Very Small Product Sets

Matt DeVos

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- $A, B \subseteq G$ are finite and nonempty,
- $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$.

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Definition

$|A + B|$ is *very small* if $|A + B| < |A| + |B|$.

Setup

- G is a group written multiplicatively,
- $A, B \subseteq G$ are finite and nonempty,
- $AB = \{ab \mid a \in A \text{ and } b \in B\}$.

Central Questions

1. How small can $|AB|$ be?
2. If $|AB|$ is small, then why?

Definition

$|AB|$ is *very small* if $|AB| < |A| + |B|$.

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If $A, B \subseteq \mathbb{Z}$ are finite and nonempty then $|A + B| \geq |A| + |B| - 1$.

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Proof:

Let $A = \{a_1 \dots a_m\}, B = \{b_1 \dots b_n\}$ with $a_1 < \dots < a_m$ and $b_1 < \dots < b_n$. Then $A + B$ contains the distinct elements

$$a_1 + b_1 < a_2 + b_1 < \dots < a_m + b_1 < a_m + b_2 < \dots < a_m + b_n.$$

□

More generally

If we start from $a_1 + b_1$ and move to $a_m + b_n$ by incrementing one index each step, we find $m + n - 1$ distinct elements.

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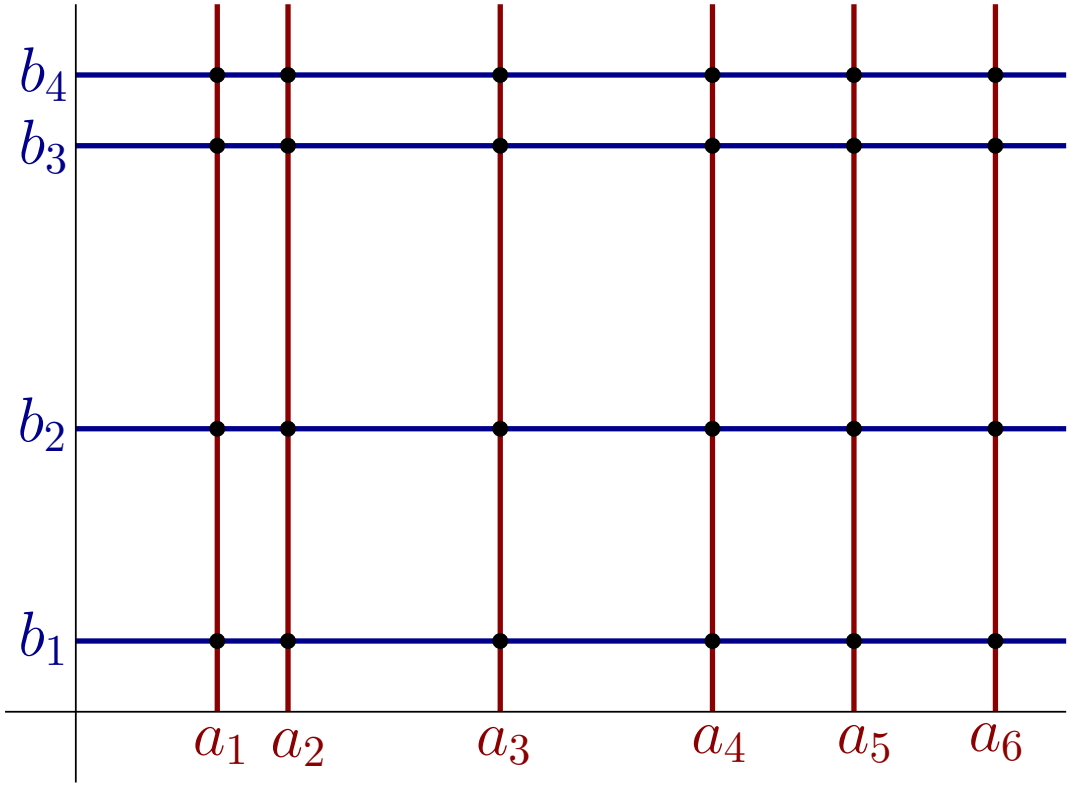

