GRAPHS OF SMALL RANK-WIDTH ARE PIVOT-MINORS OF GRAPHS OF SMALL TREE-WIDTH

O-JOUNG KWON AND SANG-IL OUM

Abstract. We prove that every graph of rank-width \( k \) is a pivot-minor of a graph of tree-width at most \( 2k \). We also prove that graphs of rank-width at most 1, equivalently distance-hereditary graphs, are exactly vertex-minors of trees, and graphs of linear rank-width at most 1 are precisely vertex-minors of paths. In addition, we show that bipartite graphs of rank-width at most 1 are exactly pivot-minors of trees and bipartite graphs of linear rank-width at most 1 are precisely pivot-minors of paths.

1. Introduction

Rank-width is a width parameter of graphs, introduced by Oum and Seymour [6], measuring how easy it is to decompose a graph into a tree-like structure where the “easiness” is measured in terms of the matrix rank function derived from edges formed by vertex partitions. Rank-width is a generalization of another, more well-known width parameter called tree-width, introduced by Robertson and Seymour [8]. It is well known that every graph of small tree-width also has small rank-width; Oum [7] showed that if a graph has tree-width \( k \), then its rank-width is at most \( k + 1 \). The converse does not hold in general, as complete graphs have rank-width 1 and arbitrary large tree-width.

Pivot-minor and vertex-minor relations are graph containment relations such that rank-width cannot increase when taking pivot-minors or vertex-minors of a graph [6]. Our main result is that for every graph \( G \) with rank-width at most \( k \) and \( |V(G)| \geq 3 \), there exists a graph \( H \) having \( G \) as a pivot-minor such that \( H \) has tree-width at most \( 2k \) and \( |V(H)| \leq (2k + 1)|V(G)| - 6k \). Furthermore, we prove that for every graph \( G \) with linear rank-width at most \( k \) and \( |V(G)| \geq 3 \), there exists a graph \( H \) having \( G \) as a pivot-minor such that \( H \) has path-width at most \( k + 1 \) and \( |V(H)| \leq (2k + 1)|V(G)| - 6k \).

As a corollary, we give new characterizations of two graph classes: graphs with rank-width at most 1 and graphs with linear rank-width at most 1. We show that a graph has rank-width at most 1 if and only if it is a vertex-minor of a tree. We also prove that a graph has linear rank-width at most 1 if and only if it is a vertex-minor of a path. Moreover, if the graph is bipartite, we prove that a vertex-minor relation can be replaced with a pivot-minor relation in both theorems. Table 1 summarizes our theorems.

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Table 1. Summary of theorems

<table>
<thead>
<tr>
<th>Condition</th>
<th>Implication</th>
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<tbody>
<tr>
<td>$G$ has rank-width $\leq k$</td>
<td>$G$ is a pivot-minor of a graph of tree-width $\leq 2k$</td>
</tr>
<tr>
<td>$G$ has linear rank-width $\leq k$</td>
<td>$G$ is a pivot-minor of a graph of path-width $\leq k + 1$</td>
</tr>
<tr>
<td>$G$ has rank-width $\leq 1$</td>
<td>$G$ is a vertex-minor of a tree</td>
</tr>
<tr>
<td>$G$ has linear rank-width $\leq 1$</td>
<td>$G$ is a vertex-minor of a path</td>
</tr>
<tr>
<td>$G$ is bipartite and has rank-width $\leq 1$</td>
<td>$G$ is a pivot-minor of a tree</td>
</tr>
<tr>
<td>$G$ is bipartite has linear rank-width $\leq 1$</td>
<td>$G$ is a pivot-minor of a path</td>
</tr>
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</table>

To prove the main theorem, we construct a graph having $G$ as a pivot-minor, called a rank-expansion. Then we prove that a rank-expansion has small tree-width.

The paper is organized as follows. We present the definition of rank-width and related operations in the next section. In Section 3, we define a rank-expansion of a graph and prove the main theorem. In Section 4, using a rank-expansion, we present new characterizations of graphs with rank-width at most 1 and graphs with linear rank-width at most 1.

2. Preliminaries

In this paper, all graphs are simple and undirected. Let $G = (V, E)$ be a graph. For $v \in V$, let $N(v)$ be the set of vertices adjacent to $v$ and $\deg(v) := |N(v)|$. And let $\delta(v)$ be the set of edges incident with $v$. For $S \subseteq V$, $G[S]$ denotes the subgraph of $G$ induced on $S$. For two sets $A$ and $B$, $A \Delta B = (A \cup B) \setminus (A \cap B)$.

A vertex partition of a graph $G$ is a pair $(A, B)$ of subsets of $V$ such that $A \cup B = V$ and $A \cap B = \emptyset$. A vertex $v \in V$ is a leaf if $\deg(v) = 1$; Otherwise we call it an inner vertex. An edge $e \in E$ is an inner edge if $e$ does not have a leaf as an end. Let $V_I(G)$ and $E_I(G)$ be the set of inner vertices of $G$ and inner edges of $G$, respectively.

For an $X \times Y$ matrix $M$ and subsets $A \subseteq X$ and $B \subseteq Y$, $M[A, B]$ denotes the $A \times B$ submatrix $(m_{i,j})_{i \in A, j \in B}$ of $M$. If $A = B$, then $M[A] = M[A, A]$ is called a principal submatrix of $M$. The adjacency matrix of a graph $G$, which is a $(0, 1)$-matrix over the binary field, will be denoted by $A(G)$.

