Vertex-minors, Monadic Second-order Logic, and a Conjecture by Seese.

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Abstract

We prove that one can express the vertex-minor relation on finite undirected graphs by formulas of monadic second-order logic (with no edge set quantification) extended with a predicate expressing that a set has even cardinality. We obtain a slight weakening of a conjecture by Seese stating that sets of graphs having a decidable satisfiability problem for monadic second-order logic have bounded clique-width. We also obtain a polynomial-time algorithm to check that the rank-width of a graph is at most k for any fixed k. The proofs use isotropic systems.

Key words: Clique-width, Rank-width, Monadic second-order logic, Seese's conjecture, Local complementation, Vertex-minor, Isotropic system

1 Introduction

The notion of *tree-width*, introduced by Robertson and Seymour [44], plays an essential role in the theory of graph minors. For instance, they proved in [45] that a set of graphs does not contain a fixed planar graph as a minor if and only if this set has bounded tree-width.

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Tree-width is also important in the theory of fixed-parameter tractablilty, see the book by Downey and Fellows [23]. In particular, many NP-complete graph problems such as 3-COLORABILITY have algorithms taking time $f(k)n$ for n-vertex graphs of tree-width at most k. Furthermore, every graph problem specified by a formula of *monadic second-order logic* has such algorithms. Monadic second-order logic, MS logic in short, is the extension of first-order logic with set variables. In this language, one can write properties of the form "there exists a set such that ...". This result actually holds for a strong version of MS logic, denoted by MS_2 logic, called *monadic second-order logic with edge set quantifications*. MS_2 logic allows to use variables denoting sets of edges in addition to variables denoting sets of vertices. (For the main definitions and results on MS logic and detailed examples of formulas, the reader is referred to the book chapter [13]. The preliminary sections of any of the articles [11,12,14,16,18] also contain definitions and examples.)

Finally, MS_2 logic is decidable on the set of graphs of tree-width at most k. There is even a kind of converse, that we will call Seese's Theorem [49], stating that if a set of graphs has a decidable satisfiability problem for MS_2 formulas, then it has bounded tree-width. The proof rests upon the result by Robertson and Seymour [45] that if a set of finite graphs has unbounded tree-width, then every square grid is isomorphic to a minor of some of its graphs.

The *clique-width* of a graph is also an important notion for the construction of polynomial-time graph algorithms. It is based on certain hierarchical graph decompositions. Every graph problem specified by a formula of MS logic (without edge set quantifications) is fixed parameter tractable when clique-width is a parameter. MS logic is also decidable on the set of graphs of clique-width at most k. These results actually hold for an extension of MS logic, called *counting monadic second-order logic* (CMS logic in short). In CMS logic, it is allowed to use predicates of the form $\mathsf{Card}_p(X)$, expressing that $|X|$ is a multiple of an integer p greater than 1. $\mathrm{C}_2\mathrm{MS}$ formulas generalize MS formulas by allowing the set predicate $\mathsf{Card}_2(X)$, for which we will write $\mathsf{Even}(X)$ for simplicity. Hence, $\mathrm{C}_2\mathrm{MS}$ is a sublanguage of CMS, strictly more expressive than MS.

The statement analogous to Seese's Theorem for MS formulas (without edge set quantifications) is a conjecture, also made by Seese in [49], of which we prove a weakening in this article. This conjecture says that if a set of graphs has a decidable satisfiability problem for MS formulas, then it has bounded clique-width. (We will explain the original form of the conjecture and its equivalence to this formulation in Section 5.4.) Its hypothesis concerns less formulas, hence is weaker than that of Seese's Theorem. Since a set of graphs has bounded clique-width if it has bounded tree-width, Seese's Theorem actually establishes another weakening of the conjecture.

We will actually prove a slight weakening of the conjecture, by assuming that the considered set of graphs has a decidable satisfiability problem for C_2MS formulas.

Our proof uses the notion of *rank-width*, introduced by Oum and Seymour [42]. It is equivalent to clique-width in the sense that a set of graphs has bounded rank-width if and only if it has bounded clique-width. Furthermore, the set of graphs of rank-width at most k is characterized by a finite set of excluded *vertex-minors*, a crucial notion that has for rank-width the good properties that minors have for tree-width.

The *local complementation* of a graph G at a vertex x consists in replacing the subgraph of G induced by neighbors of x by its complement graph. Two graphs are *locally equivalent* if one is obtained from another by a sequence of local complementations. A graph H is a *vertex-minor* of G if H is an induced subgraph of a graph that is locally equivalent to G. We prove that the vertex-minors of G can be *defined inside G by C_2MS formulas.* This is not at all obvious because local complementations relative to neighbors can interact in quite complicated ways. However, we can do so by using the notion of *isotropic system*, introduced by Bouchet [2,3]. Isotropic systems represent graphs by certain vector spaces over GF(2) and help us to handle local complementations algebraically. The corresponding computations can be formalized in $\mathrm{C_2MS}$ logic. The summations in GF(2) necessitate the use of the even cardinality set predicate.

Two main results follow from these constructions. First, the set of graphs of rank-width at most k, for every fixed k, is characterized by a $\mathrm{C_2MS}$ formula. With results by Seymour and Oum [42], this gives a polynomial-time algorithm for deciding whether a graph has rank-width at most k. By contrast, we do not know the complexity of deciding whether the clique-width of a graph is at most k for fixed $k > 3$. We remark that Oum and Seymour [42] provided an approximation algorithm suitable for proving results on fixed-parameter tractability. Recently Fellows, Rosamond, Rotics, and Szeider [27,28] have shown that the problem of deciding whether a graph has clique-width at most k is NP-complete if k is given as an input.

