4 Graphs in Additive Number Theory

In this section we will explore the use of graphs in additive combinatorics. First we will establish an important connection between the problem of small sumsets and connectivity in vertex transitive graphs. Then we consider a problem in additive combinatorics which concerns a graph with weighted edges.

Connectivity in Vertex Transitive Graphs

In this section we will turn our attention to graphs and directed graphs, and we will see that questions concerning small product sets have natural interpretations in terms of connectivity. Before we are ready for this, we'll need to introduce some terminology.

An *automorphism* of a (di)graph $\Gamma = (V, E)$ is a bijection $\phi : V \to V$ with the property that $uv \in E$ if and only if $\phi(u)\phi(v) \in E$ ($(u, v) \in E$ if and only if $(\phi(u), \phi(v)) \in E$). We say that the (di)graph Γ is *vertex transitive* if for every $u, v \in V$ there exists an automorphism ϕ for which $\phi(u) = v$. Note that in a vertex transitive (di)graph, every vertex must have the same (out-) degree, and we call this number the *degree (outdegree)* of the graph.

The most natural construction of a vertex transitive graph is known as a *Cayley Graph*, although in general these objects are actually directed graphs. Let G be a group and let $A \subseteq G$. We define $Cayley(G, A)$ to be the digraph with vertex set G and a (directed) edge (u, v) whenever $v \in uA$ (i.e. whenever you can get from u to v by multiplying on the right by an element of A). When the set A is closed under inverses, (i.e. $(u, v) \in E$ if and only if $(v, u) \in E$), we generally treat $Cayley(G, A)$ as a graph (not digraph) with an edge uv when there were directed edges in both directions. Since we have defined our edges based on right multiplication, the group G has a natural left action on the graph. More precisely, for every $g \in G$ we have a bijection $\phi_g : G \to G$ given by the rule $\phi_g(u) = gu$. It follows immediately from our definitions that ϕ_g is an automorphism of $Cayley(G, A)$ and it then follows that every Cayley graph is indeed vertex transitive.

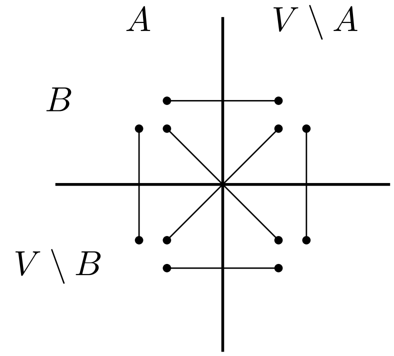
Now we return to the setting of arbitrary graphs to introduce a useful identity. Let $\Gamma = (V, E)$ be an (arbitrary) graph, and define the following function for every $A \subseteq V$

$$d(A) = |\{xy \in E \mid x \in A \text{ and } y \notin A\}|.$$

This function obeys the following useful identity[1] for every $A, B \subseteq V$

$$d(A \cap B) + d(A \cup B) \leq d(A) + d(B).$$

To see why this inequality holds, just consider the Venn diagram on the vertex set formed by the sets A and B as depicted in the figure below. The sum $d(A) + d(B)$ counts each of the horizontal and vertical edges once, and each diagonal edge twice. On the other hand, the sum $d(A \cap B) + d(A \cup B)$ counts each of the horizontal and vertical edges once, it counts one of the diagonal type edges twice, but the other type of diagonal edge makes no contribution.



We say that a graph $\Gamma = (V, E)$ is *k-edge-connected* if $\Gamma - S$ is still connected for any set $S \subseteq E$ with $|S| < k$. Equivalently, Γ is k-edge-connected if and only if $d(A) \geq k$ for every proper nonempty subset $\emptyset \subset A \subset V$. We are now ready for a classical theorem which gives the best possible bound on the edge-connectivity of a vertex transitive graph.

Theorem 4.1 (Mader) *Every finite connected simple vertex transitive graph* $\Gamma = (V, E)$ *of degree* d *is* d*-edge-connected.*

Proof: Choose a subset of vertices $\emptyset \neq A \subset V$ with the following properties.

