epsilon_1 + \bar{r}_i\epsilon_2 & \text{if } n_i \text{ even} \end{cases}
\]

and let \( R' = R \cap (\bigoplus_{4|n_i, R'_{1i}, R_{1i} \neq \mathbb{Z}/2\mathbb{Z})e_i) \). Then, the group in the numerator of (39) is generated by

\[
\{ q_i \mid Z_i = R_{1i} \text{ or } R'_{1i} \} \cup \{ 2q_i \mid R_{1i} = 2Z_i \text{ or } R'_{1i} \neq Z_i, 4|n_i \} \cup \{ 4q_i \mid R'_{1i} \neq Z_i, 4 \nmid n_i \} \cup \{ 8q_i \mid R_{1i} = 0 \} \cup \{ \sum_{i=1}^m\epsilon_i\bar{r}_iq_i \mid \bar{r}' = (r'_1, \ldots, r'_m) \in R' \}, \text{ where } \epsilon_i = \begin{cases} 2 & \text{if } n_i \text{ even,} \\ 4 & \text{if } n_i \text{ odd.} \end{cases}
\]

Therefore, the statement immediately follows by Proposition 4.3 \( \square \)

6. UNRAMIFIED INVARIANTS FOR SEMISIMPLE GROUPS \( G \) OF TYPES B, C, D

In this section, we first describe torsors for the corresponding reductive groups in Lemmas 6.1, 6.6 and 6.11. Then, using this together with Theorems 5.1, 5.5 and 5.6, we present a complete description of the corresponding cohomological invariants in Propositions 6.3, 6.7 and 6.13. Finally, using such descriptions, we show that there are no nontrivial unramified degree 3 invariants for semisimple groups of types B, C, and D (see Theorems 6.5, 6.10, 6.15). In this section, we assume that the base field \( F \) is of characteristic 0.

6.1. Type B.

**Lemma 6.1.** Let \( G = (\prod_{i=1}^m\text{Spin}_{2n_{i+1}})/\mu, m, n_i \geq 1, \) where \( \mu \simeq (\mu_2)^k \) is a central subgroup. Let \( R \) be the subgroup of \((\mu_2)^* = (\mathbb{Z}/2\mathbb{Z})^m \) whose quotient is the character group \( \mu^* \). Set \( G_{\text{red}} = (\prod_{i=1}^m\Gamma_{2n_{i+1}})/\mu, \) where \( \Gamma_{2n_{i+1}} \) is the split even Clifford group. Then, for any field extension \( K/F \) the first Galois cohomology set \( H^1(K,G_{\text{red}}) \) is
bijective to the set of $m$-tuples of quadratic forms $(\phi_1, \ldots, \phi_m)$ with $\dim \phi_i = 2n_i + 1$, $\text{disc} \phi_i = 1$ such that for all $r = e_{i_1} + \cdots + e_{i_s} \in R$, $i_1 < \cdots < i_s$,

\[
I^3(K) \ni \begin{cases} 
\frac{1}{p} \sum_{p=1}^{s} (-1)^p \phi_{i_p} & \text{if } s \text{ is even}, \\
(\frac{1}{p} \sum_{p=1}^{s} (-1)^p \phi_{i_p}) \perp (1) & \text{otherwise},
\end{cases}
\]

where $I^3(K)$ denotes the cubic power of the fundamental ideal $I(K)$ in the Witt ring of $K$.

**Proof.** Consider the natural exact sequence

\[
1 \to \left(\mathbb{G}_m\right)^m / \mu \to G_{\text{red}} \to \prod_{i=1}^{m} O_{2n_i+1}^+ \to 1,
\]

where $\Gamma_{2n_i+1}$ is the split even Clifford group and $G_{\text{red}} = (\prod_{i=1}^{m} \Gamma_{2n_i+1})/\mu$. Then, by Hilbert Theorem 90 and [24, Proposition 42], this sequence yields a bijection between the set $H^1(F, G_{\text{red}})$ and the kernel of the connecting map which factors as

\[
H^1(F, \prod_{i=1}^{m} O_{2n_i+1}^+) \to H^2(F, (\mu_2)^m) = Br_2(F)^m \supset H^2(F, (\mu_2)^m / \mu),
\]

where the first map sends an $m$-tuple of quadratic forms $(\phi_1, \ldots, \phi_m)$ with $\dim \phi_i = 2n_i + 1$, $\text{disc}(\phi_i) = 1$ to the $m$-tuple $(C_0(\phi_1), \ldots, C_0(\phi_m))$ of even Clifford algebras $C_0(\phi_i)$ associated to $\phi_i$ and the map $\alpha$ is induced by the natural surjection $(\mu_2)^m \to (\mu_2)^m / \mu$. Since $(C_0(\phi_1), \ldots, C_0(\phi_m)) \in \text{Ker}(\alpha)$ if and only if it is contained in the kernel of the composition

\[
H^2(F, (\mu_2)^m) \supset H^2(F, (\mu_2)^m / \mu) \supset H^2(F, \mathbb{G}_m)
\]

for all $r \in R = ((\mu_2)^m / \mu)^*$, we have

\[
H^1(F, G_{\text{red}}) \simeq \{(\phi_1, \ldots, \phi_m) \mid \dim \phi_i = 2n_i + 1, \text{disc} \phi_i = 1, \sum_{i=1}^{m} r_i C_0(\phi_i) = 0\}
\]

for all $r = (r_i) \in R$.

Write an element $r \in R$ as $r = e_{i_1} + \cdots + e_{i_s}$ for some $i_1 < \cdots < i_s$, so that the condition $\sum_{i=1}^{m} r_i C_0(\phi_i) = 0$ in (42) is equal to $\sum_{p=1}^{s} C_0(\phi_{i_p}) = 0$. Assume that $s$ is even. Since $\text{disc}(-\phi_{i_p} \perp \phi_{i_{p+1}}) = 1$ for any $1 \leq p \leq s/2$ and

\[
C_0(\psi) = C_0(-\psi) \text{ and } C_0(\phi) + C_0(\phi') = C(\phi \perp \phi')
\]

for any quadratic form $\psi$ and any odd-dimensional quadratic forms $\phi$ and $\phi'$, where $C(\phi \perp \phi')$ is the corresponding Clifford algebra, we have

\[
0 = \sum_{p=1}^{s} C_0(\phi_{i_p}) = C(-\phi_{i_1} \perp \phi_{i_2} \perp \cdots \perp -\phi_{i_{s-1}} \perp \phi_{i_s}),
\]
which is equivalent to \((-\phi_{i_1} \perp \phi_{i_2}) \perp \cdots \perp (-\phi_{i_{s-1}} \perp \phi_{i_s}) \in I^3(F)\) by [9, Theorem 14.3]. Now we assume that \(s\) is odd. Since \(C_0(\phi \perp \langle 1 \rangle) = C_0(\phi)\) for any odd-dimensional quadratic forms \(\phi\) and \(\text{disc}(-\phi_{i_s} \perp \langle 1 \rangle) = 1\), the same argument shows that \((-\phi_{i_1} \perp \phi_{i_2}) \perp \cdots \perp (-\phi_{i_{s-2}} \perp \phi_{i_{s-1}}) \perp (-\phi_{i_s} \perp \langle 1 \rangle) \in I^3(F)\). □

**Remark 6.2.** If we assume that \(-1 \in (F^\times)^2\), then the condition (40) in Lemma 6.1 can be simplified without sign changes as follows:

\[
H^1(K, \text{G}_{\text{red}}) \simeq \{ \phi = (\phi_1, \ldots, \phi_m) \mid \dim \phi_i = 2n_i + 1, \text{disc} \phi_i = 1, \phi[r] \in I^3(K) \}
\]

for all \(r = (r_i) \in R\), where

\[
\phi[r] := \begin{cases} \sqcap_{i=1}^m r_i \phi_i & \text{if } \sum_{i=1}^m r_i \equiv 0 \mod 2, \\ \sqcap_{i=1}^m r_i \phi_i \perp \langle 1 \rangle & \text{otherwise}. \end{cases}
\]