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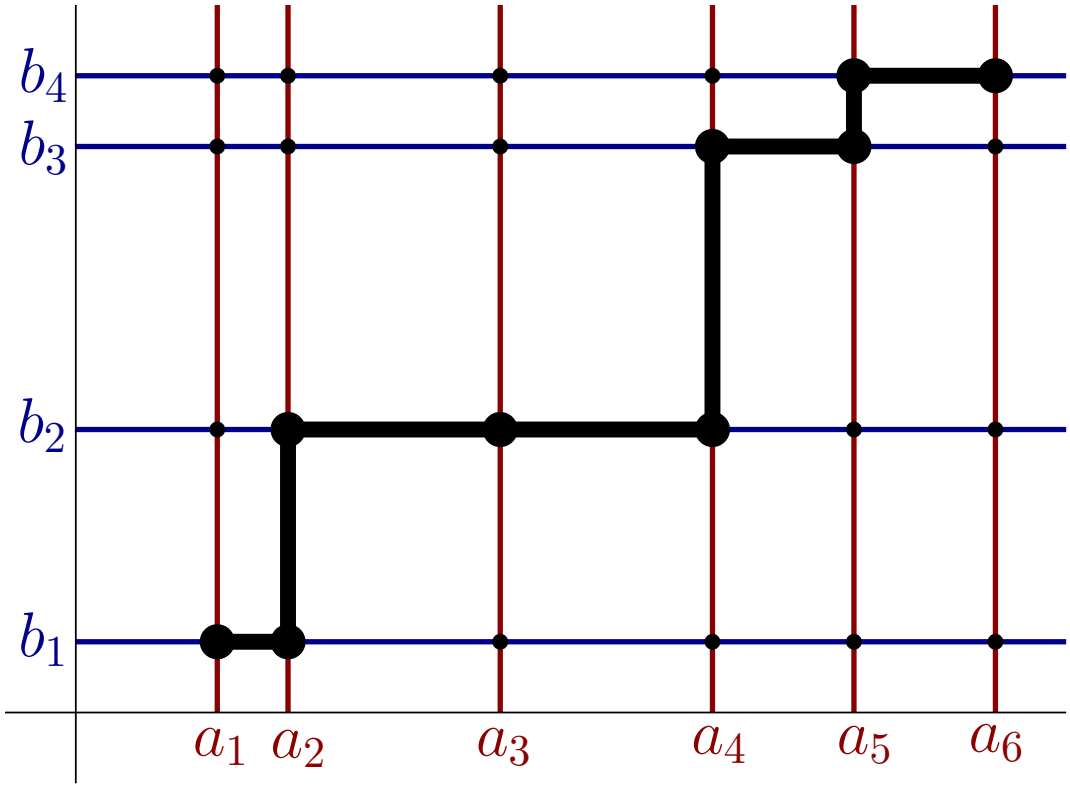
However we start from $a_1 + b_1$ and move to $a_m + b_n$ by increasing one index by one each step, we find $m + n - 1$ distinct elements.



this leads to...

Observation

If $A, B \subseteq \mathbb{Z}$ are finite and nonempty and $|A + B|$ is very small, then one of the following holds:

1. $|A| = 1$ or $|B| = 1$,
2. A and B are arithmetic progressions with a common difference.

Theorem (Cauchy-Davenport)

If p is prime and $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ are nonempty, then either $A + B = \mathbb{Z}/p\mathbb{Z}$, or $|A + B| \geq |A| + |B| - 1$.

$G = \mathbb{Z}/p\mathbb{Z}$

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Very Small Sumsets (Vosper)

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G abelian

Theorem (Kneser)

Let A, B be finite nonempty subsets of an additive abelian group G. Then there exists $H \leq G$ so that

1. $|A + B| \geq |A| + |B| - |H|$, and
2. $A + B + H = A + B$.