1. $d(A)$ is minimum.

2. A is minimal, subject to 1.

Note that $d(A) = d(V \setminus A)$ so the second criteria in our choice of A implies that $|A| \leq \frac{1}{2}|V|$.

[1]Note the similarity of this identity to the equation $(A \cap B) + (A \cup B) \subseteq A + B$ for subsets A, B of a group.

Suppose (for a contradiction) that there exists an automorphism ϕ of Γ so that $\emptyset \neq A \cap \phi(A) \neq A$. Now consider the following inequality

$$d(A \cap \phi(A)) + d(A \cup \phi(A)) \leq d(A) + d(\phi(A))$$

Observe that $A \cup \phi(A) \neq V$ since A and $\phi(A)$ are intersecting sets of size at most $\frac{1}{2}|V|$. It follows that all four of the sets which the function d is applied to in the above equation are proper nonempty subsets of V. Since ϕ is an automorphism, we must have $d(\phi(A)) = d(A)$. However, property 1 in the choice of A implies that $d(A)$ is the minimum over all proper nonempty subsets of vertices. It follows that $d(A \cap \phi(A)) = d(A \cup \phi(A)) = d(A)$. But then $A \cap \phi(A)$ contradicts the choice of A for property 2.

Let $x \in A$ and suppose that x has a neighbours inside A and b neighbours in $V \setminus A$ (so $a + b = d$). Now let $y \in A$ and choose an automorphism ϕ for which $\phi(x) = y$. It follows from the above argument that $\phi(A) = A$. However, then the vertex y must also have a neighbours inside A and b neighbours outside A. Thus, every vertex in A has exactly b edges from this vertex to $V \setminus A$ and this gives us the following equation (here we use the fact that $|A| \geq a + 1$ which follows from the fact that x has a neighbours inside A).

$$d(A) = |A|b \geq (a + 1)b \geq a + b = d$$

Since A was chosen to minimize $d(A)$ we find that Γ is d-edge-connected, as desired. □

Note that a graph with a vertex of degree d cannot be $(d+1)$-edge-connected since we can isolate this vertex by removing d edges. So, Mader's Theorem completely determines the edge-connectivity of every finite vertex transitive graph.

A graph $\Gamma = (V, E)$ is *k-connected* if $|V| > k$ and $\Gamma - S$ is connected for every $S \subseteq V$ with $|S| < k$. The following (best possible) theorem gives us a lower bound for vertex connectivity of a vertex transitive graph.

Theorem 4.2 (Mader, Watkins) *If Γ is a finite connected d-regular vertex transitive graph, then Γ is $\frac{2}{3}(d+1)$-connected.*

A digraph is *strongly connected* if there is a directed path from any vertex to any other vertex. For a digraph $\Gamma = (V, E)$, we say that Γ is *strongly k-connected* if $|V| > k$ and $\Gamma - S$ is strongly connected for every $S \subseteq V$ with $|S| < k$. The following (best possible) theorem gives us an analogous lower bound on the strong connectivity of vertex transitive digraphs.

Theorem 4.3 (Hamidoune) *If Γ is a finite connected vertex transitive graph of outdegree d, then Γ is $\frac{1}{2}(d+1)$-strongly-connected.*

For a set $A \subseteq V$ define $N^+(A) = \{v \in V \setminus A \mid \text{there exists } u \in A \text{ with } (u,v) \in E\}$. It follows from basic principles that Γ is strongly k-connected if and only if every nonempty $A \subseteq V$ satisfies either $A \cup N^+(A) = V$ or $|N^+(A)| \geq k$. Based on this we can reformulate the Cauchy-Davenport theorem in terms of the strong connectivity of Cayley graphs of prime order. To do so, let p be prime, let $B \subseteq \mathbb{Z}_p$ satisfy $0 \in B$ and $|B| \geq 2$. Then define $B' = B \setminus \{0\}$ and $\Gamma = Cayley(\mathbb{Z}_p, B')$.