Pivoting matrices. Let $M = \begin{pmatrix} X & V \backslash X \\ V \backslash X & A \ B \ C \ D \end{pmatrix}$ be a symmetric or skew-symmetric $V \times V$ matrix over a field $F$. If $A = M[X]$ is nonsingular, then we define

$$M * X = \begin{pmatrix} X & V \backslash X \\ V \backslash X & A^{-1} \ B \ -CA^{-1} \ D - CA^{-1}B \end{pmatrix}.$$ 

This operation is called a pivot. Tucker showed the following theorem.

Theorem 2.1 (Tucker [9]). Let $M[X]$ be a nonsingular principal submatrix of a square matrix $M$. Then $M * X[Y]$ is nonsingular if and only if $M[X \Delta Y]$ is nonsingular.
Vertex-minors and pivot-minors. The graph obtained from $G = (V, E)$ by applying local complementation at a vertex $v$ is $G* v = (V, E \Delta \{xy : xv, yv \in E, x \neq y\})$. The graph obtained from $G$ by pivoting an edge $uv$ is defined by $G \land uv = G * u * v * u$.

To see how we obtain the resulting graph by pivoting an edge $uv$, let $V_1 = N(u) \cap N(v)$, $V_2 = N(u) \setminus \{v\}$ and $V_3 = N(v) \setminus \{u\}$. One can easily verify that $G \land uv$ is identical to the graph obtained from $G$ by complementing adjacency of vertices between distinct sets $V_i$ and $V_j$ and swapping the vertices $u$ and $v$ [6]. See Figure 1 for example.

In fact, if $uv \in E$, then $A(G \land uv) = A(G) * \{u, v\}$. Since $\det (A(G)[\{u, v\}]) = A(G)[u, v]$. Theorem 2.1 is useful for dealing with a sequence of pivoting. In Figure 1 we can easily check that $G \land uv \land uc = G \land vc$. For $X \subseteq V$, if $A(G)[X]$ is nonsingular, then we denote $G \land X$ as the graph having the adjacency matrix $A(G) * X$.

A graph $H$ is a vertex-minor of $G$ if $H$ can be obtained from $G$ by applying a sequence of vertex deletions and local complementations. A graph $H$ is a pivot-minor of $G$ if $H$ can be obtained from $G$ by applying a sequence of vertex deletions and pivoting edges. From the definition, every pivot-minor of a graph is a vertex-minor of the graph. Note that every pivot-minor of a bipartite graph is bipartite.

Rank-width and linear rank-width. The cut-rank function $\text{cutrk}_G : 2^V \to \mathbb{Z}$ of a graph $G$ is defined by

$$\text{cutrk}_G(X) = \text{rank}(A(G)[X, V \setminus X]).$$

A tree is subcubic if it has at least two vertices and every inner vertex has degree 3. A rank-decomposition of a graph $G$ is a pair $(T, L)$, where $T$ is a subcubic tree and $L$ is a bijection from the vertices of $G$ to the leaves of $T$. For an edge $e$ in $T$, $T \setminus e$ induces a partition $(X_e, Y_e)$ of the leaves of $T$. The width of an edge $e$ is defined as $\text{cutrk}_G(L^{-1}(X_e))$. The width of a rank-decomposition $(T, L)$ is the maximum width over all edges of $T$. The rank-width of $G$, denoted by $\text{rw}(G)$, is the minimum width of all rank-decompositions of $G$. If $|V| \leq 1$, then $G$ admits no rank-decomposition and $\text{rw}(G) = 0$.

A subcubic tree is a caterpillar if it contains a path $P$ such that every vertex of a tree has distance at most 1 to some vertex of $P$. A linear rank-decomposition of a graph $G$ is a rank-decomposition $(T, L)$ of $G$, where $T$ is a caterpillar. The linear rank-width of $G$ is defined as the minimum width of all linear rank-decompositions of $G$. If $|V| \leq 1$, then $G$ admits no linear rank-decomposition and $\text{lrw}(G) = 0$. 

Figure 1. Pivoting an edge $uv$. Note that $G \land uv \land uc = G \land vc$. 

Graphs of small rank-width are pivot-minors. 

\[ \begin{align*} 
G & \quad \quad G \land uv \quad \quad G \land uv \land uc
\end{align*} \]
Note that if a graph $H$ is a vertex-minor or a pivot-minor of a graph $G$, then $\text{rw}(H) \leq \text{rw}(G)$ and $\text{lrw}(H) \leq \text{lrw}(G)$ [6]. Trivially, $\text{rw}(G) \leq \text{lrw}(G)$.

**Tree-width and path-width.** Let $T$ be a tree, and let $B = \{B_t\}_{t \in V(T)}$ be a family of vertex sets $B_t \subseteq V$ indexed by the vertices $t \in V(T)$, called *bags*. The pair $(T, B)$ is called a tree-decomposition of $G$ if it satisfies the following three conditions.

$(T1)$ $V = \bigcup_{v \in V(T)} B_v$.
$(T2)$ For every edge $uv \in E$, there exists a vertex $t$ of $T$ such that $u, v \in B_t$.
$(T3)$ For $t_1, t_2$ and $t_3 \in V(T)$, $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever $t_2$ is on the path from $t_1$ to $t_3$.

The *width* of a tree-decomposition $(T, B)$ is $\max\{|B_t| : t \in V(T)\}$. The tree-width of $G$, denoted by $\text{tw}(G)$, is the minimum width of all tree-decompositions of $G$. A path-decomposition of a graph $G$ is a tree-decomposition $(T, B)$ where $T$ is a path. The path-width of $G$, denoted by $\text{pw}(G)$, is the minimum width of all path-decompositions of $G$.