The second result is the above discussed weakening of Seese's Conjecture. This latter result extends to countable graphs.

This article is organized as follows. Sections 2, 3, 4 review definitions, notation and results on graphs, matroids, isotropic systems, and the relationships between these different notions. Section 5 reviews monadic second-order logic and its use for expressing properties and transformations of graphs, matroids, and isotropic systems. The various forms of Seese's Conjecture are recalled in this section. In section 6, we show how the notion of a vertex-minor can be formalized in C_2MS logic. This formalization is done via a logical formalization of isotropic systems and their so-called *fundamental graphs*. The application to the recognition of graphs of given rank-width follows then. We apply these constructions in Section 7 to prove our weakening of Seese's Conjecture. In Section 8 we give an alternative proof of it based on binary matroids and using results by Geelen, Gerards, and Whittle [30] and Hliněný and Seese [34]. Section 9 is a conclusion.

2 Graphs, clique-width and rank-width

In this section, we review the notion of clique-width, and give a survey of results about rank-width, which will be necessary to understand this paper. We assume graphs are undirected, simple (no loops and parallel edges), and finite, except at the end of Section 7 where we discuss countable graphs.

2.1 Definitions of clique-width and rank-width

A *graph* is defined as a pair (V, E) where V is the set of vertices and E is the set of edges. We write $V(G)$ and $E(G)$, or sometimes V_G and E_G to specify the graph under consideration.

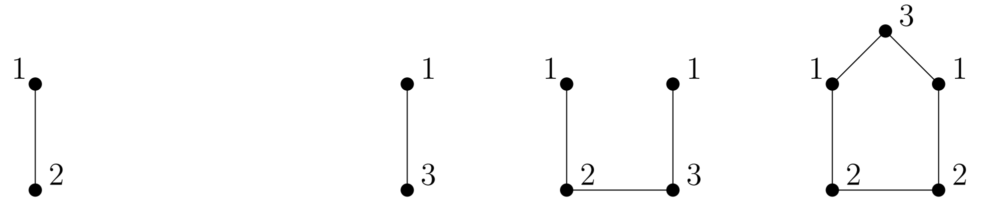
Clique-width is, like tree-width and branch-width, a graph complexity measure. It is defined in terms of algebraic expressions denoting graphs up to isomorphism. The operations used in these expressions have been introduced in [19] for denoting hypergraphs. Their restriction to graphs yields the notion of clique-width which has been defined and investigated first in Courcelle and Olariu [20], and then in subsequent papers among which we quote Corneil et al. [8].

Let k be a positive integer. A *k-graph* is a graph given with a function lab from its vertices to $[k] = \{1, ..., k\}$. Hence it is defined as a triple (V, E, lab). We call $lab(v)$ the *label* of a vertex v. We have the following operations on k-graphs.

(1) For each $i \in [k]$, we define a constant $\mathbf{i}$ for denoting a k-graph having one vertex labeled by i.
(2) For distinct $i, j \in [k]$, we define a unary function $\eta_{i,j}$ such that

$$\eta_{i,j}(V, E, lab) = (V, E', lab)$$

where E' is E augmented with the set of all edges joining a vertex labeled by i to a vertex labeled by j.



$A = \eta_{1,2}(\mathbf{1} \oplus \mathbf{2})$ $\quad B = \eta_{1,3}(\mathbf{1} \oplus \mathbf{3})$ $\quad C = \eta_{2,3}(A \oplus B)$ $\quad \eta_{1,3}(\rho_{3\to 2}(C) \oplus \mathbf{3})$

Fig. 1. Illustration of $\eta_{1,3}(\rho_{3\to 2}(\eta_{2,3}(\eta_{1,2}(\mathbf{1} \oplus \mathbf{2}) \oplus \eta_{1,3}(\mathbf{1} \oplus \mathbf{3}))) \oplus \mathbf{3})$

(3) We let $\rho_{i\to j}$ be a unary function such that

$$\rho_{i\to j}(V, E, lab) = (V, E, lab')$$

where

$$lab'(v) = \begin{cases} j & \text{if } lab(v) = i, \\ lab(v) & \text{otherwise.} \end{cases}$$

This mapping relabels every vertex labeled by i into j.

(4) Finally, we use the binary operation $\oplus$ that makes the union of two disjoint copies of its arguments. (Hence $G \oplus G \neq G$ unless G is empty, and the number of vertices of $G \oplus G$ is twice that of G.)

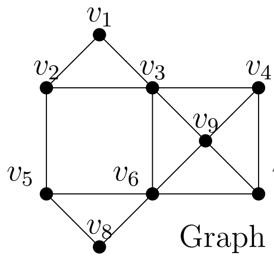
A well-formed expression t over these symbols is called a *k-expression*. Its *value* is a k-graph $G = val(t)$. The set of vertices of $val(t)$ can be defined as the set of occurrences of the constant symbols in t. However, we will also consider that an expression t designates all k-graphs isomorphic to $val(t)$. A graph is considered as a 1-graph whose vertices are (necessarily) labeled by 1. The *clique-width* of a graph G, denoted by $\mathrm{cwd}(G)$, is the minimum k such that $G = val(t)$ for some k-expression t.