**Proposition 6.3.** Let \(G = (\prod_{i=1}^m \text{Spin}_{2n_i+1})/\mu\) defined over an algebraically closed field \(F\), where \(m, n_i \geq 1, \mu \simeq (\mu_2)^k\) is a central subgroup. Let \(R\) be the subgroup of \((\mu_2^n)^*\) whose quotient is the character group \(\mu^*\). Set \(G_{\text{red}} = (\prod_{i=1}^m \text{Spin}_{2n_i+1})/\mu\), where \(\Gamma_{2n_i+1}\) is the split even Clifford group. Then, every normalized invariant in \(\text{Inv}^3(G_{\text{red}})\) is of the form \(e_3(\phi[r])\) for some \(r \in R\), where \(\phi[r]\) is the quadratic form defined in Remark 6.2 and \(e_3 : I^3(K) \to H^3(K)\) denotes the Arason invariant for a field extension \(K/F\). Moreover, we have

\[
\text{Inv}^3(G_{\text{red}})_{\text{norm}} \simeq \frac{R}{\langle e_i, e_j + e_k \in R \mid e_j, e_k \not\in R, n_i \leq 2, n_j = n_k = 1 \rangle}.
\]

**Proof.** Observe that \(\text{Inv}^3(G_{\text{red}})_{\text{norm}} = \text{Inv}^3(G_{\text{red}})_{\text{ind}}\) as \(F\) is algebraically closed. Since \(\phi[r] \in I^3(K)\) for any \(r \in R\), the Arason invariant gives a normalized invariant of \(G_{\text{red}}\) of order dividing \(2\) that sends an \(m\)-tuple \(\phi \in H^1(K, G_{\text{red}})\) to \(e_3(\phi[r]) \in H^3(K)\).

Let \(r \in R_1' + R_2'\), where \(R_1'\) and \(R_2'\) are subgroups of \(R\) defined in Proposition 4.1. Then, as every 4 and 6-dimensional quadratic forms in \(I^3(K)\) are hyperbolic, the invariant \(e_3(\phi[r])\) vanishes.

Now we show that the invariant \(e_3(\phi[r])\) is nontrivial for any \(r \in R \setminus (R_1' + R_2')\). Let \(G'_{\text{red}} = (\Gamma_3)^m/\mu\). If \(R\) is a subgroup such that every element \(r \in R\) has at least 3 nonzero components, then by [14, Lemma 4.3] and the exceptional isomorphism \(A_1 = B_1\), any invariant of \(G'_{\text{red}}\) is nontrivial. Hence, it follows from the map

\[
\text{Inv}^3(G_{\text{red}}) \to \text{Inv}^3(G'_{\text{red}})
\]

induced by the standard embedding \(\Gamma_3 \to \Gamma_{2n_i+1}\) that every invariant \(e_3(\phi[r])\) is nontrivial. Otherwise, by the proof of Lemma 6.4 each invariant \(e_3(\phi[r])\) is nontrivial, thus the statements follow from Theorem 5.1 □

Recall from Section 3 the following subspaces of \(R\).

\[R_1 = \langle e_i \mid e_i \in R \rangle \text{ and } R_2 = \langle e_i + e_j \mid e_i + e_j \in R, e_i, e_j \not\in R_1 \rangle.\]

We shall need the following key lemma.
Lemma 6.4. Let $G = \prod_{i=1}^{m} \text{Spin}_{2n_i+1}/\mu$ defined over an algebraically closed field $F$, where $m, n_i \geq 1$, $\mu$ is a central subgroup. Set $G_{\text{red}} = \prod_{i=1}^{m} \Gamma_{2n_i+1}/\mu$. Then, every normalized invariant in $\text{Inv}^3(G_{\text{red}})$ is ramified if either $n_i \geq 3$ for some $i$ with $e_i \in R_1$ or $n_j + n_k \geq 3$ for some $j$ and $k$ such that $e_j + e_k \in R_2$.

Proof. Let $R_3$ be a complementary subspace of $R_1 + R_2$ in $R$. Then, by Proposition 6.3 any normalized invariant $\alpha$ in $\text{Inv}^3(G_{\text{red}})$ can be written as

$$
\alpha(\phi) = e_3(\phi[r_1]) + e_3(\phi[r_2]) + e_3(\phi[r_3])
$$

for some $r_i \in R_i$, $1 \leq i \leq 3$, where $\phi = (\phi_1, \ldots, \phi_m)$ denotes a $G_{\text{red}}$-torsor.

Suppose that $e_3(\phi[r_1]) \neq 0$ for some $r_1 \in R_1$. Then, we may assume that $r_1 = e_1$ with $n_1 \geq 3$. Choose a division quaternion algebra $(x, y)$ over a field extension $K/F$. Find $\phi[e_1] = \phi_1$ such that $\phi_1 \perp \langle 1 \rangle = \langle \langle x, y, z \rangle \rangle \perp h$ over $K((z))$ and set $\phi_1 = h \perp (1)$ for all $2 \leq i \leq m$, where $h$ denotes a hyperbolic form. Then, we have $\partial_i(\alpha(\phi)) = (x, y) \neq 0$, thus $\alpha(\phi)$ ramifies.

Now we may assume that $\alpha(\phi) = e_3(\phi[r_2]) + e_3(\phi[r_3])$ with $e_3(\phi[r_2]) \neq 0$. To show that $\alpha(\phi)$ ramifies, we shall choose bases of $R_2$ and a complementary subspace of $R_2$. For simplicity, we will write $e(i_1, \ldots, i_k)$ for $e_{i_1} + \ldots + e_{i_k}$. We first select $e(i_p, i_{pq}) \in R_2$, where $i_p, i_{pq}$ $(1 \leq p \leq k, 1 \leq q \leq m_p)$ are all distinct integers for some $m_1, \ldots, m_k$, so that $B_2 := \{e(i_p, i_{pq})\}$ is a basis of $R_2$. In particular, if $n_{ip_{pq}} = 1$ for some $p$ and $q$, say $n_{i_{11}} = 1$, then we replace the subset $\{e(i_1, i_{i1}) | 1 \leq q \leq m_1\}$ of $B_2$ by $\{e(i_{11}, i_1), e(i_{11}, i_{1q}) | 2 \leq q \leq m_1\}$ so that we may assume that $n_{i_p} = 1$. We set

$$
I_2 = \{i_p | 1 \leq p \leq k\} \text{ and } I' = \{i_p, i_{pq} | 1 \leq p \leq k, 1 \leq q \leq m_p\}.
$$

We select a basis $B_3$ of a complementary subspace of $R_2$. First, we find any basis $D_3$ of $R_3$. Then, we modify each element $d$ of $D_3$ by adding $e(i_p, i_{pq})$ to it whenever either $e(i_{pq})$ or $e(i_p, i_{pq})$ appears in $d$. Hence, we obtain a basis $C_3 := \{e(k_1, \ldots, k_l)\}$ of a complementary subspace of $R_2$ such that the intersection

$$
(\bigcup \{k_1, \ldots, k_l | e(k_1, \ldots, k_l) \in C_3\}) \cap I',
$$

where the union is over all elements of $C_3$, is a subset of $I_2$. We denote by $J_2$ the intersection. We can divide all elements of the basis $C_3$ into two types: either $e(i_p)$ for some $i_p \in J_2$ appears in $e(k_1, \ldots, k_l) \in C_3$ (the first type) or not (the second type).

We first select basis elements from the first type elements as follows. We choose any element $b(i_1)$ in $C_3$ of the first type such that $e(i_1)$ appears in the element (if there is no element of the first type, we skip the selection of elements of the first type). We write $b(i_1) := e(i_1) + b'(i_1)$, where $e(i_1)$ does not appear in $b'(i_1)$. We modify every element of the first type by adding $b(i_1)$ to the element whenever $e(i_1)$ appears in the element. For simplicity, we shall use the same notation $C_3$ for the modified basis of $C_3$. Then, $e(i_1)$ appears only in $b(i_1)$ among the elements of $C_3$. Now we choose another element $b(i_2)$ of the first type in which $e(i_2)$ appears for some $i_2 \in J_2$. We write $b(i_2) := e(i_2) + b'(i_2)$, where $e(i_2)$ does not appear in $b'(i_2)$. As $e(i_1)$ appears only in $b(i_1)$, both $e(i_1)$ and $e(i_2)$ do not appear in $b(i_2)$. Again, we modify every element of the first type by adding $b(i_2)$ to the element whenever $e(i_2)$ appears in the
chosen basis elements $b(i_p) := e(i_p) + b'(i_p)$ for all $i_p$ in some subset $J'_2 \subseteq J_2$ such that all the terms $e(i_p)$ do not appear in $b'(i_p)$.