### 3. Rank-expansions and pivot-minors of graphs with small tree-width

In this section, for a graph $G$ with rank-width $k$, we construct a graph having tree-width at most $2k$ such that it has $G$ as a pivot-minor.

**Theorem 3.1.** Let $k$ be a non-negative integer. Let $G$ be a graph of rank-width at most $k$ and $|V(G)| \geq 3$. Then there exists a graph $H$ having a pivot-minor isomorphic to $G$ such that tree-width of $H$ is at most $2k$ and $|V(H)| \leq (2k + 1)|V(G)| - 6k$.

**Theorem 3.2.** Let $k$ be a non-negative integer. Let $G$ be a graph of linear rank-width at most $k$ and $|V(G)| \geq 3$. Then there exists a graph $H$ having a pivot-minor isomorphic to $G$ such that path-width of $H$ is at most $k + 1$ and $|V(H)| \leq (2k + 1)|V(G)| - 6k$.

We need the following lemma.

**Lemma 3.3.** Let $G$ be a graph and $(A_1, B_1), (A_2, B_2)$ be two vertex partitions such that $A_2 \subseteq A_1$. Let $S \subseteq A_1$ be a set corresponding to a basis of row vectors in $A(G)[A_1, B_1]$. Then there exists a subset of $A_2$ representing a basis of row vectors in $A(G)[A_2, B_2]$ containing $S \cap A_2$.

**Proof.** Because $A_2 \subseteq A_1$, rows in $A(G)[S \cap A_2, B_2]$ are independent. Therefore we can extend $S \cap A_2$ to a basis of rows in $A(G)[A_2, B_2]$. $\square$

To prove Theorems 3.1 and 3.2, we construct a rank-expansion of a graph. Let $G$ be a connected graph and $(T, L)$ be a rank-decomposition of $G$. We fix a leaf $x \in V(T)$. For $e \in E(T)$, let $T_e$ be the component of $T \setminus e$ which does not contain $x$, and let $A_e = L^{-1}(V(T_e))$, $B_e = V(G) \setminus A_e$ and $M_e = A(G)[A_e, B_e]$. For each $a \in A_e$, let $R^a_e = M_e[\{a\}, B_e]$ the row vector of $M_e$.

First, for each edge $e = uv \in E(T)$, we orient the edge towards $v$ if $v \in V(T_e)$. We choose a vertex set $U_e \subseteq A_e$ such that $\{R^a_e\}_{a \in U_e}$ forms a basis of row vectors in $M_e$ and $(U_e \cap A_f) \subseteq U_f$ if the tail of an edge $f$ is the head of $e$. Since $R^a_e$ can be uniquely expressed as a linear combination of vectors of $\{R^a_e\}_{a \in U_e}$ for each $a \in A_e$, there exists a unique $A_e \times U_e$ matrix $P_e$ such that $P_eA(G)[U_e, B_e] = A(G)[A_e, B_e]$.

If the tail of an edge $f$ is the head of an edge $e$, then let $C_f = P_e[U_f, U_e]$. 


Let $H$ be a rank-expansion $R(G, T, L, x, \{U_f\}_{f \in E(T)})$ of a graph $G$ such that

$V(H) = \bigcup_{v \in V_I} \bigcup_{e \in \delta(v)} (U_e \times \{e\} \times \{v\})$

$E(H) = \{(a, e, v), (a, e, w) : e = vw \in E_I(T), a \in U_e\}$

$\cup \{(a, e, v), (b, f, v) : v \in V_I(T), e, f \in E(T), v \text{ is the head of } e \text{ and the tail of } f,$

$a \in U_f, b \in U_e \text{ and } C_f(a, b) \neq 0\}$

$\cup \{(a, f_1, v), (b, f_2, v) : v \text{ is the tail of both } f_1 \text{ and } f_2 \in E(T),

a \in U_{f_1}, b \in U_{f_2} \text{ and } ab \in E(G)\}.$

For $v \in V_I(T)$, let $S_v = \bigcup_{e \in \delta(v)} U_e \times \{e\} \times \{v\} \subseteq V(H)$. For $e = vw \in E_I(T)$, let $\overline{\pi} = \{(a, e, v), (a, e, w) : a \in U_e\} \subseteq V(H)$. For $W \subseteq E_I(T)$, let $\overline{W} = \bigcup_{f \in W} \overline{f} \subseteq V(H)$. If $e \in E_I(T)$ is directed from $w$ to $v$, let $L_e = S_w \cap \overline{\pi}$ and $R_e = S_w \cap \overline{\pi}$. For a vertex $a$ in $V(G)$ and $e = \{L(a), v\} \in E(T)$, let $\overline{\pi}$ be the unique vertex in $U_e \times \{e\} \times \{v\}$ and let $\overline{\pi} = \overline{\pi}$.

We discuss the number of vertices in the rank-expansion $H$. We easily observe that $|E_I(T)| = |V(G)| - 3$. So if $rw(G) \leq k$, then $|\overline{\pi}| \leq 2k$ for each $e \in E_I(T)$, and we deduce that $|V(H)| \leq 2k|E_I(T)| + |V(G)| = 2k(|V(G)| - 3) + |V(G)| = (2k + 1)|V(G)| - 6k$.

First, we prove that every rank-expansion of a graph has the given graph as a pivot-minor. To obtain $G$ as a pivot-minor of $H$, we will pivot $\bigcup_{v \in E_I(T)} \overline{\pi}$ to $H$.