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Very Small Sumsets (Kemperman)

G arbitrary

Theorem (D.)

Let A, B be finite nonempty subsets of an arbitrary multiplicative group G. Then there exists $H \leq G$ so that

1. $|AB| \geq |A| + |B| - |H|$,
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Note

We prove this by classifying the very small product sets.

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Theorem (Cauchy-Davenport)

If p is prime and $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ are nonempty, then

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Shifting

For an element $g \in G$ and a set $C \subseteq G$ we define $g + C = \{g\} + C$ and $C + g = C + \{g\}$, and we call sets of this form *shifts* of C.

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$$(A \cap (A + g)) + (B \cup (-g + B)) \subseteq A + B$$

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Consider a counterexample A, B for which

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$(A \cup (A + g)) + (B \cap (-g + B)) \subseteq A + B$.
One of these pairs contradicts our choice of A, B. $\square$

G arbitrary

General Technique

In an arbitrary (multiplicative) group we still have the identities
$(A \cap (Ag)) \cdot (B \cup (g^{-1}B)) \subseteq AB$ and
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3. BC is very small.

(this follows from the fact that BC is disjoint from $G \setminus A^{-1}$, so $|BC| \leq |G| - |A^{-1}| < |B| + |C|$)

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Def:
A *Trio* is a triple (A, B, C) of subsets of G with $1 \notin ABC$. We call it *very small* if $|A| + |B| + |C| > |G|$.

Graphs

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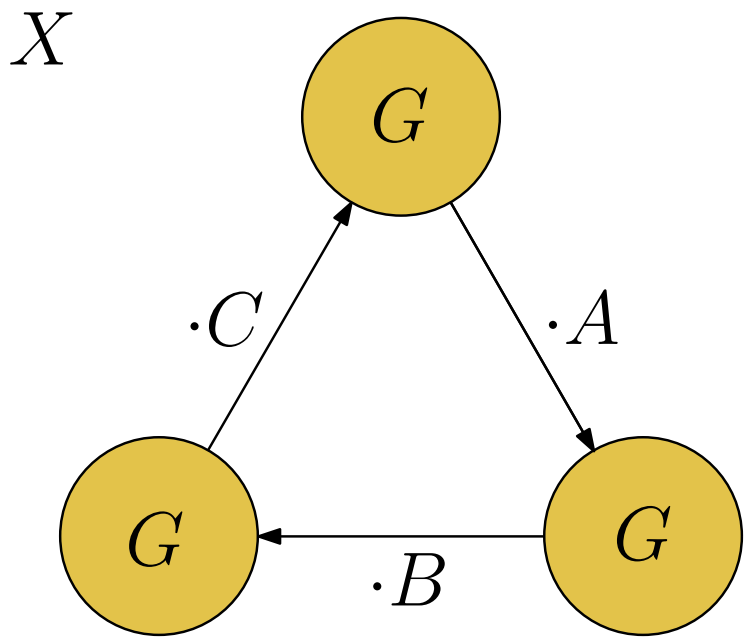

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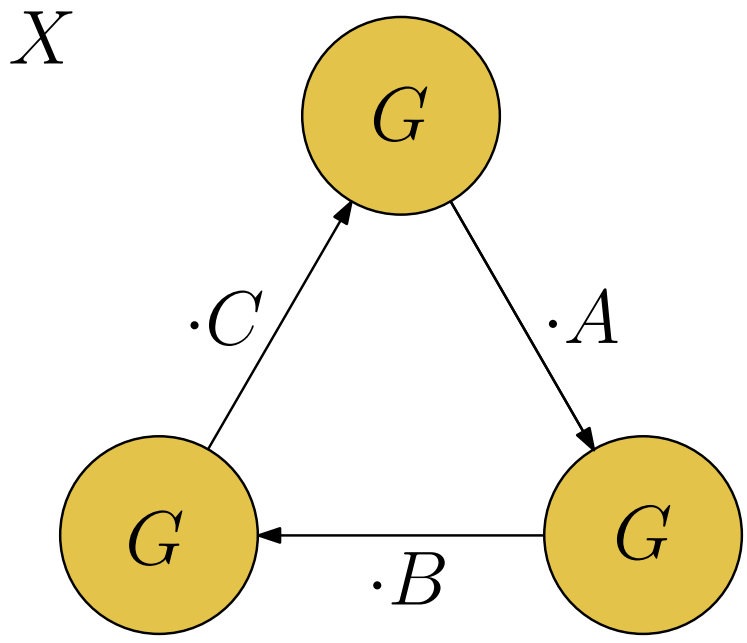