Similarly, we select basis elements from the second type elements. We choose any element $b(j_1)$ of the second type with $j_1 \not\in J_2$, so that we write $b(j_1) := e(j_1) + b'(j_1)$, where $e(j_1)$ does not appear in $b'(j_1)$. We modify every element of $C_3$ (i.e., $b(i_p)$ and elements of the second type) by adding $b(j_1)$ to the element whenever $e(j_1)$ appears in the element. Then, in particular, all the terms $e(i_p)$ and $e(j_1)$ do not appear in the modified $b'(i_p)$. Now we choose another element $b(j_2)$ of the second type for some $j_2 \not\in J_2$, so that we have $b(j_2) := e(j_2) + b'(j_2)$, where both $e(j_1)$ and $e(j_2)$ do not appear in $b'(j_2)$. Again we modify every element of $C_3$ by adding $b(j_2)$ to the element whenever $e(j_2)$ appears in the element. Then, both $e(j_1)$ and $e(j_2)$ do not appear in modified $b'(j_2)$ and all the terms $e(i_p)$, $e(j_1)$, and $e(j_2)$ do not appear in the modified $b'(i_p)$. Applying the same procedure to all elements of the second type, we obtain the following basis $B_3$ of a complementary subspace of $R_2$:

$$b(i_p) := e(i_p) + b'(i_p), \quad b(j_1) := e(j_1) + b'(j_1), \quad \ldots, \quad b(j_s) := e(j_s) + b'(j_s)$$

for all $i_p \in J'_2$ and some distinct $j_1, \ldots, j_s \not\in J_2$ such that all the terms $e(i_p)$ and $e(j_r)$ do not appear in $b'(i_p), b'(j_r)$ for all $1 \leq r \leq s$, thus

$$B_3 = \{b(i_p), b(j_r) \mid i_p \in J'_2, 1 \leq r \leq s\}.$$

Using the basis $B_2 \cup B_3$, we rewrite the invariant $\alpha(\phi) = e_3(\phi[r_2]) + e_3(\phi[r_3])$ as

$$\alpha(\phi) = \sum_{b \in B_2'} e_3(\phi[b]) + \sum_{b \in B_3'} e_3(\phi[b])$$

for some subsets $\emptyset \neq B'_2 \subseteq B_2$ and $B'_3 \subseteq B_3$. Now we show that the invariant $\alpha(\phi)$ in (44) ramifies. It is convenient to split the proof into two cases.

**Case 1:** $\exists e(i_p, i_{pq}) \in B'_2$ with $n_{i_p} + n_{i_{pq}} \geq 3$ such that $i_p \not\in J'_2$. Let $e(i_u, i_{uv}) \in B'_2$ be such an element for some $1 \leq u \leq k$ and $1 \leq v \leq m_u$ and let $I = \{1, \ldots, m\}$. We take a division quaternion algebra $(x, y)$ over a field extension $K/F$. Then, choose $\phi_i$ for all $i \in I$ such that

$$\phi[e(i_u)] = \phi[e(i_{uv})] = \langle x, y, xy \rangle \perp h, \quad \phi[e(i_{uv})] = \langle 1, z, xz, yz, xyz \rangle \perp h$$

for all $1 \leq q \neq v \leq m_u$,

$$\phi[e(i_p)] = \phi[e(i_{pq})] = \begin{cases} \langle x, y, xy \rangle \perp h & \text{if } e(i_u) \text{ appears in } b(i_p), \\ \langle 1 \rangle \perp h & \text{otherwise}, \end{cases}$$

for all $i_p \in J'_2$ and all $q$ with $e(i_p, i_{pq}) \in B_2$, and $\phi_i = \langle 1 \rangle \perp h$ for the remaining $i \in I$ over $K((z))$, where $h$ denotes a hyperbolic form depending on the dimension of each $\phi_i$. Then, we have

$$\phi[e(i_u, i_{uv})] = \langle \langle x, y, z \rangle \rangle, \quad \phi[e(i_u, i_{uv})] = \langle \langle x, y, 1 \rangle \rangle$$
for all 1 \leq q \neq v \leq m_u,

\[ \phi[e(i_p, i_{pq})] = \phi[b(i_p)] = \langle x, y, 1 \rangle \]

for all \( p \in J'_2 \) and all \( q \) with \( e(i_p, i_{pq}) \in B_2 \) such that \( e(i_u) \) appears in \( b(i_p) \), and \( \phi[b] = 0 \) for all remaining \( b \in B_2 \cup B_3 \) in the Witt ring of \( K((z)) \). Therefore, we obtain \( \partial_x(\alpha(\phi)) = (x, y) \neq 0 \). Hence, \( \alpha(\phi) \) ramifies.

**Case 2:** \( \exists e(i_p, i_{pq}) \in B'_2 \) with \( n_{i_p} + n_{i_{pq}} \geq 3 \) such that \( i_p \in J'_2 \). Let \( e(i_u, i_{uv}) \in B'_2 \) be such an element as in the previous case. Observe that by construction of \( B_3 \) there exists

\[ k_1 \in I\{i_p, j_r \mid i_p \in J'_2, 1 \leq r \leq s \} \]

such that \( e(k_1) \) appears in \( b'(i_u) \). We first choose \( \phi[e(i_{uv})] \) as in (45) and \( \phi[e(k_1)] = \langle x, y, xy \rangle \perp h \). Then, we choose \( \phi_i \) for \( i \in I \{i_{uv}, k_1 \} \) such that

\[ \phi[e(i)] = \begin{cases} 
\langle x, y, xy \rangle \perp h & \text{if } e(k_1) \text{ appears in } b(i), \\
\langle 1 \rangle \perp h & \text{otherwise}
\end{cases} \]

for all \( i \in \{i_p, j_r \mid i_p \in J'_2, 1 \leq r \leq s \} \),

\[ \phi[e(i_{pq})] = \begin{cases} 
\langle x, y, xy \rangle \perp h & \text{if } i_p = k_1 \text{ or } e(k_1) \text{ appears in } b(i_p), \\
\langle 1 \rangle \perp h & \text{otherwise}
\end{cases} \]

for all \( q \) such that \( e(i_p, i_{pq}) \in B_2 \), and \( \phi_i = \langle 1 \rangle \perp h \) for the remaining \( i \in I \{i_{uv}, k_1 \} \) over \( K((z)) \). Therefore, we obtain (46).

\[ \phi[b(i)] = \phi[e(i_p, i_{pq})] = \langle x, y, 1 \rangle \]

for all \( i \in \{i_p, j_r \mid i_p \in J'_2, 1 \leq r \leq s \} \) such that \( e(k_1) \) appears in \( b(i) \) and for all \( e(i_p, i_{pq}) \in B_2 \) such that \( i_p = k_1 \) or \( e(k_1) \) appears in \( b(i_p) \), and \( \phi[b] = 0 \) for all remaining \( b \in B_2 \cup B_3 \) in the Witt ring of \( K((z)) \). Hence, \( \partial_x(\alpha(\phi)) = (x, y) \neq 0 \), thus \( \alpha(\phi) \) ramifies.

We present the second main result on the group of unramified degree 3 invariants for type \( B \).

**Theorem 6.5.** Let \( G = (\prod_{i=1}^m \text{Spin}_{2n_i+1})/\mu \) defined over an algebraically closed field \( F \), \( m, n_i \geq 1 \), where \( \mu \) is a central subgroup. Then, every unramified degree 3 invariant of \( G \) is trivial, i.e., \( \text{Inv}^3_{nr}(G) = 0 \).

**Proof.** Set \( G_{\text{red}} = (\prod_{i=1}^m \Gamma_{2n_i+1})/\mu \). Since the classifying space \( BG \) is stably birational to the classifying space \( B\text{G}_{\text{red}} \), by [1] we have \( \text{Inv}^3_{nr}(G) = \text{Inv}^3_{nr}(G_{\text{red}}) \). We shall show that \( \text{Inv}^3_{nr}(G_{\text{red}}) = 0 \). Let \( G' = (\text{Spin}_3)^m/\mu \) and \( G'_{\text{red}} = (\Gamma_3)^m/\mu \). Then, the standard embeddings \( \text{Spin}_3 \rightarrow \text{Spin}_{2n_i+1} \) and \( \Gamma_3 \rightarrow \Gamma_{2n_i+1} \) induce morphisms \( G' \rightarrow G \) and
\( G'_\text{red} \to G_{\text{red}} \), thus we have

\[
\begin{array}{c}
\text{Inv}^3(G) \longrightarrow \text{Inv}^3(G') \\
\uparrow \quad \uparrow \\
\text{Inv}^3(G_{\text{red}}) \longrightarrow \text{Inv}^3(G'_{\text{red}})
\end{array}
\]

By (43) in Proposition 6.3 and Lemma 6.4 we may assume that the bottom map in (50) is an isomorphism. By [14, Lemma 4.3] and the exceptional isomorphism \( A_1 = B_1 \), we have \( \text{Inv}_{m}^3(G'_{\text{red}}) = 0 \), thus every invariant of \( G_{\text{red}} \) is ramified.