Remark. The set of graphs of clique-width 1 is the set of graphs without edges. The set of graphs of clique-width at most two is the set of *cographs*, which are graphs having no induced path of three edges, see [20].

In this paper, the notion of rank-with, introduced by Oum and Seymour [42], is used widely. Let us review its definition. We will define the cut-rank function, rank-decompositions, and rank-width.

To describe the cut-rank function, we need a few notations. Let us denote $A(G)$ for the adjacency matrix of a graph G, that is a 0-1 $V(G) \times V(G)$ matrix where an entry is 1 if the column vertex is adjacent to the row vertex. We assume that the underlying field of $A(G)$ is $\mathrm{GF}(2)$, the field with just two elements, 0 and 1. For a $R \times C$ matrix $M = (m_{ij})_{i\in R, j\in C}$ and subsets $X \subseteq R$, $Y \subseteq C$, we denote by $M[X, Y]$ the $X \times Y$ submatrix $(m_{ij})_{i\in X, j\in Y}$ of M.

Let $\mathcal{P}(A)$ be the set of all subsets of A and let $\mathbb{Z}$ be the set of integers. The cut-rank function of a graph G is defined as the function $\mathrm{cutrk}_G : \mathcal{P}(V(G)) \to \mathbb{Z}$

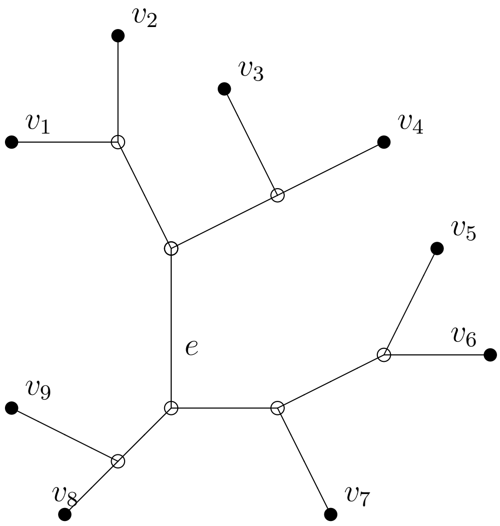


Graph G

Width of e

$= \mathrm{cutrk}_G(\{v_5, v_6, v_7, v_8, v_9\})$

$$= \mathrm{rank} \begin{array}{c} \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} = 3$$



Rank-decomposition $(T, \mathcal{L})$ of G

Fig. 2. Illustration of Rank-decompositions

such that

$$\mathrm{cutrk}_G(X) = \mathrm{rank}(A(G)[X, V(G) \setminus X]),$$

where rank is the linear rank function of matrices over GF(2).

A tree is *subcubic* if it has at least two vertices and every vertex is incident with at most three edges. A *leaf* of a tree is a vertex incident with exactly one edge. A *rank-decomposition* of a graph G is a pair $(T, \mathcal{L})$ of a subcubic tree T and a bijection $\mathcal{L} : V(G) \to \{t : t \text{ is a leaf of } T\}$. (If $|V(G)| \leq 1$ then G has no rank-decomposition.)

For each edge e of T, the connected components of $T \setminus e$ induce a partition (X_e, Y_e) of the set of leaves of T. The *width* of an edge e is defined as $\mathrm{cutrk}_G(\mathcal{L}^{-1}(X_e))$. The *width* of a rank-decomposition $(T, \mathcal{L})$ is the maximum width of all edges of T. The *rank-width* of a graph G, denoted by $\mathrm{rwd}(G)$, is the minimum k such that there is a rank-decomposition $(T, \mathcal{L})$ of width k. (We assume that $\mathrm{rwd}(G) = 0$ if $|V(G)| \leq 1$.)

Remark. Informally, its definition is a modification of that of *branch-width*, introduced by Robertson and Seymour [47]. Bouchet defined the cut-rank function under the name of *connectivity function* in [5].

The following proposition explains the most important reason why the rank-width is useful to study the clique-width.

Proposition 2.1 (Oum and Seymour [42]). For every graph G,

$$\mathrm{rwd}(G) \leq \mathrm{cwd}(G) \leq 2^{\mathrm{rwd}(G)+1} - 1.$$

Moreover, there is an $O(|V(G)|^2)$-time algorithm to convert a rank-decomposition of width k of G into a $(2^{k+1}-1)$-expression of the graph.

By this inequality, a set $\mathcal{C}$ of graphs has bounded clique-width if and only if it has bounded rank-width.

Remark. A graph G is called *distance-hereditary* if in every connected induced subgraph of G, the distance between every pair of vertices in the subgraph is equal to the distance in G. Oum [39] showed that these graphs are those of rank-width at most 1. Combined with Proposition 2.1, this gives another proof of the theorem by Golumbic and Rotics [31] stating that every distance-hereditary graph has clique-width at most three.

2.2 Algorithmic aspects

One of the main motivations to study clique-width is the fact that on graphs of clique-width at most k for fixed k, *if the input graph is given by the k-expression,* then many hard problems can be solved in polynomial time. For instance, there are polynomial-time algorithms to decide whether a graph has a Hamiltonian path or circuit [25,52], to find the chromatic number [35], and more strikingly, to solve graph problems expressible in CMS logic, see Section 5.5. This approach requires the k-expression to be given as an input. Oum and Seymour [42] removed this requirement.