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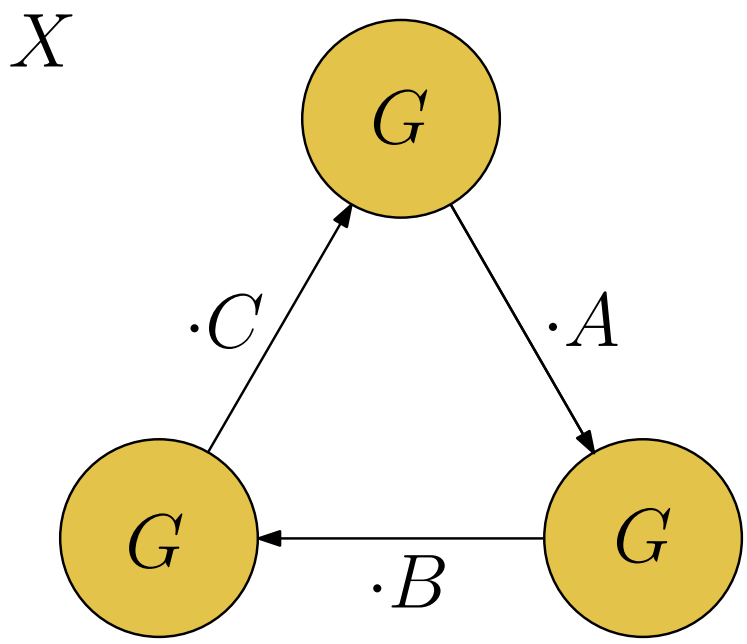

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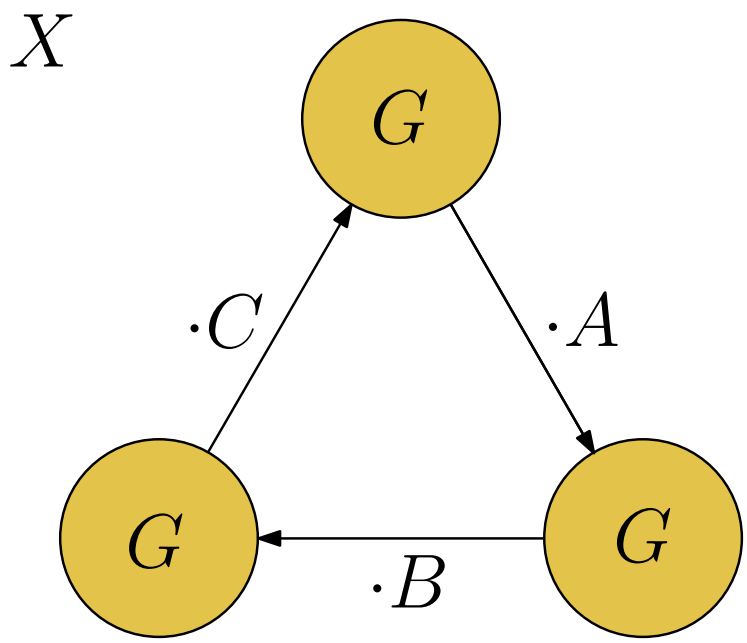

Observe

1. X has no triangle
2. The sum of the densities of the three bipartite subgraphs between blocks is $\frac{|A|}{|G|} + \frac{|B|}{|G|} + \frac{|C|}{|G|} > 1$.
3. G has a natural action on X which is transitive on each block.