6.2. Type C.

**Lemma 6.6.** Let \( G = (\prod_{i=1}^{m} \text{Sp}_{2n_i})/\mu \), \( m, n_i \geq 1 \), where \( \mu \) is a central subgroup. Let \( R \) be the subgroup of \( (\mu_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m \) whose quotient is the character group \( \mu^* \). Set \( G_{\text{red}} = (\prod_{i=1}^{m} \text{GSp}_{2n_i})/\mu \), where \( \text{GSp}_{2n_i} \) is the group of symplectic similitudes. Then, for any field extension \( K/F \) the first Galois cohomology set \( H^1(K, G_{\text{red}}) \) is bijective to the set of \( m \)-tuples \( (A_1, \sigma_1), \ldots, (A_m, \sigma_m) \) of pairs of central simple \( K \)-algebra \( A_i \) of degree \( 2n_i \) with symplectic involution \( \sigma_i \) such that for all \( r = (r_i) \in R \)

\[
r_1A_1 + \cdots + r_mA_m = 0 \quad \text{in} \quad \text{Br}(K),
\]

where \( \text{Br}(K) \) denotes the Brauer group of \( K \).

**Proof.** Consider the exact sequence

\[
1 \to \left(\mathbb{G}_m\right)^m/\mu \to G_{\text{red}} \to \prod_{i=1}^{m} \text{PGSp}_{2n_i} \to 1,
\]

where \( \text{GSp}_{2n_i} \) is the group of symplectic similitudes and \( G_{\text{red}} = (\prod_{i=1}^{m} \text{GSp}_{2n_i})/\mu \). Then, by the same argument as in the proof of Lemma 6.1 the set \( H^1(F, G_{\text{red}}) \) is bijective to the kernel of following map

\[
H^1(F, \prod_{i=1}^{m} \text{PGSp}_{2n_i}) \to \text{Br}_2(F)^m \to H^2(F, (\mu_2)^m/\mu),
\]

where the first map sends an \( m \)-tuple \( ((A_1, \sigma_1), \ldots, (A_m, \sigma_m)) \) of simple algebra \( A_i \) of degree \( 2n_i \) with symplectic involution \( \sigma_i \) to the \( m \)-tuple \( (A_1, \ldots, A_m) \) and the map \( \alpha \) is induced by the natural surjection \( (\mu_2)^m \to (\mu_2)^m/\mu \). Since \( (A_1, \ldots, A_m) \in \text{Ker}(\alpha) \) if and only if it is contained in the kernel of the map in (41) for all \( r \in R \), thus we have

\[
H^1(F, G_{\text{red}}) \simeq \{((A_1, \sigma_1), \ldots, (A_m, \sigma_m)) \mid \text{deg } A_i = 2n_i, \sum_{i=1}^{m} r_iA_i = 0\}
\]

for all \( r = (r_i) \in R \).
Let \((A, \sigma)\) be a pair of central simple \(F\)-algebra \(A\) of degree \(2n\) with involution \(\sigma\) of the first kind. The trace form \(T_\sigma : A \to F\) given by \(T_\sigma(a) = \text{Trd}(\sigma(a)a)\), where \(\text{Trd}\) denotes the reduced trace. We denote by \(T_\sigma^+\) the restriction of \(T_\sigma\) to \(\text{Sym}(A, \sigma)\). Set
\[
\phi[r] := \bot_{i=1}^m r_i \phi_i, \quad \text{where } \phi_i = \begin{cases} T_{\sigma_i} & \text{if } n_i \equiv 1 \mod 2, \\ T_{\sigma_i}^+ & \text{if } n_i \equiv 2 \mod 4. \end{cases}
\]
for all \(r = (r_i) \in R \cap (\bigoplus_{4|n_i}(\mathbb{Z}/2\mathbb{Z})e_i)\). For all \(i \in I\) such that \(4|n_i\), we simply write \(\Delta\) for the Garibaldi-Parimala-Tignol invariant \(\Delta(A, \sigma_i)\) defined in \([\text{III} \text{ Theorem A}]\). Then, this degree 3 invariant induces the following invariants of \(G_{\text{red}}\)
\[
\Delta_i : H^1(K, G_{\text{red}}) \to H^1(K, \text{PGSp}_{2n_i}) \xrightarrow{\Delta} H^3(K),
\]
where the first map in \((53)\) is the projection and \(K/F\) is a field extension. We show that every invariant of semisimple group of type \(C\) is generated by the Arason invariants associated to \(\phi[r]\) and the Garibaldi-Parimala-Tignol invariants \(\Delta_i\).

**Proposition 6.7.** Let \(G = (\prod_{i=1}^m \text{Sp}_{2n_i})/\mu\) defined over an algebraically closed field \(F\), where \(m, n_i \geq 1\), \(\mu\) is a central subgroup. Let \(R\) be the subgroup of \((\mu^m_2)^\ast\) whose quotient is the character group \(\mu^\ast\). Set \(G_{\text{red}} = (\prod_{i=1}^m \text{GSp}_{2n_i})/\mu\). Then, every normalized invariant in \(\text{Inv}^3(G_{\text{red}})\) is of the form
\[
\sum_{r \in R'} e_3(\phi[r]) + \sum_{i \in I'} \Delta_i
\]
for some \(R' \subset R \cap (\bigoplus_{4|n_i}(\mathbb{Z}/2\mathbb{Z})e_i)\) and some subset \(I' \subset \{i \in I \mid 4|n_i\}\), where \(\phi[r]\) is the quadratic form defined in \((52)\) and \(e_3 : F^3(K) \to H^3(K)\) denotes the Arason invariant for a field extension \(K/F\). Moreover, we have
\[
\text{Inv}^3(G_{\text{red}})_{\text{norm}} \cong \bigoplus_{4|n_i}(\mathbb{Z}/2\mathbb{Z})e_i \bigoplus (R \cap (\bigoplus_{4|n_i}(\mathbb{Z}/2\mathbb{Z})e_i)) /
\langle e_i, e_j + e_k \in R \mid e_j, e_k \notin R, n_j \equiv n_k \equiv 1 \mod 2 \rangle.
\]

**Proof.** Since \(F\) is algebraically closed, we get \(\text{Inv}^3(G_{\text{red}})_{\text{norm}} = \text{Inv}^3(G_{\text{red}})_{\text{ind}}\). Let \(i\) be an integer such that \(n_i \equiv 0 \mod 4\). If \(e_i \in R\), then, as every symplectic involution on a split algebra is hyperbolic, by Lemma 6.6 and \([\text{III} \text{ Theorem A}]\) the invariant \(\Delta_i\) defined in \((53)\) vanishes. Now assume that \(e_i \notin R\). Let \(Q = (x, y)\) be a division quaternion algebra over a field extension \(K/F\) and let \(b = (1, z) \perp h\) be a symmetric bilinear form on \(E^{n_i}\), where \(h\) denotes a hyperbolic form and \(E = K((z))\). Consider the linear system as in \((51)\) with the coefficients given by a basis of \(R\). As \(e_i \notin R\), it follows by the rank theorem (or Rouché-Capelli theorem) that there exists a \(G_{\text{red}}\)-torsor \(\eta = ((A_1, \sigma_1), \ldots, (A_m, \sigma_m))\) over \(E\) such that
\[
(A_i, \sigma_i) = (M_{n_i}(Q), \sigma_b \otimes \gamma) \text{ and } (A_j, \sigma_j) = (M_{2n_j}(E), \sigma_\omega) \text{ or } (M_{n_j}(Q), t \otimes \gamma)
\]
for all \(1 \leq j \neq i \leq m\), where \(\gamma\) is the canonical involution on \(Q\), \(\sigma_b\) is the adjoint involution on \(\text{End}(E^{n_i}) = M_{n_i}(E)\) with respect to \(b\), \(\sigma_\omega\) is the adjoint involution
with respect to the standard symplectic bilinear form $\omega$, and $t$ denotes the transpose involution on $M_{n_j}(E)$. Then, by [Example 3.1] we have

\begin{equation}
(57) \quad \Delta_i(\eta) = (Q) \cup (z),
\end{equation}

thus, $\partial_2(\alpha(\eta)) = (x, y) \neq 0$. Therefore, we have a nontrivial invariant $\Delta_i$ of order 2 for any $i$ such that $n_i \equiv 0 \mod 4$ and $e_i \not\in R$.