Theorem 2.2 (Oum and Seymour [42]). Let k be fixed. There is an $O(n^9 \log n)$-time algorithm that either confirms that an n-vertex input graph has rank-width greater than k or outputs a rank-decomposition of width at most $3k+1$.

Combined with Proposition 2.1, the above algorithm can give a (8^k-1)-expression, which can be used as an input to algorithms based on the given k-expression. We remark that Oum [41] improved the running time of Theorem 2.2 to $O(n^3)$.

So we have an "approximation" algorithm saying that either the input graph has clique-width at most $f(k)$ or its clique-width is greater than k, where $f(k)=8^k-1$. How about recognizing graphs of clique-width at most k? It is easy when $k=1$. When $k=2$, there is a linear-time algorithm by Corneil, Perl, and Stewart [9] that recognize cographs, which are the graphs of clique-width at most two. When $k=3$, there is a polynomial-time algorithm by Corneil et al. [8]. The complexity of deciding $\mathrm{cwd}(G) \leq k$ is still unknown for $k>3$. However we will describe a polynomial-time algorithm to recognize graphs of rank-width at most k for a fixed k in Section 6.

The minor relation on graphs is essential for understanding the structure of many classes of graphs such as the class of graphs embeddable on surfaces without crossings and the class of graphs of tree-width at most k. Robertson and Seymour [48] proved that every minor-closed class of graphs is characterized by finitely many excluded minors. Their theorem extends Kuratowski's theorem for planar graphs.

It will be interesting to find a graph relation meaningful with clique-width and rank-width. Courcelle and Olariu [20] showed that the clique-width of an induced subgraph of a graph G is at most the clique-width of the graph G. But the induced subgraph relation is not rich enough to yield theorems similar to those with the minor relation. For example, the cycles form an infinite list of graphs of clique-width at most four in which none of them is an induced subgraph of another.

For sets A and B, $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Let $G = (V, E)$ be a graph and $v \in V$. The graph obtained by *local complementation* at v is defined by $G * v = (V, E\Delta\{xy : xv, yv \in E, x \neq y\})$. A graph H is *locally equivalent* to G if H is obtained from G by a sequence of local complementations. A graph H is a *vertex-minor* of G if H is obtained from G by a sequence of vertex deletions and local complementations.

From the definition, it is easy to show the following lemma.

Lemma 2.3. Let H and G be graphs and v be a vertex of H.

(1) If $H * v$ is an induced subgraph of G, then H is an induced subgraph of $G * v$.
(2) A graph H is a vertex-minor of G if and only if H is an induced subgraph of a graph that is locally equivalent to G.
(3) A graph locally equivalent to a vertex-minor of G is also a vertex-minor of G.

Bouchet [5] showed that cut-rank is preserved by local complementations. Therefore, rank-width is preserved too. So, we deduce the following proposition.

Proposition 2.4 (Oum [39]). If H is a vertex-minor of G, then $\mathrm{rwd}(H) \leq \mathrm{rwd}(G)$.

The following theorem is an analogy of the theorem by Robertson and Seymour [46] on minors and tree-width of graphs and of the theorem by Geelen, Gerards, and Whittle [29] on minors and branch-width of matroids.

Theorem 2.5 (Oum [40]). For every infinite sequence G_1, G_2, G_3, ... of graphs having bounded clique-width, there exist i and j such that $i < j$ and G_i is isomorphic to a vertex-minor of G_j. In other words, we say that a set of graphs of bounded clique-width is *well-quasi-ordered* by the vertex-minor relation up to isomorphism.

From the previous theorem, we obtain the following corollary, which has a more direct proof by Oum [39].

Corollary 2.6 (Oum [39,40]). For every integer k, there is a finite set $\mathcal{C}_k$ of graphs such that for every graph G, $\mathrm{rwd}(G) \leq k$ if and only if no vertex-minors of G are isomorphic to a graph in $\mathcal{C}_k$.

If $\mathcal{C}_k$ contains two graphs H and H', and H' is locally equivalent to a graph isomorphic to H, then one can replace $\mathcal{C}_k$ by $\mathcal{C}'_k = \mathcal{C}_k \setminus \{H'\}$. Hence, in Corollary 2.6, we may assume that $\mathcal{C}_k$ contains no two isomorphic graphs and no two locally equivalent graphs (up to isomorphism).

3 Matroids

In this section, we review the concept of a matroid, its connections with bipartite graphs, and the grid theorem for matroids.

3.1 Matroid and branch-width

A pair $\mathcal{M} = (E, \mathcal{I})$ of a finite set E and a set $\mathcal{I}$ of *independent* subsets of E is called a *matroid* if

(i) $\emptyset \in \mathcal{I}$,
(ii) if $B \in \mathcal{I}$ and $A \subseteq B$, then $A \in \mathcal{I}$,
(iii) for every subset Z of E, the maximal independent subsets of Z have the same size $r(Z)$.

We call r the *rank* function of a matroid $\mathcal{M}$. For more about matroids, we refer to the book by Oxley [43].

A matroid $\mathcal{M} = (E, \mathcal{I})$ is called *binary* if there exists a matrix N over GF(2) such that E is a set of column vectors of N and

$$\mathcal{I} = \{X \subseteq E : X \text{ is linearly independent as a set of vectors}\}.$$

For a matroid $\mathcal{M} = (E, \mathcal{I})$, the *dual matroid* $\mathcal{M}^* = (E, \mathcal{I}')$ of $\mathcal{M}$ is defined as follows: X is independent in $\mathcal{M}^*$ if and only if there is a maximally independent set B in $\mathcal{M}$ such that $B \cap X = \emptyset$.