Lemma 3.4. Let $G$ be a graph and $uv \in E(G)$. If $\deg(u) = 1$, then $G \setminus uv \setminus \{u, v\} = G \setminus \{u, v\}$.

Proof. It is clear from the definition. \hfill \qed

For convenience, let $\det(A(H)_0) = 1$.

Lemma 3.5. Let $W \subseteq E_I(T)$. Then $A(H)|_{\overline{W}}$ is nonsingular.

Proof. We proceed by induction on $|W|$. If $W$ is empty, then it is trivial. If $|W| \geq 1$, then $W$ induces a forest in $T$, and therefore there must be an edge $f \in W$ which has a leaf in $T[W]$. By induction hypothesis, $A(H)|_{\overline{W} \setminus \{f\}}$ is nonsingular. Since
every edge in $H[\bar{f}]$ is incident with a leaf in $H[\bar{W}]$, by Lemma 3.4 pivoting all edges in $\bar{f}$ does not change the graph $H[\bar{W} \setminus \{f\}]$. So, $A(H[\bar{W} \setminus \{f\}] = A(H)[\bar{W} \setminus \{f\}]$ and therefore, by Theorem 2.1 $A(H)[\bar{W}] = A(H)[\bar{W} \setminus \{f\}] = A(H)[\bar{W}]$ is nonsingular.

**Lemma 3.6.** Let $a, b \in V(G)$ and let $P$ be a path from $L(a)$ to $L(b)$ in $T$. Then for $E(P) \cap E(T) \subseteq W \subseteq E(T)$, $A(H)[W \cup \{a, b\}]$ is nonsingular if and only if $A(H)[W, P]$ is nonsingular.

**Proof.** We use induction on $|W|$. If $W = E(P) \cap E(T)$, then it is trivial, because $E(P) = E(P)$. So we may assume that $|W| > |E(P) \cap E(T)|$. Since $P$ is a maximal path in $T$, the subgraph of $T$ having the edge set $W \cup E(P)$ must have at least 3 leaves. Thus there is an edge $f$ in $W \cup E(P)$ incident with a leaf in $T[W \cup E(P)]$ other than $L(a)$ and $L(b)$. Since every edge in $\bar{f}$ is incident with a leaf in $H[\bar{W}]$, by Lemma 3.4 $A(H)[W \cup \{\pi, \beta\} \setminus \bar{f}] = A(H)[W \setminus \{f\}] = A(H)[W \setminus \{f\} \cup \{\pi, \beta\}]$. By induction hypothesis and Theorem 2.1 we deduce that

$$A(H)[W \setminus \{f\} \cup \{\pi, \beta\}]$$

is nonsingular.

$$A(H)[W \setminus \{f\} \cup \{\pi, \beta\}]$$

is nonsingular if and only if $A(H)[W, P]$ is nonsingular.

$$A(H)[W \cup \{a, b\}]$$

is nonsingular.

**Lemma 3.7.** Let $P = (e_{n+1}, e_n, \ldots, e_0)$ be the directed path from $w$ to $v$ in $T$. Then $C_{e_1} C_{e_2} \cdots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}].$

**Proof.** We proceed by induction on $n$. If $n = 1$, then by definition, $C_{e_1} A(G)[U_{e_2}, B_{e_2}] = P_{e_2}[U_{e_1}, U_{e_2}] A(G)[U_{e_1}, B_{e_2}] = A(G)[U_{e_1}, B_{e_2}]$. We may assume that $n \geq 2$. By induction hypothesis, $C_{e_1} C_{e_2} \cdots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = A(G)[U_{e_2}, B_{e_{n+1}}]$. Since $C_{e_1} A(G)[U_{e_2}, B_{e_2}] = A(G)[U_{e_1}, B_{e_2}]$ and $B_{e_{n+1}} \subseteq B_{e_2}$, $C_{e_1} A(G)[U_{e_2}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}]$. Therefore, we conclude that $C_{e_1} C_{e_2} \cdots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = C_{e_1} A(G)[U_{e_2}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}].$
Lemma 3.8.
\[
\begin{bmatrix}
0 & C_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_n \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I
\end{bmatrix}
= (-1)^n \det(C_1 C_2 \cdots C_{n+1}).
\]

Proof. By elementary row operation,
\[
\begin{bmatrix}
0 & C_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_n \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I
\end{bmatrix}
= \det
\begin{bmatrix}
0 & 0 & -C_1 C_2 & 0 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_n \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I
\end{bmatrix}
= \det
\begin{bmatrix}
0 & 0 & 0 & (-1)^2 C_1 C_2 C_3 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_n \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I
\end{bmatrix}
= \det
\begin{bmatrix}
(-1)^n C_1 C_2 \cdots C_{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_n \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I
\end{bmatrix}
= (-1)^n \det(C_1 C_2 \cdots C_{n+1}). \qed
Proposition 3.9. Let \( k \geq 1 \). Let \( G \) be a connected graph with rank-width \( k \) and \( |V(G)| \geq 3 \). Then a rank-expansion of \( G \) has a pivot-minor isomorphic to \( G \).

Proof. Let \((T, L)\) be a rank decomposition of a graph \( G \) and let \( x \) be a leaf in \( T \). We orient each edge \( f \) away from \( x \). For each \( f \in E(T) \), if \( m \) is the width of \( f \), we choose a basis \( U_f = \{ u_f^1, u_f^2, \ldots, u_f^m \} \subseteq A_f \) of rows in the matrix \( A(G)[A_f, B_f] \) such that \((U_e \cap A_f) \subseteq U_f\) if the head of an edge \( e \) is the tail of \( f \). Since \( G \) is connected, \(|U_f| \geq 1\). Let \( H \) be a rank-expansion \( R(G, T, L, x, \{U_f\}_{f \in E(T)}) \) of a graph \( G \). By Lemma 3.4, for every \( W \subseteq E_I(T), \ A(H)[W] \) is nonsingular. We will prove that for \( a, b \in V(G), \ \bar{a} \bar{b} \in E(H \cap E_I(T)) \) if and only if \( ab \in E(G) \).