Let $r \in R \cap (\bigoplus_{j=1}^{m} (\mathbb{Z}/2\mathbb{Z})e_i)$. Since each quadratic form $\phi_i$ in (52) has even dimension and trivial discriminant, we obtain $\phi[r] \in I^2(K)$ for each $r$. By [Theorem 1] the Hasse invariant of $\phi_i$ in (52) coincides with the class of $A_i$ in $\text{Br}(K)$, thus by the relation in (51), we have $\phi[r] \in I^3(K)$ for each $r \in R \cap (\bigoplus_{j=i}^{n_i} (\mathbb{Z}/2\mathbb{Z})e_i)$. Therefore, the Arason invariant induces a normalized invariant $e_3(\phi[r])$ of order dividing 2 that sends an $m$-tuple in (51) to $e_3(\phi[r]) \in H^3(K)$.

Let $r \in R''_1 + R''_2$, where $R''_1 = \langle e_i \in R \mid 4 \nmid n_i \rangle$ and $R''_2$ is the subgroup of $R$ defined in Proposition 6.7. For any $e_i \in R''_1$ and any $e_j + e_k \in R''_2$, we have

\begin{equation}
(58) \quad \phi_i = T_{a_i} = h \quad \text{and} \quad \phi_j \perp \phi_k = T_{a_j} \perp T_{a_k} = \langle(a, b, 1) \rangle \perp h',
\end{equation}

where $A_j = A_k = (a, b)$ in $\text{Br}(K)$, $h$ and $h'$ denote hyperbolic forms, thus both invariants $e_3(\phi[e_i])$ and $e_3(\phi[e_j + e_k])$ vanish. Therefore, the invariant $e_3(\phi[r])$ vanishes.

To complete the proof, by Theorem 5.5 it suffices to show that the invariant $e_3(\phi[r])$ is nontrivial for any $r \in R \cap (\bigoplus_{j=i}^{n_i} (\mathbb{Z}/2\mathbb{Z})e_i) \setminus (R''_1 + R''_2)$. Let $G'_{\text{red}} = (\mathbf{GSp}_2)^m/\mu$. Then, the rest of the proof of Proposition 6.7 still works if we replace the exceptional isomorphism $A_1 = B_1$, the standard embedding $\Gamma_3 \to \Gamma_{2n_1, +1}$, and Lemma 6.4 in the proof of Proposition 6.3 by the exceptional isomorphism $A_1 = C_1$, the standard embedding $\mathbf{GSp}_2 \to \mathbf{GSp}_{2n_1}$, and Lemma 6.9 respectively.

\begin{remark}
If $m = 2$, $n_1 \equiv n_2 \equiv 0 \mod 2$, and $\mu \subseteq \mu_2^2$ is the diagonal subgroup, then the invariant in Proposition 6.7 coincides with the invariant defined in [3].
\end{remark}

We present the following analogue of Lemma 6.4, which plays the same role for the triviality of unramified invariants as Lemma 6.4 plays for the groups of type $B$.

\begin{lemma}
Let $G = (\prod_{i=1}^{m} \mathbf{Sp}_{2n_i})/\mu$ defined over an algebraically closed field $F$, where $m, n_i \geq 1$, $\mu$ is a central subgroup. Set $G'_{\text{red}} = (\prod_{i=1}^{m} \mathbf{GSp}_{2n_i})/\mu$. Then, every normalized invariant in $\text{Inv}^3(G'_{\text{red}})$ is ramified if either $n_i$ is divisible by 4 for some $i$ with $e_i \not\in R_1$ or $n_jn_k \equiv 1 \mod 2$ for some $j$ and $k$ such that $e_j + e_k \in R \cap (\bigoplus_{j=1}^{n_i} (\mathbb{Z}/2\mathbb{Z})e_i)$.
\end{lemma}

\begin{proof}
Let $\alpha$ be a normalized invariant in $\text{Inv}^3(G'_{\text{red}})$ be written as in (54) for some subspace $R' \subseteq R \cap (\bigoplus_{j=1}^{n_i} (\mathbb{Z}/2\mathbb{Z})e_i)$ and subset $I' \subseteq \{i \in I \mid n_i \equiv 0 \mod 4, e_i \not\in R\}$.

Assume that there exist $i \in I'$. Let $\eta = ((A_1, \sigma_1), \ldots, (A_m, \sigma_m))$ be a $G_{\text{red}}$-torsor as in the proof of Proposition 6.7. Then, by [56], [Example 3.1], and [Theorem A] we have

$$\Delta_j(\eta) = 0$$

\end{proof}
for all $j \neq i$ such that $n_j \equiv 0 \mod 4$. Since
\[ \phi_j = \begin{cases} h & \text{if } (A_j, \sigma_j) = (M_{2n_j}(E), \sigma), \\ \langle\langle x, y \rangle\rangle \perp h & \text{if } (A_j, \sigma_j) = (M_n(Q), t \otimes \gamma), \end{cases} \]
where $h$ denotes a hyperbolic form and the pairs of the form $(M_n(Q), t \otimes \gamma)$ appear an even number of times in the relation of (51) for any $r \in R'$, we have $e_3(\phi[r]) = 0$ for any $r \in R'$. Therefore, by (57) we have $\partial_2(\alpha(\eta)) = (x, y) \neq 0$, thus the invariant $\alpha$ ramifies.

We may assume that $n_i \not\equiv 0 \mod 4$ for all $1 \leq i \leq m$, thus
\[ \alpha(\eta) = e_3(\phi[r_2]) + e_3(\phi[r_3]) \]
for some $r_2 \in R_2$ and $r_3 \in R_3$, where $R_1$ and $R_2$ are subspaces of $R$ in (12). $R_3$ is a complementary subspace of $R_1 + R_2$ in $R$, and $\eta$ is a $G_{\text{red}}$-torsor. Then, we choose bases $B_2 = \{e(i_p, i_{pq})\}$ of $R_2$ with $n_{pq} \geq n_p$ and $B_3$ of a complementary subspace of $R_2$ as in Lemma 6.4 and write the symbol $d$ for the corresponding degree of the matrix algebras in the rest of the proof. Now we choose $\eta = ((A_i, \sigma_i))$ for $i \in I$ such that
\[ (A_i, \sigma_i) = (M_d(Q), t \otimes \gamma), \quad (A_{iuv}, \sigma_{iuv}) = (M_d(Q_1 \otimes Q_2), t \otimes \gamma_1' \otimes \gamma_2) \]
for $i = i_u, i_{uv}$ and all $1 \leq q \neq u \leq m_u$, where $t$ denotes the transpose involution on a matrix algebra and $\gamma_1'$ is an orthogonal involution on $Q_1$ given by the composition of $\gamma_1$ and the inner automorphism induced by one of the generators of pure quaternions in $Q_1$,
\[ (A_{ip}, \sigma_{ip}), (A_{ipq}, \sigma_{ipq}) = \begin{cases} (M_d(Q), t \otimes \gamma) & \text{if } e(i_u) \text{ appears in } b(i_p), \\ (M_d(E), \sigma) & \text{otherwise}, \end{cases} \]
for all $i_p \in J'_2$ and all $q$ with $e(i_p, i_{pq}) \in B_2$, and
\[ (A_i, \sigma_i) = (M_d(E), \sigma) \]
for the remaining $i \in I$. Then, we have
\[ \phi[e(i_u)] = \phi[e(i_{uv})] = \langle\langle x, y \rangle\rangle \perp h, \quad \phi[e(i_{uv})] = \langle z, xz, yz, xzy \rangle \perp h \]
for all $1 \leq q \neq u \leq m_u$, thus we obtain (48), (47), and $\phi[b] = 0$ for all remaining $b \in B_2 \cup B_3$ in the Witt ring of $E$. Hence, $\partial_2(\alpha(\eta)) = (x, y) \neq 0$, i.e., $\alpha$ ramifies.