$$\begin{aligned}
\text{Cauchy-Davenport} &\Leftrightarrow A + B \geq \min\{p, |A| + |B| - 1\} \text{ for every } \emptyset \neq A \subseteq \mathbb{Z}_p \\
&\Leftrightarrow A \cup (A + B') = \mathbb{Z}_p \text{ or } |(A + B') \setminus A| \geq |B'| \text{ for every } \emptyset \neq A \subseteq \mathbb{Z}_p \\
&\Leftrightarrow A \cup N^+(A) = V(\Gamma) \text{ or } |N^+(A)| \geq |B'| \text{ for every } A \subseteq V(\Gamma) \\
&\Leftrightarrow \Gamma \text{ is } |B'|\text{-strongly connected.}
\end{aligned}$$

So, in short, the Cauchy-Davenport Theorem is equivalent to the statement that every loopless Cauchy graph on the group $\mathbb{Z}_p$ has strong connectivity equal to its outdegree.

The Schrijver-Seymour Theorem

Next we turn our attention to a problem in additive combinatorics where the sums we are interested in are determined by the structure of a graph.[2] Namely, we will consider a graph $\Gamma = (V, E)$ equipped with a weighting $w : E \to G$ (where G, as usual, is an abelian group). For any set of edges $S \subseteq E$ we define $w(S) = \sum_{e \in S} w(e)$. Our focus here will be on the set of all possible weights of spanning trees of G which we define as

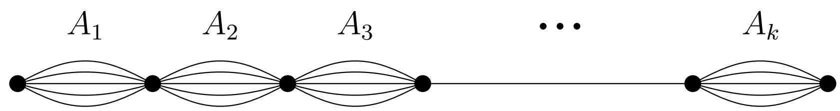
$$w(\Gamma) = \{w(E(T)) \mid T \text{ is a spanning tree of } \Gamma\}$$

This might appear to be a strange set, but in fact there are a number of nice theorems from this setting. For instance, here is a classical theorem which generalized an earlier result of Bialstocki-Dierker (who proved the special case when $G = \mathbb{Z}_p$).

Theorem 4.4 (Füredi-Kleitman) *Let G be an abelian group of order n and consider the complete graph K_{n+1}. For every weighting $w : E(K_{n+1}) \to G$ we have $0 \in w(K_{n+1})$.*

[2]In fact, everything from this subsection generalizes naturally to the broader concept of matroids.

The Schrijver-Seymour Theorem, which will be our main goal gives a natural lower bound on $w(\Gamma)$ in the case when $G = \mathbb{Z}_p$. To see why this bound is natural, let us consider a graph Γ obtained from a path of length k by adding parallel edges. We will assume that the set of weights on the edges parallel with the first edge is A_1, the second A_2, and so on to A_k (so here $A_i \subseteq \mathbb{Z}_p$).



Each spanning tree consists of one edge from each parallel class, so in this case we have $w(\Gamma) = A_1 + A_2 + \ldots + A_k$. We can repeatedly apply the Cauchy-Davenport Theorem to get a good bound on the size of $w(\Gamma)$ in this case. It tells us that this set will either be the entire group $\mathbb{Z}_p$ or will have size at least $(\sum_{i=1}^{k} |A_i|) - (k-1) = \sum_{i=1}^{k} |A_i| - (|V(\Gamma)| - 2)$. In fact, the Schrijver-Seymour Theorem will give the same lower bound in this case. However, we will need to introduce some further terminology before we state it.