For $e \in E(\mathcal{M})$, $\mathcal{M} \setminus e$ is a matroid $(E \setminus \{e\}, \mathcal{I}')$ such that X is independent in $\mathcal{M} \setminus e$ if $X \subseteq E \setminus \{e\}$ is independent in $\mathcal{M}$. This operation is called the *deletion* of e. $\mathcal{M}/e$ is defined by $(\mathcal{M}^* \setminus e)^*$. This operation is called the *contraction* of e. A matroid $\mathcal{N}$ is called a *minor* of $\mathcal{M}$ if $\mathcal{N}$ can be obtained from $\mathcal{M}$ by applying a sequence of deletions and contractions.

The *connectivity* $\lambda_{\mathcal{M}}(X)$ of $\mathcal{M} = (E, \mathcal{I})$ is defined as $r(X) + r(E \setminus X) - r(E) + 1$.

A *branch-decomposition* of a matroid $\mathcal{M} = (E, \mathcal{I})$ is a pair $(T, \mathcal{L})$ of a subcubic tree T and a bijection $\mathcal{L} : E \to \{t : t \text{ is a leaf of } T\}$. (If $|E| \leq 1$ then $\mathcal{M}$ has no branch-decomposition.) For each edge e of T, the connected components of $T \setminus e$ induce a partition (X_e, Y_e) of the set of leaves of T. The *width* of an edge e is defined as $\lambda_{\mathcal{M}}(\mathcal{L}^{-1}(X_e))$. The *width* of a branch-decomposition $(T, \mathcal{L})$ is the maximum width of all edges of T. The *branch-width* of a matroid $\mathcal{M}$ is the minimum k such that there is a branch-decomposition $(T, \mathcal{L})$ of width k. (We assume that the branch-width of $\mathcal{M}$ is 1 if $|E| \leq 1$.)

3.2 Bipartite graphs and binary matroids

Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$. Let M be the $A \times B$ submatrix $A(G)[A, B]$ of the adjacency matrix of G. Let $Bin(G, A, B)$ be the binary matroid on V, represented by the $A \times V$ matrix $\begin{pmatrix} I_A & M \end{pmatrix}$, where I_A is the $A \times A$ identity matrix. If $\mathcal{M} = Bin(G, A, B)$, then G is called a *fundamental graph* of $\mathcal{M}$.

It is straightforward to prove the following.

Proposition 3.1 (Oum [39]). Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$. Let $\mathcal{M} = Bin(G, A, B)$. Then, for every subset X of V, we have

$$\lambda_{\mathcal{M}}(X) = \mathrm{cutrk}_G(X) + 1,$$

and therefore the branch-width of $\mathcal{M}$ is exactly one more than the rank-width of G.

We recall that $G * u$ denotes the local complementation of G at the vertex u, as defined in Section 2.3.

Proposition 3.2 (Oum [39]). Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$. Let $\mathcal{M} = Bin(G, A, B)$. Then,

(1) $Bin(G, B, A) = \mathcal{M}^*$,
(2) For $uv \in E(G)$, $Bin(G * u * v * u, A\Delta\{u, v\}, B\Delta\{u, v\}) = \mathcal{M}$.
(3) $Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}) = \begin{cases} \mathcal{M} / v & \text{if } v \in A, \\ \mathcal{M} \setminus v & \text{if } v \in B. \end{cases}$

From (2) and (3), we deduce the following corollary.

Corollary 3.3 (Oum [39]). Let $\mathcal{N}$, $\mathcal{M}$ be binary matroids, and H, G be fundamental graphs of $\mathcal{N}$, $\mathcal{M}$ respectively. If $\mathcal{N}$ is a minor of $\mathcal{M}$, then H is a vertex-minor of G.

3.3 Grid theorem

From Proposition 3.1, theorems about the branch-width of binary matroids give corollaries about the rank-width of bipartite graphs. One of the theorems about branch-width of binary matroids was proved by Geelen, Gerards, and Whittle [30]. Here is the restatement of their theorem in the context of binary matroids.

Theorem 3.4 (Grid theorem for matroids). For every positive integer k, there is an integer l such that if $\mathcal{M}$ is a binary matroid with branch-width at least l, then $\mathcal{M}$ contains a minor isomorphic to the cycle matroid of the $k \times k$ grid.

Oum [39] showed the following corollary from the above theorem. We define a graph S_k, for $k > 1$ as follows. Let $A = \{a_i : 1 \leq i \leq k^2 - 1\}$ and $B = \{b_i : 1 \leq i \leq k^2 - k\}$. The graph S_k is a bipartite graph with $V(S_k) = A \cup B$ such that a_i and b_j are adjacent if and only if $i \leq j < i + k$ (see Fig. 8).

Corollary 3.5 (Oum [39]). For every positive integer k, there is an integer l such that if a bipartite graph G has rank-width at least l, then it contains a vertex-minor isomorphic to S_k

This corollary will be used in the Section 7.

4 Isotropic systems

Bouchet [2] introduced the notion of isotropic system and developed it in subsequent articles. Isotropic systems represent in an algebraic way the equivalence classes of graphs by local equivalence. So far they have been used in very few circumstances, but they provide a really powerful tool to study locally equivalent graphs, vertex-minors, and related notions.