Case 2: $\exists e(i_p, i_{pq}) \in B'_2$ with $n_{pq} \equiv 1 \mod 4$ such that $i_p \in J'_2$. Let $e(i_u, i_{uv}) \in B'_2$ be such an element. We choose $k_1$ as in (48) and then choose $(A_{k_1}, \sigma_{k_1})$ and
\((A_{iuv}, \sigma_{iuv})\) as in (59). Then, we choose \((A_i, \sigma_i)\) for \(i \in I \setminus \{i_{uv}, k_1\}\) such that

\[
(A_i, \sigma_i) = \begin{cases} 
(M_d(Q), t \otimes \gamma) & \text{if } e(k_1) \text{ appears in } b(i), \\
(M_d(E), \sigma_\omega) & \text{otherwise}
\end{cases}
\]

for all \(i \in \{i_p, j_r \mid i_p \in J'_2, 1 \leq r \leq s\}\),

\[
(A_{i_{pq}}, \sigma_{i_{pq}}) = \begin{cases} 
(M_d(Q), t \otimes \gamma) & \text{if } i_p = k_1 \text{ or } e(k_1) \text{ appears in } b(i_p), \\
(M_d(E), \sigma_\omega) & \text{otherwise}
\end{cases}
\]

for all \(q\) such that \(e(i_p, i_{pq}) \in B_2\), and

\[
(A_i, \sigma_i) = (M_d(E), \sigma_\omega)
\]

for the remaining \(i \in I \setminus \{i_{uv}, k_1\}\). Therefore, we obtain (46), (49), and \(\phi[b] = 0\) for all remaining \(b \in B_2 \cup B_3\) in the Witt ring of \(E\). Therefore, \(\partial_2(\alpha(\eta)) = (x, y) \neq 0\), thus \(\alpha\) ramifies.

We show that the same result in Theorem 6.5 holds for the groups of type \(C\).

**Theorem 6.10.** Let \(G = (\prod_{i=1}^m \text{Sp}_{2n_i})/\mu\) defined over an algebraically closed field \(F\), \(m, n_i \geq 1\), where \(\mu\) is a central subgroup. Then, every unramified degree 3 invariant of \(G\) is trivial, i.e., \(\text{Inv}^3_m(G) = 0\).

**Proof.** Let \(G_\text{red} = (\prod_{i=1}^m \text{GSp}_{2n_i})/\mu, G'_\text{red} = (\text{GSp}_2)^m/\mu, \) and \(G' = (\text{Sp}_2)^m/\mu\). Then, the proof of Theorem 6.5 still works if we replace Proposition 6.3, Lemma 6.4, and the exceptional isomorphism \(A_1 = B_1\) in the proof by Proposition 6.7, Lemma 6.9 and the exceptional isomorphism \(A_1 = C_1\), respectively.

**6.3. Type \(D\).**

**Lemma 6.11.** Let \(G = (\prod_{i=1}^m \text{Spin}_{2n_i})/\mu\), \(m, \geq 1, n_i \geq 3\), where \(\mu\) is a central subgroup. Let \(R\) be the subgroup of \(Z\) such that \(\mu^* = Z/R\). Set \(G_\text{red} = (\prod_{i=1}^m \Omega_{2n_i})/\mu\), where \(\Omega_{2n_i}\) is the extended Clifford group. Then, for any field extension \(K/F\) the first Galois cohomology set \(H^1(K, G_\text{red})\) is bijective to the set of \(m\)-tuples \(((A_1, \sigma_1, f_1), \ldots, (A_m, \sigma_m, f_m))\) of triples consisting of a central simple \(K\)-algebra \(A_i\) of degree \(2n_i\) with orthogonal involution \(\sigma_i\) of trivial discriminant and a \(K\)-algebra isomorphism \(f_i : Z(C(A_i, \sigma_i)) \cong K \times K\), where \(Z(C(A_i, \sigma_i))\) denotes the center of the Clifford algebra \(C(A_i, \sigma_i)\), satisfying

\[
B_1 + \cdots + B_m = 0 \text{ in } \text{Br}(K)
\]

for all \(\sum_{i=1}^m r'_i \in R\) with

\[
r'_i = \begin{cases} 
 r_ie_i & \text{if } n_i \text{ odd,} \\
r_{i1}e_{i1} + r_{i2}e_{i2} & \text{if } n_i \text{ even,}
\end{cases}
\]

where

\[
B_i := \begin{cases} 
 r_1C_{i1} \text{ or } r_1C_{i2} & \text{if } n_i \text{ odd,} \\
r_{i1}C_{i1} + r_{i2}C_{i2} \text{ or } r_{i1}C_{i2} + r_{i2}C_{i1} & \text{if } n_i \text{ even}
\end{cases}
\]
depending on the choice of two isomorphisms \( f_i \) for each triple \((A_i, \sigma_i, f_i)\), \( C_{i1} \) and \( C_{i2} \) denote simple \( K \)-algebras such that \( C(A_i, \sigma_i) = C_{i1} \times C_{i2} \), and \( \text{Br}(K) \) denotes the Brauer group of \( K \).

**Proof.** Consider the exact sequence

\[
1 \to (\mathbb{G}_m)^{2m}/\mu \to G_{\text{red}} \to \prod_{i=1}^m \text{PGO}^+_{2n_i} \to 1,
\]

where \( \text{PGO}^+_{2n_i} \) and \( \Omega_{2n_i} \) denote the projective orthogonal group and the extended Clifford group (§13), respectively and \( G_{\text{red}} = (\prod_{i=1}^m \Omega_{2n_i})/\mu \). Applying the same argument as in the proof of Lemma 6.1 we see that the set \( H^1(K, G_{\text{red}}) \) is bijective to the kernel of following map

\[
H^1(K, \prod_{i=1}^m \text{PGO}^+_{2n_i}) \xrightarrow{\beta} \text{Br}(Z(\prod_{i=1}^m \text{Spin}_{2n_i})) \xrightarrow{\alpha} H^2(K, Z(\prod_{i=1}^m \text{Spin}_{2n_i})/\mu),
\]

where the map \( \beta \) sends an \( m \)-tuple \((A_i, \sigma_i, f_i)\) of triples consisting of a central simple \( K \)-algebra \( A_i \) of degree \( 2n_i \) with orthogonal involution \( \sigma_i \) of trivial discriminant and a \( K \)-algebra isomorphism \( f_i : Z(C(A_i, \sigma_i)) \cong K \times K \) to the \( m \)-tuple \((B_1', \ldots, B_m')\) with

\[
B_i' := \begin{cases} 
C_{i1} & \text{if } n_i \text{ odd}, \\
C_{i2} \text{ or } (C_{i1}, C_{i2}) & \text{if } n_i \text{ even},
\end{cases}
\]

depending on the choice of two isomorphisms \( f_i \) for each triple \((A_i, \sigma_i, f_i)\) (i.e., For odd (resp. even) \( n_i \), the image of \((A_i, \sigma_i, f_i)\) under \( \beta \) is \( C_{i1} \) (resp. \((C_{i1}, C_{i2})\)) if and only if the image of \((A_i, \sigma_i, f_i')\) for another isomorphism \( f_i' \) of degree \( 2 \) under \( \beta \) is \( C_{i2} \) (resp. \((C_{i2}, C_{i1})\))) and the map \( \alpha \) is induced by the natural surjection \( Z(\prod_{i=1}^m \text{Spin}_{2n_i}) \to Z(\prod_{i=1}^m \text{Spin}_{2n_i})/\mu \). As \((B_1', \ldots, B_m') \in \text{Ker}(\alpha)\) if and only if it is contained in the kernel of the composition

\[
H^2(K, Z(\prod_{i=1}^m \text{Spin}_{2n_i})/\mu) \xrightarrow{r} H^2(K, \mathbb{G}_m)
\]

for all \( r \in R = (Z(\prod_{i=1}^m \text{Spin}_{2n_i})/\mu)^* \), we obtain

\[
H^1(K, G_{\text{red}}) \cong \{ (\langle A_i, \sigma_i, f_i \rangle) \mid \sum_{i=1}^m B_i = 0 \text{ in } \text{Br}(K) \}
\]

for all \( \sum_{i=1}^m r'_i \in R \).  

Recall from Theorem 5.6 the following subsets

\[
I_1 = \{ i \mid Z_i = R_{1i} \text{ or } R'_{1i}; n_i \neq 3 \} \quad \text{and}
I_2 = \{ i \mid R'_{1i} = 0, 4|n_i \} \cup \{ i \mid R'_{1i} = (Z/2Z)e_{1i} \text{ or } (Z/2Z)e_{2i}, n_i \geq 6, 4|n_i \} =: I_{21} \cup I_{22}.
\]

Let \( i \in I_1 \). Then, from Lemma 6.11 we see that both \( K \)-algebras \( A_i \) and \( C(A_i, \sigma_i) \) split, thus we have \((A_i, \sigma_i, f_i) \cong (M_{2n_i}(K), \sigma_{\psi_i})\) for some adjoint involution \( \sigma_{\psi_i} \) with
respect to a quadratic form $\psi_i$ such that $\psi_i \in I^3(K)$. Hence, the Arason invariant $e_3$ induces the following invariant

$$e_{3,i} : H^1(K, G_{\text{red}}) \rightarrow H^3(K)$$

given by $e_{3,i}((A_1, \sigma_1, f_1), \ldots, (A_m, \sigma_m, f_m)) = e_3(\phi_i)$.