Let \( a, b \) be distinct vertices in \( V(G) \). We consider the path \( P \) from \( L(a) \) to \( L(b) \) in \( T \). By Lemma 3.6, \( \pi \) is adjacent to \( b \) in \( H \cap E_I(T) \) if and only if \( \pi \) is adjacent to \( b \) in \( H[E(P)] \cap (E(P) \cap E_I(T)) \). Then, by Theorem 2.1

\[
\bar{a} \bar{b} \in E(H \cap E_I(T)) \iff \bar{a} \bar{b} \in E(H[E(P)] \cap (E(P) \cap E_I(T)))
\]

\[
\iff A \left( H[E(P)] \cap (E(P) \cap E_I(T)) \right) \{\pi, \bar{b}\} \text{ is nonsingular}
\]

\[
\iff A \left( H[E(P)] \right) \{\pi, \bar{b}\} \Delta \{\pi, \bar{b}\} \text{ is nonsingular}
\]

\[
\iff A(H[E(P)]) \text{ is nonsingular.}
\]

Thus, it is enough to show that \( \text{det}(A(H[E(P)]) = A(G)(a, b) \).

If \( L(b) = x \), then \( P = (e_n+1, e_n, \ldots, e_1, e_0) \) is a directed path from \( L(b) \) to \( L(a) \). The submatrix of \( A(H) \) induced by \( E(P) \) is

\[
\begin{pmatrix}
\bar{b} & L_{e_1} & L_{e_2} & \cdots & L_{e_{n-1}} & L_{e_n} & \pi & R_{e_1} & R_{e_2} & \cdots & R_{e_{n-1}} & R_{e_n} \\
0 & C_{e_0} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_{e_1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & C_{e_0} & I & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & C_{e_1} & I & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & I & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & I & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & C_{e_{n-1}} & I \\
\end{pmatrix}
\]

\[
= \left( \begin{array}{c|c}
C & 0 \\
\hline
0 & C^{t} \\
\end{array} \right).
\]

Note that \( \text{det}(A(H)[E(P)]) = \text{det}(C) \text{det}(C^{t}) = \text{det}(C)^{2} \). By Lemma 3.8, \( \text{det}(C) = (-1)^{n} \text{det}(C_{e_0} C_{e_1} \ldots C_{e_n}) \). Since \( |U_{e_{n+1}}| = |B_{e_{n+1}}| = 1 \) and \( \text{rank}(A(G)[U_{e}, B_{e}]) = \)
where $L$ is the length of $a$. Let $R = (e_{n-1}, \ldots, e_0)$ be the edges of $P$ from $y$ to $L(a)$ and $P_\ast = (f_{m-1}, \ldots, f_0)$ be the edges of $P$ from $y$ to $L(b)$.

Therefore $\det(A(H)(E(P))) = A(G)(a, b)$, as required.

Now we assume that $L(a) \neq x$ and $L(b) \neq x$. Then there exists a vertex $y$ in $V(P)$ such that it has a shortest distance to $x$. Let $P_1 = (e_n, e_{n-1}, \ldots, e_0)$ be the edges of $P$ from $y$ to $L(a)$ and $P_2 = (f_0, f_{m-1}, \ldots, f_0)$ be the edges of $P$ from $y$ to $L(b)$.

Let $M = A(H)[R_{e_n}, R_{f_m}]$. By the construction of a rank-expansion, $M = A(G)[U_{e_n}, U_{f_m}]$. The submatrix of $A(H)$ induced by $E(P)$ is

$$
\begin{pmatrix}
\{\bar{b}\} \cup \bigcup_{i=1}^{n} R_{e_i} \cup \bigcup_{i=1}^{m} L_{f_i} \\
\{\bar{b}\} \cup \bigcup_{i=1}^{n} L_{e_i} \cup \bigcup_{i=1}^{m} R_{f_i} \\
0 & C \\
\end{pmatrix}
$$

where $C$ is

$$
\begin{pmatrix}
\bar{b} & L_{e_1} & L_{e_2} & \cdots & L_{e_{n-1}} & L_{e_n} & R_{f_m} & R_{f_{m-1}} & \cdots & R_{f_1} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & I & C_{e_1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & I & C_{e_{n-1}} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & I & M & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & I & C_{f_{m-1}} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I \\
C_{f_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I \\
\end{pmatrix}
$$

It is enough to show that $C_{e_0}C_{e_1} \cdots C_{e_{n-1}}MC_{f_{m-1}} \cdots C_{f_0} = A(G)(a, b)$.

Since $M = A(G)[U_{e_n}, U_{f_m}] \leq A(G)[U_{e_n}, B_{e_n}]$, by Lemma 3.7 we have

$$
C_{e_0}C_{e_1} \cdots C_{e_{n-1}}MC_{f_{m-1}} \cdots C_{f_0} = C_{e_0}C_{e_1} \cdots C_{e_{n-1}}A(G)[U_{e_n}, U_{f_m}]C_{f_m-1} \cdots C_{f_0} = A(G)[U_{e_n}, U_{f_m}]C_{f_m-1} \cdots C_{f_0} = (C_{f_0}C_{f_1} \cdots C_{f_m-1}(A(G)[U_{f_m}, U_{e_n}]))^{t} = A(G)[U_{f_0}, U_{e_0}^{t}] = A(G)(a, b).
$$

So, $\det(A(H)(E(P))) = A(G)(a, b)$, as claimed. Therefore, $\bar{b} \in E(H \cap E(T))$ if and only if $ab \in E(G)$. We conclude that a rank-expansion of $G$ has a pivot-minor isomorphic to $G$.