4.1 Definition

Let K be the two-dimensional vector space over GF(2). We may write $K = \{0, \alpha, \beta, \gamma\}$ with $0 = \alpha + \alpha = \beta + \beta = \gamma + \gamma = \alpha + \beta + \gamma$. We define a bilinear form $\langle\, , \rangle$ by

$$\langle x, y \rangle = \begin{cases} 1 & \text{if } x \neq y, x \neq 0, \text{ and } y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For a finite set V, the set K^V of functions from V to K form a vector space over GF(2) with a bilinear form $\langle\, , \rangle$ defined as follows:

$$\text{for } a, b \in K^V, \quad \langle a, b \rangle = \sum_{v \in V} \langle a(v), b(v) \rangle.$$

An *isotropic system* is a pair $S = (V, L)$ of a finite set V and a subspace L of K^V such that $\dim(L) = |V|$ and $\langle x, y \rangle = 0$ for all $x, y \in L$.

A vector a in K^V is *complete* if $a(v) \neq 0$ for all $v \in V$. Two vectors a, $b \in K^V$ are *supplementary* if $\langle a(v), b(v) \rangle = 1$ for all $v \in V$. For $a \in K^V$ and $P \subseteq V$, we define the *restriction* $a[P] \in K^V$ of the vector a to P as a vector in K^V such that

$$(a[P])\,(v) = \begin{cases} a(v) & \text{if } v \in P, \\ 0 & \text{otherwise.} \end{cases}$$

4.2 Fundamental base and fundamental graphs

Bouchet [3] studied a connection between isotropic systems and graphs. A vector x of K^V is called an *Eulerian vector* of an isotropic system $S = (V, L)$ if x is complete and $x[P] \notin L$ for all nonempty subset P of V.

Proposition 4.1 (Bouchet [3]). Let $S = (V, L)$ be an isotropic system. For every complete vector c of K^V, there is an Eulerian vector a of S, supplementary to c.

Proposition 4.2 (Bouchet [3, (4.3)]). Let a be an Eulerian vector of an isotropic system $S = (V, L)$. For every $v \in V$, there exists a *unique* vector $b_v \in L$ such that

(i) $b_v(v) \neq 0$,
(ii) $b_v(w) = 0$ or $a(w)$ for all $w \neq v$.

Furthermore, the family $\{b_v\}_{v\in V}$ is a basis of L. The unique family $\{b_v\}_{v\in V}$ is called the *fundamental basis* of S with respect to an Eulerian vector a.

Remark. In his paper [3, (4.3)], Bouchet wrote a weaker statement, saying that the family $\{b_v\}$ is uniquely determined. But, in his proof, he proved the stronger one, which is the above statement. This stronger statement is helpful for Proposition 6.3.

We can construct graphs from isotropic systems as follows. The *fundamental graph* of S with respect to an Eulerian vector a is defined as a graph G such that $V(G) = V$ and v and w are adjacent in G if and only if $v \neq w$ and $b_v(w) \neq 0$, where $\{b_v : v \in V\}$ is the fundamental basis of S with respect to a. The fundamental graph G is undirected because $\langle b_v, b_w\rangle = 0$ implies that $b_v(w) \neq 0$ if and only if $b_w(v) \neq 0$.

Now we discuss how to construct isotropic systems from graphs. Let $n_G(v)$ be the set of neighbors of a vertex v of a graph G. For a graph $G = (V, E)$ and supplementary vectors a, b in K^V, let $S(G, a, b)$ be an isotropic system (V, L) such that L is a vector space spanned by $\{a[n_G(v)] + b[\{v\}] : v \in V\}$. If an isotropic system S is equal to $S(G, a, b)$, then the triple (G, a, b) is called the *graphic presentation* of the isotropic system S.

Proposition 4.3 (Bouchet [3]). For an isotropic system $S = (V, L)$, let a be the Eulerian vector, let G be the fundamental graph with respect to an Eulerian vector a, and let $\{b_v : v \in V\}$ be the corresponding fundamental basis. If we let $b \in K^V$ such that $b(v) = b_v(v)$ for all $v \in V$, then (G, a, b) is a graphic presentation of S.

Conversely, if $G = (V, E)$ is a graph and a, b are supplementary vectors in K^V, then $S = S(G, a, b)$ is an isotropic system such that the vector a is Eulerian and G is the fundamental graph of S with respect to the Eulerian vector a.

4.3 Isomorphism and locally equivalent graphs

Let G be a fundamental graph of an isotropic system S. Bouchet [3] proved that all fundamental graphs of S are locally equivalent to G and moreover every graph locally equivalent to G is a fundamental graph of S.

What can we say about two isotropic systems sharing the same fundamental graph? Let us clarify the notion of isomorphism of isotropic systems. A permutation π of K is *linear* if $\pi(0) = 0$. Let V be a finite set and $\Pi = (\pi_v)_{v\in V}$ be a family of linear permutations of K. For every vector a in K^V, we let $\Pi(a)$ be the vector defined by $(\Pi(a))(v) = \pi_v(a(v))$ for all $v \in V$. The mapping Π is a linear automorphism of K^V. If $S = (V, L)$ is an isotropic system, then $(V, \Pi(L))$ is an isotropic system, denoted by $\Pi(S)$ and said to be *strongly isomorphic to* S.