Now let $i \in I_2$. Then, the invariant $\Delta'$ of $\text{PGO}_{2n}^+$ ([19, Theorem 4.7]) gives the following invariant of $G_{\text{red}}$

$$\Delta'_i : \begin{cases} 
H^1(K, G_{\text{red}}) \rightarrow H^1(K, \text{PGO}_{2n}^+) \overset{\Delta'}{\rightarrow} H^3(K) & \text{if } i \in I_{21}, \\
H^1(K, G_{\text{red}}) \rightarrow H^1(K, \text{HSpin}_{2n_i}) \overset{\Delta'}{\rightarrow} H^3(K) & \text{if } i \in I_{22},
\end{cases}$$

where the first map in (67) is the projection for each case.

We shall need the following analogue of [11, Example 3.1].

**Lemma 6.12.** Let $Q$ be a quaternion algebra over $F$ and let $(A, \sigma, f) \in H^1(F, \text{PGO}_{2n}^+)$ such that $n \equiv 0 \pmod{2}$ and $(A, \sigma) = (M_n(F) \otimes Q, \sigma_1 \otimes \sigma_2)$ for some orthogonal involutions $\sigma_1$ and $\sigma_2$ on $M_n(F)$ and $Q$, respectively. Then, we have

$$\Delta'(A, \sigma, f) = Q \cup (\text{disc } \sigma_1).$$

**Proof.** Let $t$ be the transpose involution on $M_n(F)$. Since $\sigma_1 = \text{Int}(x) \circ t$ for some $t$-symmetric invertible element $x$, where $\text{Int}(x)$ denotes the inner automorphism induced by $x$, we have

$$\text{disc}(\sigma_1) = \text{Nrd}_{M_n(F)}(x) = \sqrt{\text{Nrd}_A(x \otimes 1)}$$

and $\sigma = \text{Int}(x \otimes 1) \circ (t \otimes \sigma_2)$, where $\text{Nrd}$ denotes the reduced norm. As $x \otimes 1$ is a $\sigma$-symmetric invertible element, the result follows from [19, §4b]. \qed

**Proposition 6.13.** Let $G = (\prod_{i=1}^m \text{Spin}_{2n_i})/\mu$ defined over an algebraically closed field $F$, where $m \geq 1$, $n_i \geq 3$, $\mu$ is a central subgroup. Set $G_{\text{red}} = (\prod_{i=1}^m \Omega_{2n_i})/\mu$, where $\Omega_{2n_i}$ is the extended Clifford group. Then, every normalized invariant in $\text{Inv}^3(G_{\text{red}})$ is of the form

$$\sum_{i \in I'_1} e_{3,i} + \sum_{i \in I'_2} \Delta'_i + \sum_{r \in R''} e_3(\phi[r])$$

for some subsets $I'_1 \subseteq I_1$, $I'_2 \subseteq I_2$, and $R'' \subseteq R'$, where $R'$ denotes the group as defined in Theorem 5.6, $\phi[r]$ is the quadratic form defined in (52) and $e_3 : I^3(K) \rightarrow H^3(K)$ denotes the Arason invariant for a field extension $K/F$. Moreover, we have

$$\text{Inv}^3(G_{\text{red}})_{\text{norm}} \simeq \bigoplus_{i \in I'_1 \cup I'_2} (\mathbb{Z}/2\mathbb{Z}) \bar{e}_i \bigoplus_{R'} \bigoplus_{n_j \equiv n_k \equiv 1 \pmod{2}} R'.$$

**Proof.** Since $F$ is algebraically closed, we obtain $\text{Inv}^3(G_{\text{red}})_{\text{norm}} = \text{Inv}^3(G_{\text{red}})_{\text{ind}}$. We first show that the invariant $\Delta'_i$ is nontrivial for all $j \in I_2$. Choose a field extension $E/K$ containing variables $x_{i_1}, x_{i_2}, x_i, y_i$, division quaternion $K$-algebras

$$Q_{i_1} = (x_{i_1}, y_{i_1}), Q_{i_2} = (x_{i_2}, y_{i_2})$$
for all $i \in I$ such that $n_i \equiv 0 \pmod{2}$, and cyclic division $K$-algebras

$$P_i = (x_i, y_i)_4$$

of exponent 4 for all $i \in I$ such that $n_i \equiv 1 \pmod{2}$. Let

$$Q_i = \begin{cases} (x_{i1}x_{i2}, y_i) & \text{if } n_i \text{ even,} \\ (x_i, y_i) & \text{if } n_i \text{ odd,} \end{cases}$$

so that $Q_i = \begin{cases} Q_{i1} + Q_{i2} & \text{if } n_i \text{ even,} \\ 2P_i & \text{if } n_i \text{ odd} \end{cases}$

in $Br(K)$. For $r \in R$, let

$$D_{1r} = \bigotimes_{2|n_i} (Q_{i1}^{2r} \otimes Q_{i2}^{2r}), \quad D_{2r} = \bigotimes_{2|n_i} P_i^{r}, \quad \text{and } D_r = D_{1r} \otimes D_{2r}.$$ 

Let $L$ be the function field of the product $\prod_{r \in R} \text{SB}(D_r)$ of Severi-Brauer varieties $\text{SB}(D_r)$ of $D_r$ over $K$. For all $i$ such that $n_i \equiv 1 \pmod{2}$, consider the exterior square $\lambda^2 P_i$ of $P_i$ with its canonical involution $\rho_i$ [12, §10]. By the exceptional isomorphism $A_3 = D_3$ ([12, 15.32]), we have

$$C(\lambda^2 P_i, \rho_i) = P_i \times P_i^{op},$$

where $P_i^{op}$ denotes the opposite algebra of $P_i$. Let $\chi_i$ be a skew-hermitian form over $Q_i$ such that $(M_3(Q_i), \sigma_{\chi_i}) = (\lambda^2 P_i, \rho_i)$, where $\sigma_{\chi_i}$ is the adjoint involution with respect to $\chi_i$. Let $\psi_i = \chi_i \perp h$ be a skew-hermitian form over $Q_i$ of rank $n_i$, where $h$ denotes a hyperbolic form (if $n_i = 3$, then $\psi_i = \chi_i$). We denote by $\sigma_{\psi_i}$ the adjoint involution on $M_{n_i}(Q_i)$ with respect to $\psi_i$. Let

$$(A_i, \sigma_i) = \begin{cases} (M_{n_i}(L) \otimes Q_i, \sigma_{i1} \otimes \sigma_{i2}) & \text{if } n_i \text{ even,} \\ (M_{n_i}(Q_i), \sigma_{\psi_i}) & \text{if } n_i \text{ odd} \end{cases}$$

for some orthogonal involutions $\sigma_{i1}$ on $M_{n_i}(L)$ and $\sigma_{i2}$ on $Q_i$ such that $\text{disc}(\sigma_{i1}) = x_{i1}$ and $\text{disc}(\sigma_{i2}) = y_i$. Then, by [8, Theorem 1.1] and [7, Corollary 3] together with (69), we obtain

$$C(A_i, \sigma_i) = \begin{cases} M_{2n_i-2}(Q_{i1}) \times M_{2n_i-2}(Q_{i2}) & \text{if } n_i \text{ even,} \\ M_{2n_i-3}(P_i) \times M_{2n_i-3}(P_i)^{op} & \text{if } n_i \text{ odd,} \end{cases}$$

thus by a theorem of Amitsur we have a $G_{\text{red}}(L)$-torsor $\eta = ((A_i, \sigma_i, f_i))$. Finally, by Lemma [12, 16.12] we get $\Delta_j(\eta) = (x_{ij1}, x_{ij2}, y_{ij}) \neq 0$.