In the next proposition, we show that a rank-expansion has tree-width at most $2k$ when $\text{rw}(G) \leq k$. 


\[
\begin{align*}
U_v &= \{a_4, a_5, a_7\} \\
U_{f_1} &= \{a_4, a_5\} \\
U_{f_2} &= \{a_6, a_7\}
\end{align*}
\]

Figure 4. A rank-expansion of the graph \(G\) in Figure 2. By the construction of a rank-expansion, every vertex in \(L_e\) has exactly one neighbor in \(R_{f_1} \cup R_{f_1} \setminus \{(a_6, f_2, v)\}\) in the subgraph \(H[S_e]\).

**Proposition 3.10.** Let \(k \geq 1\). Let \(G\) be a connected graph with \(|V(G)| \geq 3\). If \(G\) has rank-width \(k\), then \(G\) has a rank-expansion of tree-width at most \(2k\). Moreover, if \(G\) has linear rank-width \(k\), then \(G\) has a rank-expansion of path-width at most \(k + 1\).

**Proof.** Let \((T, L)\) be a rank-decomposition of \(G\) of width \(k\). We fix a leaf \(x \in V(T)\) and orient each edge \(f\) away from \(x\). For each \(f \in E(T)\), if \(m\) is the width of \(f\), we choose a basis \(U_f = \{u_{f1}^1, u_{f1}^2, \ldots, u_{fm}^m\} \subseteq A_f\) of rows in the matrix \(A(G)[A_f, B_f]\) such that \((U_e \cap A_f) \subseteq U_f\) if the head of an edge \(e\) is the tail of \(f\). Since \(G\) is connected, \(|U_f| \geq 1\). Let \(H\) be a rank-expansion \(R(G, T, L, x, \{U_f\}_{f \in E(T)})\) of a graph \(G\).

Let \(T'\) be a tree obtained from \(T[V_f(T)]\) by replacing each edge from \(v\) to \(w\) with a path \(wz_y^1z_y^2 \ldots z_y^{U_e}z_y^{U_e}p_y^{w}p_y^{a} \ldots p_y^{u_{f1}^2}v\). Let \(y\) be the neighbor of \(x\) in \(T\) and let \(B(y) = S_y\). For \(v \in V_f(T) \setminus \{y\}\), let \(e = vw\) be the edge incoming to \(v\) and \(f_1, f_2\) be edges outgoing from \(v\). Let \(R_v = \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a \notin U_e\}\). Since \((U_e \cap A_f) \subseteq U_f\) for each \(i \in \{1, 2\}\), each vertex in \(L_e\) has exactly one neighbor in \(R_{f_1} \cup R_{f_2} \setminus R_v\). Let \(B(v) = R_{f_1} \cup R_{f_2}\) and \(B(z_y^{U_e}) = R_{v} \cup \{(u_{f1}^1, e, v)\}\). \(B(p_y^{a}) = R_v \cup L_v \cup \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a = u_{f1}^1\}\). And for each \(2 \leq i \leq |U_e|\), we define

\[
\begin{align*}
B(z_y^{U_e}) &= B(z_y^{U_e-1}) \setminus \{(u_{f1}^{i-1}, e, v)\} \cup \{(u_{f1}^{i}, e, v)\} \\
B(p_y^{a}) &= B(p_y^{a-1}) \setminus \{(u_{f1}^{i-1}, e, v)\} \cup \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a = u_{f1}^i\}.
\end{align*}
\]

Now we show that the pair \((T', \{B(v)\}_{v \in V(T')}\) is a tree-decomposition of \(H\). Note that for each \(v \in V_f(T) \setminus \{y\}\) with the incoming edge \(e\), \(\bigcup_{v} E(H[B(z_y)]) = E(H[V_t])\) and \(\bigcup_{v} E(H[B(p_y^{a})]) = E(H[S_v])\). Therefore all vertices and all edges in \(H\) are covered by \(B(v)\) for some \(v \in V(T')\). So the first and second axioms of a tree-decomposition are satisfied.

For the third axiom, it suffices to show that for every \(t \in V(H)\), \(T'[[z : B(z) \ni t]]\) is a subtree of \(T'\). Let \(t = (u_{f1}^1, e, v) \in V(H)\) for some \(e = vw \in E(T)\) and \(1 \leq j \leq |U_e|\). If \(v\) is the head of \(e\), \(T'[[z : B(z) \ni t]] = T'[[z_y^{U_e} \ldots z_y^{U_e}, p_y^{a} \ldots p_y^{u_{f1}^j}]]\), and it forms a path. Suppose \(v\) is the tail of \(e\). Let \(f\) be the edge incoming to \(v\), and if \(a \in U_f\), then let \(h\) be the integer such that \(a = u_{f1}^h\), if otherwise, let \(h = 1\). Then \(T'[\{z : B(z) \ni t]\} = T'[\{p_y^{a} \ldots p_y^{a}, v, z_y^{U_e}, \ldots, z_y^{U_e}\}]\). It also forms a path, thus \((T', \{B(v)\}_{v \in V(T')}\) is a tree-decomposition of \(H\).
Since $|B(y)| \leq 2k + 1$ and for each $v \in V_1(T) \setminus \{y\}$ with the incoming edge $e$, $|B(z_i^y)| = |B(z_i^v)| = |R_e| + 1 \leq k + 1$, $|B(p_i^v)| = |B(p_i^y)| = |R'_e| + |L_e| + 1 \leq (2k - |U_e|) + |U_e| + 1 = 2k + 1$ and $|B(v)| \leq 2k$, the resulting tree-decomposition has width at most $2k$.