Let $G = (V, E)$ be a graph, let a and b be supplementary vectors in K^V, and let $S = S(G, a, b)$ be an isotropic system. Then it is easy to see that $\Pi(S) = S(G, \Pi(a), \Pi(b))$ is another isotropic system having G as a fundamental graph. The following lemma states a converse.

Lemma 4.4. Two isotropic systems with same fundamental graph are strongly isomorphic.

Proof. We first prove the following fact. If x, x', y, y' belong to $K \setminus \{0\}$, with $x \neq y, x' \neq y'$, then there exists a unique linear permutation of K mapping x to x' and y to y'. Without loss of generality, we can assume that $x = \alpha$ and $y = \beta$. By applying, if necessary, a linear permutation, we can also assume that $x' = \alpha$. There are two cases to consider. Either $y' = \beta$ or $y' = \gamma$. In both cases we get a unique linear permutation.

Now consider $S = (G, a, b)$ and $S' = (G, a', b')$. By applying the above observation to $a(v), a'(v), b(v), b'(v)$ for each v in V, we can find a unique Π such that $\Pi(a) = \Pi(a')$ and $\Pi(b) = \Pi(b')$. Hence $S' = \Pi(S)$. $\square$

We can consider two strongly isomorphic isotropic systems as the same mathematical object, because the three elements of $K \setminus \{0\}$ are indistinguishable.

Two isotropic systems $S = (V, L)$ and $S' = (V', L')$ are called *isomorphic* if there exist a bijection $h : V' \to V$ and a family $\Pi = (\pi_v)_{v \in V}$ of linear permutations of K such that $L' = \{b \in K^{V'}$: there exists $a \in L$ such that $b(v') = \pi_{h(v')}(a(h(v')))$ for all $v' \in V'\}$. Intuitively, h induces a bijection between L' and $\Pi(L)$. Hence S and S' are isomorphic if and only if the fundamental graphs of S are isomorphic to the fundamental graphs of S'. Therefore, up to isomorphism, isotropic systems represent classes of locally equivalent graphs.

5 Monadic second-order logic

We review background results on monadic second-order (MS) logic and transformations of structures expressed in this language and its extensions. We discuss the links between clique-width and MS logic, and we present Seese's Conjecture. For the main definitions and results on MS logic and some examples of formulas, the reader is referred to the book chapter [13], or the preliminary sections of any of the articles [11,12,14,16,18]. However all necessary definitions are given in full in the present section.

Let $R = \{A, B, C, \ldots\}$ be a finite set of *relation symbols* and *set predicates*, each of them given with a nonnegative integer $\rho(A)$ called its *arity*. We denote by $\mathcal{STR}(R)$ the set of R-structures $S = \langle D_S, (A_S)_{A \in R} \rangle$ where $A_S \subseteq D_S^{\rho(A)}$ if $A \in R$ is a relation symbol, and $A_S \subseteq (\mathcal{P}(D_S))^{\rho(A)}$ if A is a set predicate. Unless otherwise specified, structures will be finite, which means that their *domains* D_S will be finite.

A graph G without parallel edges can be defined as an $\{\mathsf{edg}\}$-structure $G = \langle V, \mathsf{edg} \rangle$ where V is the set of vertices of G and $\mathsf{edg} \subseteq V \times V$ is a binary relation representing the edges. Since we will consider simple undirected graphs, the relation edg will be symmetric and anti-reflexive ($\mathsf{edg}(x, x)$ will never hold).

Remark. We write $G = \langle V, \mathsf{edg} \rangle$ and not $G = (V, E)$ to stress the fact that, in this logical representation, the edges are defined by a binary relation on V and not as a set of objects apart from V, as in the case of MS_2 logic mentioned in the introduction where quantified variables may denote sets of edges.

A matroid $\mathcal{M}$ can be represented by a structure $\mathcal{M} = \langle E, \mathsf{Indep} \rangle$ where $\mathsf{Indep}(F)$ holds if and only if F is an independent set of $\mathcal{M}$. See Hliněný [32,33] about MS logic for matroids. An isotropic system $S = (V, L)$ can be represented by a structure $\langle V, \mathsf{Member} \rangle$ where $\mathsf{Member}(X, Y, Z)$ holds if and only if X, Y, Z are pairwise disjoint subsets of V and L contains a vector $a \in K^V$ such that for each $v \in V$,

$$a(v) = \begin{cases} \alpha & \text{if } v \in X, \\ \beta & \text{if } v \in Y, \\ \gamma & \text{if } v \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

We denote also by S the $\{\mathsf{Member}\}$-structure representing an isotropic system S. We will use subscripts G, $\mathcal{M}$, S in notation like V_G, edg_G, $\mathsf{Indep}_{\mathcal{M}}$, Member_S if it is necessary to make precise the relevant graph, matroid or isotropic system.

We recall that *monadic second-order logic* (*MS* logic for short) is the extension of first-order logic by variables denoting subsets of the domains of the considered structures, and new atomic formulas of the form $x \in X$ expressing the membership of x in a set X. (Uppercase letters will denote set variables, lowercase letters will denote ordinary first-order variables). If A is an n-ary set predicate, then we will use atomic formulas of the form $A(X_1, \ldots, X_n)$. We will denote by $MS(R, W)$ the set of MS formulas written with the set R of relation and set predicate symbols and having their free variables in a set W consisting of individual as well as of set variables.