Graphs

New problem

We intend to classify all tripartite graphs X which satisfy

1. X has no triangle
2. The sum of the densities of the three bipartite subgraphs between blocks is > 1.
3. The subgroup of $Aut(X)$ which fixes each block setwise still acts transitively on each block.

Example

Consider a finite map on a surface with vertices V, edges E, and faces F for which the automorphism group acts transitively on V, E, and F.

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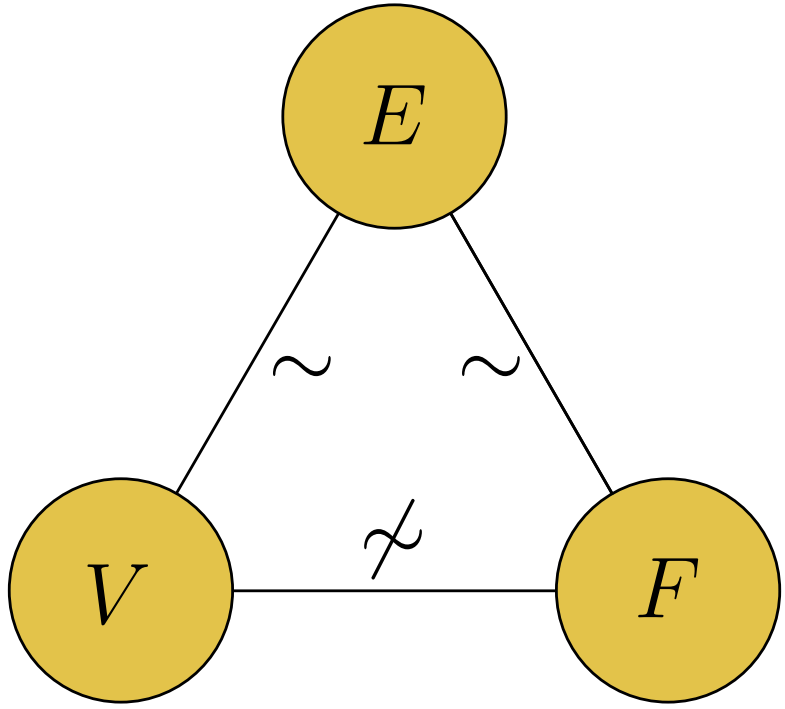

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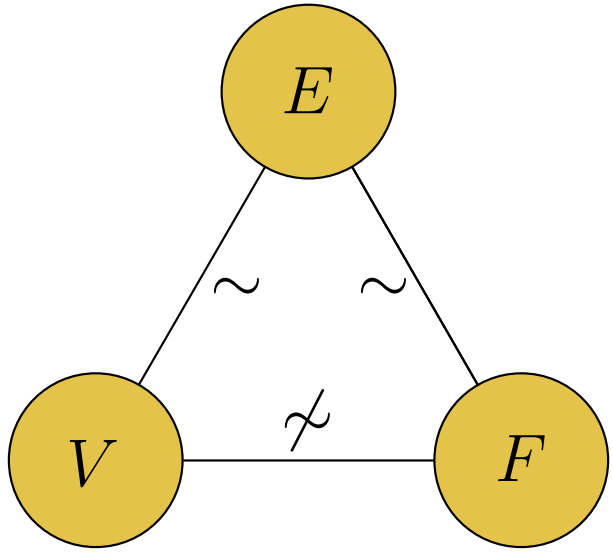