Suppose that $G$ has linear rank-width at most $k$. Here, we choose $x \in V(T)$ such that $x$ is an end of a longest path in $T$, and let $y$ be the neighbor of $x$. For $v \in V_1(T)$ with outgoing edges $f_1$ and $f_2$, $|U_{f_1}| = 1$ or $|U_{f_2}| = 1$ because every inner vertex of $T$ is incident with a leaf. Therefore, for each $v \in V_1(T) \setminus \{y\}$ and $1 \leq i \leq |U_e|$, $|B(p_i^v)| \leq (k + 1 - |U_e|) + |U_e| + 1 = k + 2$ and $|B(v)| \leq k + 1$, and $|B(y)| \leq k + 2$. Moreover, since $T[V_1(T)]$ is a path, $T'$ is also a path. Therefore $(T', \{B(v)\}_{v \in V(T')})$ is a path-decomposition of $H$ with path-width at most $k + 1$.

**Proof of Theorem 3.11.** If $k = 0$, then it is trivial. We assume that $k \geq 1$. We proceed by induction on the number of vertices.

Suppose $G$ is connected. Since $G$ has rank-width at most $k$ and $|V(G)| \geq 3$, by Proposition 3.10, there is a rank-expansion $H$ of $G$ such that $\text{tw}(H) \leq 2k$, and $|V(H)| \leq (2k + 1)|V(G)| - 6k$. By Proposition 3.9, $H$ has a pivot-minor isomorphic to $G$.

If $G$ is disconnected, then we choose a largest component $Y$ of $G$. Since $k \geq 1$, the component $Y$ has at least 2 vertices. If $|V(Y)| = 2$, then $G$ has rank-width 1 and tree-width 1, and $|V(G)| \leq (2 + 1)|V(G)| - 6$ since $|V(G)| \geq 3$. We assume that $|V(Y)| \geq 3$. Then by induction hypothesis, there is a graph $H_1$ such that $Y$ is isomorphic to a pivot-minor of $H_1$ and $\text{tw}(H_1) \leq 2k$ and $|V(H_1)| \leq (2k + 1)|V(Y)| - 6k$.

If $G \setminus V(Y)$ has tree-width at most 1, then $G$ is isomorphic to a pivot-minor of the disjoint union of two graphs $H_1$ and $G \setminus V(Y)$, and the tree-width of it is equal to the tree-width of $H_1$. Since $|V(H_1)| + |V(G) \setminus V(Y)| \leq (2k + 1)|V(Y)| - 6k + |V(G) \setminus V(Y)| \leq (2k + 1)|V(Y)| - 6k$, we obtain the result. If tree-width of $G \setminus V(Y)$ is at least 2, then $|V(G) \setminus V(Y)| \geq 3$. Therefore, by induction hypothesis, there is a graph $H_2$ such that $G \setminus V(Y)$ is isomorphic to a pivot-minor of $H_2$ and $\text{tw}(H_2) \leq 2k$ and $|V(H_2)| \leq (2k + 1)|V(G) \setminus V(Y)| - 6k$. So $G$ is isomorphic to a pivot-minor of the disjoint union of two graphs $H_1$ and $H_2$, and the tree-width of
it is at most $2k$, and $|V(H_1)| + |V(H_2)| \leq (2k + 1)|V(G)| - 6k$. Thus, we conclude the theorem.

**Proof of Theorem 3.2.** We can easily obtain the proof of Theorem 3.2 from the proof of Theorem 3.1.

4. **Graphs with rank-width or linear rank-width at most 1**

Distance-hereditary graphs are introduced by Bandelt and Mulder [2]. A graph $G$ is **distance-hereditary** if for every connected induced subgraph $H$ of $G$ and vertices $a, b$ in $H$, the distance between $a$ and $b$ in $H$ is the same as in $G$. Oum [6] showed that distance-hereditary graphs are exactly graphs of rank-width at most 1. Recently, Ganian [5] obtains a similar characterization of graphs of linear rank-width 1. In this section, we obtain another characterization for these classes in terms of vertex-minor relation.

Note that every tree has rank-width at most 1 and every path has linear rank-width at most 1.

**Theorem 4.1.** Let $G$ be a graph. The following are equivalent:

1. $G$ has rank-width at most 1.
2. $G$ is distance-hereditary.
3. $G$ has no vertex-minor isomorphic to $C_5$.
4. $G$ is a vertex-minor of a tree.

**Proof.** ((1) $\Leftrightarrow$ (2)) is proved by Oum [6], and ((2) $\Leftrightarrow$ (3)) follows from the Bouchet's theorem [3, 4]. Since every tree has rank-width at most 1, ((4) $\Rightarrow$ (1)) is trivial. We want to prove that (1) implies (4).

Let $G$ be a graph of rank-width at most 1. We may assume that $G$ is connected. If $|V(G)| \leq 2$, then $G$ itself is a tree. So we may assume that $|V(G)| \geq 3$. Let $(T, L)$ be a rank-decomposition of $G$ of width 1. From Proposition 3.9, a rank-expansion $H$ with the rank-decomposition $(T, L)$ has $G$ as a pivot-minor.