Now, let $r = (\bar{r}_1, \ldots, \bar{r}_m) \in R'$. Then, from Lemma [6.11] we have

$$B_i = A_i = \begin{cases} 2\bar{r}_iC_{i1} = 2\bar{r}_iC_{i2} & \text{if } n_i \text{ odd,} \\ \bar{r}_iC_{i1} + \bar{r}_iC_{i2} & \text{if } n_i \text{ even,} \end{cases}$$

in $Br(K)$, thus the relation in (65) is equivalent to

$$\bar{r}_1A_1 + \cdots + \bar{r}_mA_m = 0.$$ (70)

As each quadratic form $\phi_i$ in [52] has even dimension and trivial discriminant, we have $\phi_i[r] \in T^3(K)$ for each $r \in R'$. By [20, Theorem 1] the Hasse invariant of $\phi_i$ in [52] coincides with the class of $A_i$ in $Br(K)$, thus by the relation in (70), we have $\phi_i[r] \in T^3(K)$ for each $r \in R'$. Therefore, the Arason invariant induces a
normalized invariant $\mathbf{e}_3(\phi[r])$ of order dividing 2 that sends an $m$-tuple in (65) to $\mathbf{e}_3(\phi[r]) \in H^3(K)$.

Let $r \in R'' + R''$, where $R'' = \langle \bar{e}_i, \bar{e}_i \in R' \rangle$ and $R'' = \langle \bar{e}_j + \bar{e}_k \in R', \bar{e}_j, \bar{e}_k \notin R', n_j \equiv n_k \equiv 1 \mod 2 \rangle$. Then, by (58) both invariants $\mathbf{e}_3(\phi[\bar{e}_i])$ and $\mathbf{e}_3(\phi[\bar{e}_j + \bar{e}_k])$ vanish for any $\bar{e}_i \in R''$ and any $\bar{e}_j + \bar{e}_k \in R''$, thus $\mathbf{e}_3(\phi[r])$ vanishes.

As before, by Theorem 5.6 it is enough to show that the invariant $\mathbf{e}_3(\phi[r])$ is nontrivial for any $r \in R' \setminus (R'_1 + R''_2)$. Let $G^{r}_\text{red} = (\Omega_6^r \setminus \mu)$. Then, the same arguments as in the proof of Proposition 6.3 work if we replace [14, Lemma 4.3], the exceptional isomorphism $A_1 = B_1$, the standard embedding $\Gamma_3 \to \Gamma_{2n+1}$, and Lemma 6.4 in the proof of Proposition 6.3 by [14, Lemma 4.2], the exceptional isomorphism $A_3 = D_3$, the standard embedding $\Omega_6 \to \Omega_{2n}$, and Lemma 6.14 respectively. □

We shall present the following analogue of Lemmas 6.4 and 6.9.

**Lemma 6.14.** Let $G = (\prod_{i=1}^m \text{Spin}_{2n_i})/\mu$ defined over an algebraically closed field $F$, where $m \geq 1$, $n_i \geq 3$, $\mu$ is a nontrivial subgroup. Set $G_{\text{red}} = (\prod_{i=1}^m \Omega_{2n_i})/\mu$. Then, every normalized invariant in $\text{Inv}^3(G_{\text{red}})$ is ramified if either $n_i \geq 4$ for some $i \in I_1 \cup I_2$ or $n_j n_k \equiv 1 \mod 2$ for some $j$ and $k$ such that $\bar{e}_j + \bar{e}_k \in R'$.

**Proof.** Let $\alpha$ be a normalized invariant in $\text{Inv}^3(G_{\text{red}})$ be written as in (68) for some subsets $I'_1 \subseteq I_1$, $I'_2 \subseteq I_2$ and $R'' \subseteq R'$.

First, assume that there exists $j \in I'_1$. Let $Q = (x, y)$ be a division quaternion algebra over a field extension $K/F$ and let $\psi_i = \langle (x, y, z) \rangle \perp h$ be a quadratic form over $E := K((z))$, where $h$ denotes a hyperbolic form. Choose a $G_{\text{red}}$-torsor $\eta = ((A_1, \sigma_1, f_1), \ldots, (A_m, \sigma_m, f_m))$ such that

$$(A_j, \sigma_j, f_j) = (M_{2n_j}(E), \sigma_{\psi_j}) \text{ and } (A_i, \sigma_i, f_i) = (M_{2n_i}(E), t)$$

for all $1 \leq i \neq j \leq m$, where $\sigma_{\psi_j}$ denotes the adjoint involution on $M_{2n_j}(E)$ with respect to $\psi_j$ and $t$ denotes the transpose involution on $M_{2n_i}(E)$. Then, we have

$$\sum_{i \in I'_1} \mathbf{e}_{3,i}(\eta) = (x, y, z), \quad \sum_{i \in I'_2} \Delta_i(\eta) = \sum_{r \in R''} \mathbf{e}_3(\phi[r])(\eta) = 0.$$

Therefore, we have $\partial_2(\alpha(\eta)) = (x, y) \neq 0$. Hence, the invariant $\alpha$ ramifies.

We assume that $I'_1 = \emptyset$ and $I'_2 \neq \emptyset$, i.e., $\alpha(\eta) = \sum_{i \in I'_2} \Delta_i + \sum_{r \in R''} \mathbf{e}_3(\phi[r])$. Let $j \in I'_2$ and let $\eta = ((A_1, \sigma_1, f_1), \ldots, (A_m, \sigma_m, f_m))$ be a $G_{\text{red}}$-torsor over $L$ as in the proof of Proposition 6.13. Then, over $L((y_j))$ we have

$$\partial_{y_j}(\alpha(\eta)) = \partial_{y_j}(\Delta_j(\eta)) = \partial_{y_j}((x_{j1}, x_{j2}, y_j)) = (x_{j1}, x_{j2}) \neq 0,$$

thus the invariant $\alpha$ ramifies.

Now we may assume that $n_i \equiv \not\equiv 0 \mod 4$ and $R'_i$, $R_i \neq Z_i$ for all $1 \leq i \leq m$, thus

$$\alpha(\eta) = \mathbf{e}_3(\phi[r_2]) + \mathbf{e}_3(\phi[r_3])$$

for some $r_2 \in R_2$ and $r_3 \in R_3$, where $R_2$ denotes the subspace of $R$ generated by $\bar{e}_i + \bar{e}_j$ for all $1 \leq i \neq j \leq m$, $R_3$ denotes a complementary subspace of $R_3$ in $R$, and $\eta$ is a $G_{\text{red}}$-torsor. For simplicity, we write $e(i_1, \ldots, i_k)$ for $\bar{e}_{i_1} + \cdots + \bar{e}_{i_k}$. Choose bases
$B_2 = \{ e(i_{pq}) \}$ of $R_2$ with $n_{ip} \geq n_{ip}$ and $B_3$ of a complementary subspace of $R_2$ as in Lemma 6.4 so that the invariant $\alpha$ is written as in (44).

To show that the invariant $\alpha(\eta)$ ramifies, we now proceed as in the proof of Lemma 6.9 with the following simple modifications. Let $(Q, \gamma), (Q_1, \gamma_1), (Q_2, \gamma_2)$ be the quaternions with canonical involutions as in the proof of Lemma 6.9 and let $\sigma$ be an orthogonal involution on $Q$ given by the composition of $\gamma$ and the inner automorphism induced by one of the generators of pure quaternions in $Q$. Then, the same proof as in Lemma 6.9 still works if we choose $\eta = (\eta_i, \sigma_i, f_i)$ satisfying (59), (60), (61) for Case 1 and (62), (63), and (64) for Case 2, after replacing the involutions $\gamma'_1, \gamma'_2,$ and $\sigma$ in those equations by $\gamma_1, \sigma,$ and $t$, respectively.

Finally, we prove the second main result on the group of unramified degree 3 invariants for type $D$.

**Theorem 6.15.** Let $G = (\prod_{i=1}^{m} \text{Spin}_{2n_i})/\mu$ defined over an algebraically closed field $F$, $m \geq 1$, $n_i \geq 3$, where $\mu$ is a central subgroup. Then, every unramified degree 3 invariant of $G$ is trivial, i.e., $\text{Inv}^3_{nr}(G) = 0$.

**Proof.** Let $G'_{\text{red}} = (\prod_{i=1}^{m} \Omega_{2n_i})/\mu$, $G''_{\text{red}} = (\Omega_{4})^m/\mu$, and $G' = (\text{SL}_4)^m/\mu$. Then, by the same argument as in the proof of Theorem 6.5 together with Proposition 6.13 and Lemma 6.14 we may assume that the bottom map in (50) is an isomorphism. By [14, Lemma 4.2], we have $\text{Inv}^3_{nr}(G''_{\text{red}}) = 0$. Hence, every invariant of $G_{\text{red}}$ is ramified.

**References**


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