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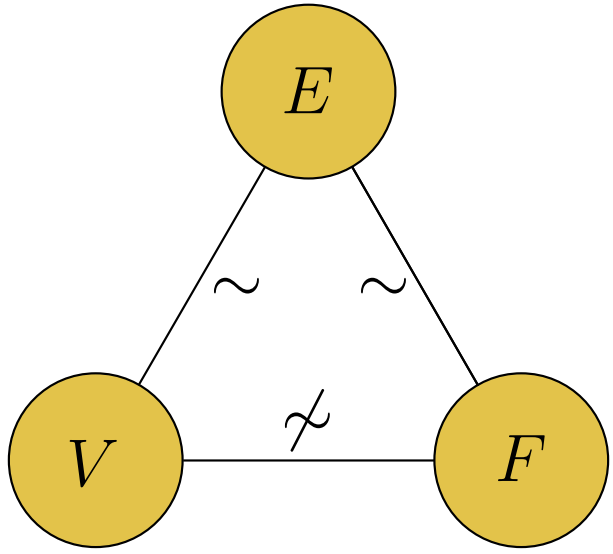

Observe:

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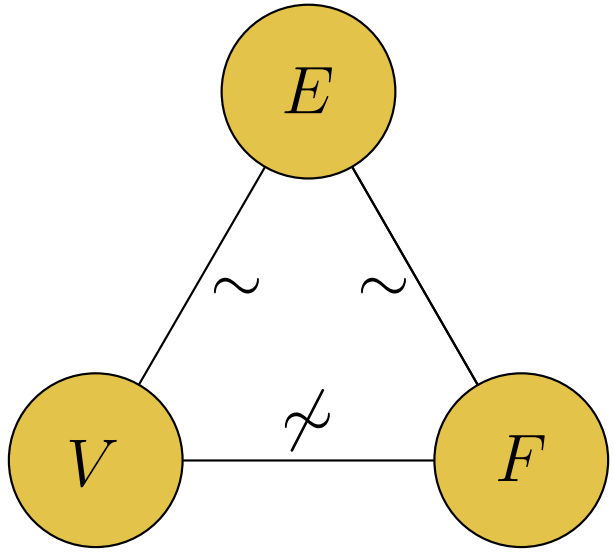
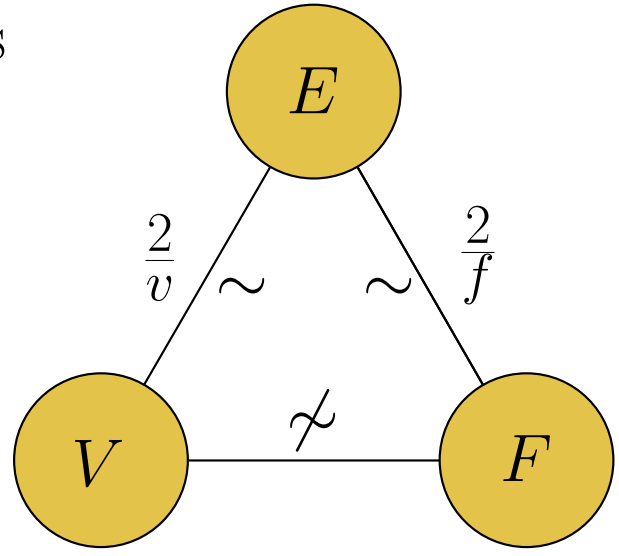


Observe:

- X has no triangle.
- The automorphism group of the map acts on X and is transitive on V, E, and F.
- next we compute compute densities..