The width of each edge in $T$ is 1. Thus for $v \in V_I(T)$, the subgraph $H[S_v]$ is a path of length 2 or a triangle because $G$ is connected. Also for $e \in E_I(T)$, $H[e]$ consists of an edge. Therefore $H$ is connected and does not have cycles of length at least 4.

Let $Q$ be a tree obtained from $H$ by replacing each triangle $abc$ with $K_{1,3}$ by adding a new vertex $d$, making $d$ adjacent to $a$, $b$, $c$ and deleting $ab$, $bc$, $ca$. Clearly $H$ is a vertex-minor of the tree $Q$ because we can obtain the graph $H$ from $Q$ by applying local complementation on those new vertices and deleting them. Therefore $G$ is a vertex-minor of a tree, as required.

We also obtain a characterization of graphs with linear rank-width at most 1. Obstruction sets for graphs of linear rank-width 1 are $C_5$, $N$ and $Q$ [1], depicted in Figure 6.

**Lemma 4.2.** Every subcubic caterpillar is a pivot-minor of a path.

**Proof.** Let $H$ be a subcubic caterpillar. By the definition of a caterpillar, there is a path $P$ in $H$ such that every vertex in $V(H) \setminus V(P)$ is a leaf. We choose such path $P = p_1p_2 \ldots p_m$ in $H$ with maximum length. We construct a path $Q$ from $P$ by replacing each edge $p_ip_{i+1}$ with a path $p_ia.bp_{i+1}$. We can obtain a pivot-minor of $P$ isomorphic to $Q$ by pivoting each edge $a_ib_i$ and deleting all $a_i$ and deleting $b_i$ if $p_i$ is not adjacent to a leaf in $H$. □
Theorem 4.3. Let $G$ be a graph. The following are equivalent:

(1) $G$ has linear rank-width at most 1.

(2) $G$ has no vertex-minor isomorphic to $C_5$, $N$ or $Q$.

(3) $G$ is a vertex-minor of a path.

Proof. ((1) $\iff$ (2)) is proved by Adler, Farley and Proskurowski [1]. Since every path has linear rank-width at most 1, ((3) $\Rightarrow$ (1)) is trivial. Let us prove that (1) implies (3).

Let $G$ be a graph of linear rank-width at most 1. We may assume that $G$ is connected and $|V(G)| \geq 3$. Let $H$ be a rank-expansion of $G$ with a linear rank-decomposition $(T, L)$ of width 1. Note that $T$ is a caterpillar.

Since $(T, L)$ is a linear rank-decomposition of width 1, for each triangle in $H$, one of those vertices is of degree 2 in $H$. Let $P$ be a caterpillar obtained from $H$ by replacing each triangle with a path of length 2 whose internal vertex has degree 2 in $H$. We can obtain $H$ from $P$ by applying local complementation on the inner vertex of those paths of length 2, $H$ is a vertex-minor of $P$. And by Lemma 4.2, $P$ is a pivot-minor of a path. Therefore $G$ is a vertex-minor of a path. □

In Theorems 4.1 and 4.2, if a given graph is bipartite, we do not need to apply local complementation at some vertices. To prove it, we need the following lemma.

Lemma 4.4. Let $G$ be a connected bipartite graph with rank-width 1 and $|V(G)| \geq 3$. Let $(T, L)$ be a rank-decomposition of width 1. Then a rank-expansion of $G$ with respect to $(T, L)$ is a tree.

Proof. Let $x \in V(T)$ be a leaf and $H$ be a rank-expansion $R(G, T, L, x, \{U_f\}_{f \in E(T)})$ of $G$.

Suppose that $H$ has a triangle. Then there exists a vertex $v \in V_l(T)$ such that $H[S_v]$ is the triangle. Let $e_1, e_2$ and $e_3$ be edges incident with $v$ and assume that $e_1$
is the incoming edge. Let $U_{e_1} = \{a\}$, $U_{e_2} = \{b\}$ and $U_{e_3} = \{c\}$. By the construction of a rank-expansion, $bc \in E(G)$ and $R^e_{a} = R^e_{b} = R^e_{c}$. Since $R^e_{a}$ is a non-zero vector, there is a vertex $x \in V(G)$ such that $x$ is adjacent to all of $a$, $b$ and $c$. Therefore $xbc$ is a triangle in $G$, contradiction. □

**Theorem 4.5.** Let $G$ be a graph. Then $G$ is bipartite and has rank-width at most 1 if and only if $G$ is a pivot-minor of a tree.

**Proof.** We may assume that $G$ is connected. Since every tree has rank-width at most 1, backward direction is trivial. If $G$ is bipartite and has rank-width at most 1, then by Lemma 4.4, we have a rank-expansion of $G$ which is a tree. Hence, $G$ is a pivot-minor of a tree. □

**Theorem 4.6.** Let $G$ be a graph. Then $G$ is bipartite and has linear rank-width 1 if and only if $G$ is a pivot-minor of a path.

**Proof.** We may assume that $G$ is connected. Similarly, backward direction is trivial. Suppose $G$ is bipartite and has linear rank-width 1. Let $H$ be a rank-expansion of $G$ with a linear rank-decomposition $(T, L)$ of width 1. By Lemma 4.4, the graph $H$ is a tree, and since $T$ is a caterpillar, $H$ is also a caterpillar. By Lemma 4.2, $H$ is a pivot-minor of a path, and so is $G$. □

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