The binary incidence matrix of the graph $\Gamma = (V, E)$ we shall denote by $\{R_{(v,e)}\}_{(v,e) \in V \times E}$. This matrix has $R_{(v,e)} = 1$ if the vertex v is incident with the edge e and $R_{(v,e)} = 0$ otherwise. We will regard R as a matrix over the two element field $\mathbb{F}_2$. Let C be the edge set of a cycle of G and consider the corresponding set of columns of R. An easy check reveals that these columns sum to zero, so they are linearly dependent. Next suppose that S is the edge set of a forest in G and consider the corresponding columns of R. If $e \in S$ is a leaf of the forest, then the column associated with e does not lie in the span of the columns associated with those edges in $S \setminus \{e\}$. So, an easy inductive argument shows that the columns corresponding to S are linearly independent. With this we are ready to introduce a key definition for an arbitrary set $S \subseteq E$ which we will state in a few equivalent forms. (here we let $comp(\Gamma)$ denote the number of components of the graph Γ).

$$\begin{aligned}\text{rank}(S) := &\text{ the rank of the set of columns of } R \text{ associated with } S\\ = &\text{ the size of the largest forest in } S\\ = &\ |V| - comp(V, S)\end{aligned}$$

We are finally ready to state and then prove our main result. It is worthwhile to verify that the lower bound from this theorem for the path with parallel edges graph we considered above is precisely the same as that given by (repeated applications of) the Cauchy-Davenport Theorem.

Theorem 4.5 (Schrijver-Seymour) *Let $\Gamma = (V, E)$ be a connected graph, let p be prime and let $w : E \to \mathbb{Z}_p$. Then*

$$|w(\Gamma)| \geq \min\{p, \Big(\sum_{g \in \mathbb{Z}_p} \mathrm{rank}(w^{-1}(g))\Big) - (|V| - 2)\}.$$

Proof: We proceed by induction on $|V|$. As a base case, when $|V| = 1$ the bound asserts that $|w(\Gamma)| \geq 1$ and this is true, since the trivial spanning tree has weight 0. For the inductive step we choose an edge $e = uv$ and define the following set

$$A = \{g \in \mathbb{Z}_p \mid \text{there is a path from } u \text{ to } v \text{ of edges with weight } g\}.$$

Note that A is not empty since it contains $w(e)$. Next we will consider the effect of contracting the edge e on the ranks of our sets. Let $g \in \mathbb{Z}_p$ and consider the set of edges $w^{-1}(g)$. If $g \notin A$ then there is no path from u to v of edges of weight g and we have $\mathrm{rank}_{\Gamma/e}(w^{-1}(g)) = \mathrm{rank}_{\Gamma}(w^{-1}(g))$. On the other hand, if $g \in A$ then such a path does exist and in this case $\mathrm{rank}_{\Gamma/e}(w^{-1}(g)) = \mathrm{rank}_{\Gamma}(w^{-1}(g)) - 1$. This gives us the equation

$$|A| = \Big(\sum_{g \in \mathbb{Z}_p} \mathrm{rank}_{\Gamma}(w^{-1}(g))\Big) - \Big(\sum_{g \in \mathbb{Z}_p} \mathrm{rank}_{\Gamma/\mathrm{e}}(w^{-1}(g))\Big).$$

Now let us turn our attention to the weights of spanning trees. Let T be the edge set of a spanning tree in Γ/e. Then in the original graph Γ, the subgraph (V, T) has two components, each containing one of u, v. For any $g \in A$ there is a path from u to v of edges with weight g, and one of these edges may be added to T to form a spanning tree (in Γ). It follows from this that $w(T) + A \subseteq w(\Gamma)$. More generally we have $w(\Gamma) \supseteq w(\Gamma/e) + A$. This together with Cauchy-Davenport and induction then yields

$$\begin{aligned}
|w(\Gamma)| &\geq |w(\Gamma/e) + A| \\
&\geq \min\{p, |w(\Gamma/e)| + |A| - 1\} \\
&\geq \min\{p, \Big(\sum_{g \in \mathbb{Z}_p} \mathrm{rank}_{\Gamma/\mathrm{e}}(w^{-1}(g))\Big) - (|V| - 3) + |A| - 1\} \\
&= \min\{p, \Big(\sum_{g \in \mathbb{Z}_p} \mathrm{rank}_{\Gamma}(w^{-1}(g))\Big) - (|V| - 2)\} \qquad \square
\end{aligned}$$