Example

densities



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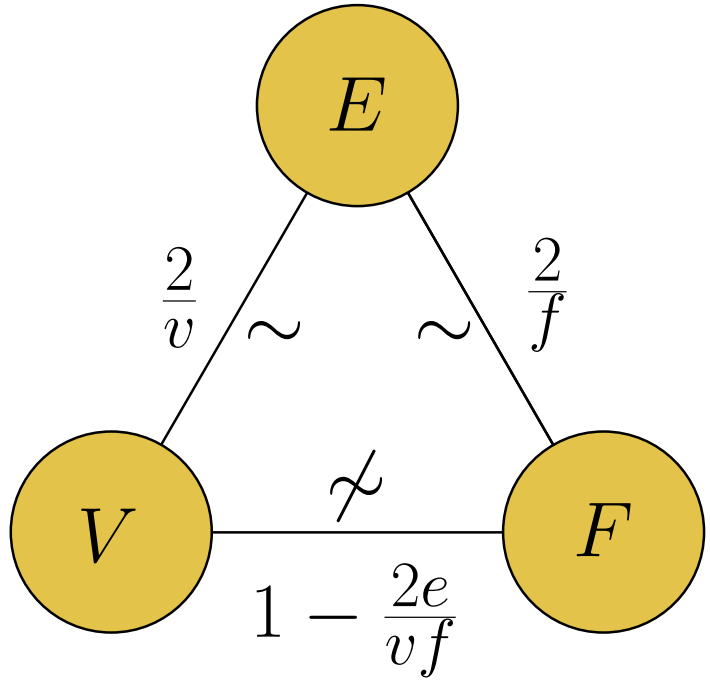

- The number of vertex-face incidences is $2e$
- The density of the vertex-face incidence bipartite graph is $\frac{2e}{vf}$
- The density of the vertex-face nonincidence bipartite graph is $1 - \frac{2e}{vf}$

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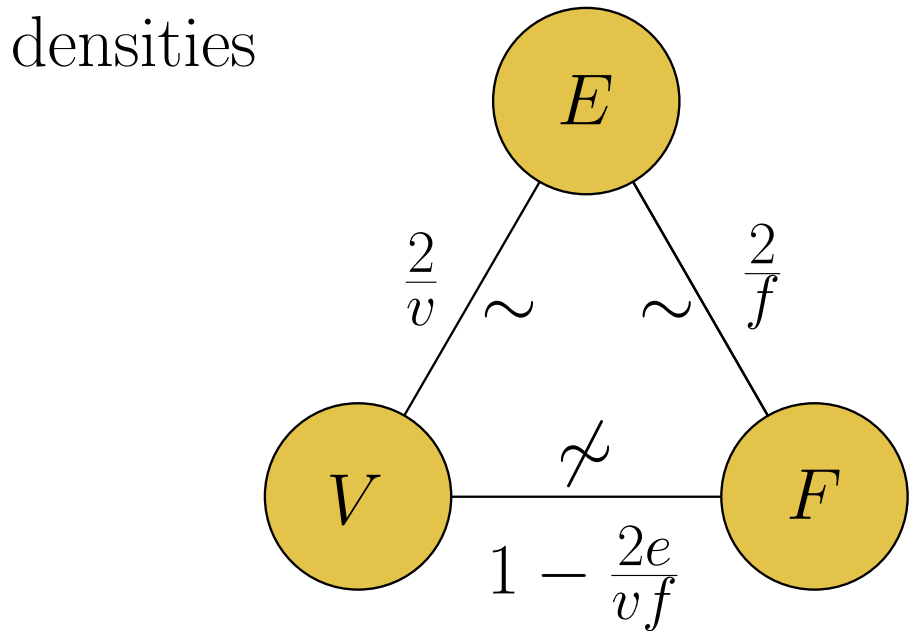
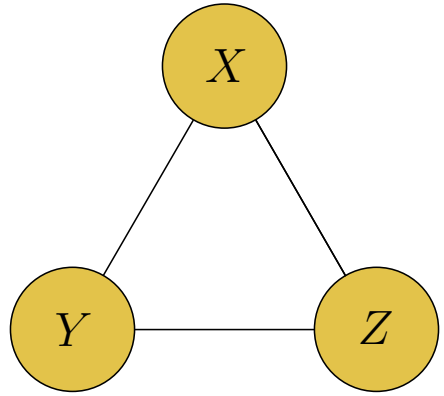


So the sum of the densities of the three bipartite graphs is

$$\frac{2}{f} + \left(1 - \frac{2e}{vf}\right) + \frac{2}{v} = 1 + \frac{2}{vf}(v - e + f)$$

and X satisfies our conditions precisely when $v - e + f > 0$

Back to Groups



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- If $G \leq Aut(\Gamma)$ acts regularly on each block, then Γ may be constructed from a very small product set in G.

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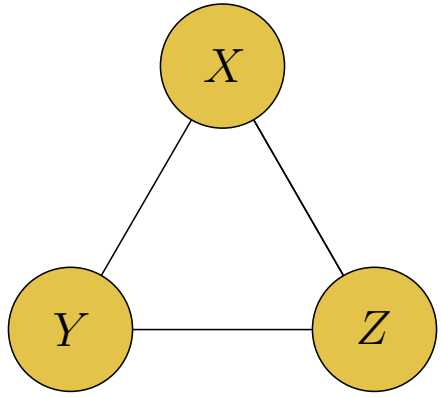

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- If $G \leq Aut(\Gamma)$ acts regularly on each block, then Γ may be constructed from a very small product set in G.
- (Sabidussi) In general we can construct Γ using a very small product set in G and then identifying clones.

Theorem and Proof

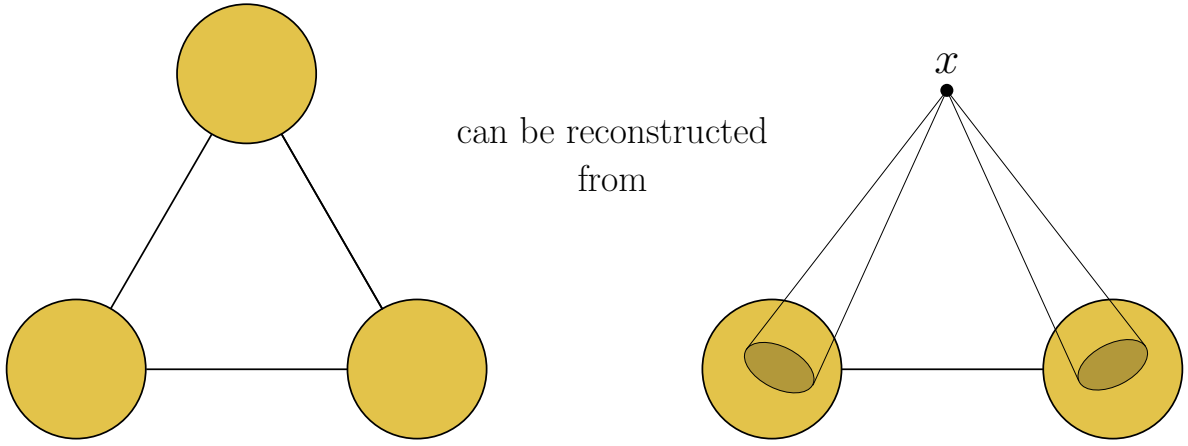
Theorem

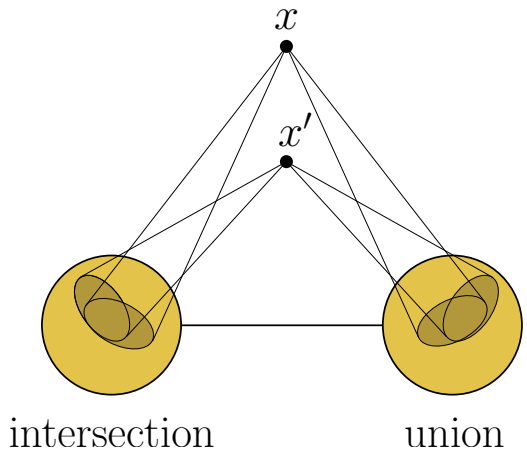
The structure theorem is extremely complex with several types of recursive structures, a few families of terminal structures, and a handful of sporadic structures based on the Platonic Solids and the regular maps on the projective plane.

Proof

The proof is very long (the paper is 147 pages!), but it relies on a delicate induction, some technical stability lemmas, and the following idea...

Theorem and Proof





The End

Thanks for your attention!