As a typical and useful example of MS formula, we give a formula with free variables x and y expressing that (x, y) belongs to the reflexive and transitive closure of a binary relation A:

$$\forall X\Big(x \in X \wedge \forall u, v[(u \in X \wedge A(u, v)) \Longrightarrow v \in X] \Longrightarrow y \in X\Big)$$

If the relation A is not given in the structure but defined by an MS formula, then one replaces $A(u, v)$ by this formula with appropriate substitutions of variables.

We will use an extension of MS logic, referred by $\mathrm{C_2MS}$ logic and called *modulo-2 counting monadic second-order logic*, using the set predicate $\mathsf{Even}(X)$ expressing that $|X|$ is even. Since we consider structures with finite domains, that a set X has odd cardinality can be expressed by the formula $\neg\,\mathsf{Even}(X)$. An even larger extension called *counting monadic second-order logic*, referred by CMS logic, uses set predicates $\mathsf{Card}_p(X)$ meaning that $|X|$ is a multiple of an integer $p > 1$. We will denote by $C_2MS(R, W)$ and $CMS(R, W)$ instead of $MS(R, W)$ the corresponding sets of formulas that can use modulo 2 and modulo p cardinality predicates (for all p) respectively.

We have a strict inclusion of languages considered as sets of formulas: $MS \subset C_2MS \subset CMS$. The corresponding hierarchy of expressive powers is strict. It can be proved that no MS formula $\varphi(X)$ can express, in every structure, that a set X has even cardinality [10], and similarly, that the property that the cardinality of X is a multiple of three cannot be expressed by a $\mathrm{C_2MS}$ formula. (The argument by Courcelle [10] can be adapted.) However, for particular classes $\mathcal{C}$ of structures, if there exists an MS formula defining a linear ordering of each structure in $\mathcal{C}$ (the formal definition will be given in Section 7), then the Card_p predicates can be expressed by MS formulas and so, CMS is no longer more expressive than MS. For instance $\mathsf{Even}(X)$ can be expressed as follows: the set X is partitioned into two sets Y and Z such that the least element of X is in Y, the largest one is in Z and the successor of an element in Y (respectively in Z) is in Z (respectively in Y). Courcelle [12] investigated linear orders by MS formulas.

Let $\mathcal{C}$ be a set of (finite) relational structures that represent graphs, matroids, isotropic systems, or other combinatorial objects like hypergraphs and partial orders. The *MS satisfiability problem for* $\mathcal{C}$ is the following decision problem:

for every closed MS formula φ,
we ask whether there exists a structure in $\mathcal{C}$ that satisfies φ.

This decision problem does not concern particular properties like planarity of a graph, but *all properties expressible in monadic second-order logic.* Note that $\mathcal{C}$ is fixed and the input is any formula of MS logic. This problem is trivially decidable if $\mathcal{C}$ is finite, because we assume that relational structures are finite and the validity of a formula in a single finite structure can be decided, simply by applying the definition. If $\mathcal{C}$ is the set of all finite trees, then the MS satisfiability problem is decidable, as a consequence of deep results relating MS logic and tree-automata due to Doner [22] and Thatcher and Wright [50]; these results are presented in the book chapter by Thomas [51].

Seese [49] conjectured that roughly speaking, if a set of graphs has a decidable MS satisfiability problem, then it is, in a precise sense, definable from finite trees by MS formulas. This conjecture can be formulated for extensions of MS logic, like C_2MS or CMS logic. Note that the condition "the C_2MS satisfiability problem for $\mathcal{C}$ is decidable" is *a priori* stronger than "the MS satisfiability problem for $\mathcal{C}$ is decidable", because the intended algorithm must take more formulas as input in the former case. Hliněný and Seese [34] stated that there exists a set of countable trees having a decidable MS satisfiability problem but an undecidable C_2MS satisfiability problem. Actually there also exists a set of finite trees with the same property [private email exchange with Seese].

5.2 Transductions of relational structures

We now define some transformations of relational structures that can be formalized in MS logic (or its extensions). They are called *MS transductions*, because they generalize transformations of words and trees called *transductions* in formal language theory. They are similar to polynomial reductions which make it possible to compare algorithmic problems, because if a set of structures has a decidable MS satisfiability problem, then so has its image under an MS transduction. They make it possible to transfer decidability results from a set of structures to another one.

The basic idea is to specify a structure T inside a given structure S in terms of subsets of D_S specified by set variables called *parameters*, and by means of a fixed sequence of MS (or CMS) formulas. In particular, we will be able to describe all vertex-minors of a graph inside this graph, by means of C_2MS formulas and appropriately chosen sets of vertices taken as values of parameters.

Actually, the general definition of an MS transduction allows to define T inside a structure built from a fixed number of disjoint copies of the given structure S. For the most general definition, we refer the reader to articles by Courcelle [11,13,16]. We only define formally the special transductions that will be useful for the main proofs.

We let R and Q be two finite sets of relation symbols. Let W be a finite set of set variables, called *parameters*. In order to describe a transformation of R-structures into Q-structures in MS logic, we define a definition scheme as follows. A *definition scheme* is a tuple of formulas of the form

$\Delta = (\varphi, \psi, (\theta_A)_{A \in Q})$ where

(a) $\varphi \in MS(R, W)$,
(b) $\psi \in MS(R, W \cup \{x_1\})$,
(c) $\theta_A \in MS(R, W \cup \{x_1, \cdots, x_{\rho(A)}\})$ for each relation symbol A,
(d) $\theta_A \in MS(R, W \cup \{X_1, \cdots, X_{\rho(A)}\})$ for each set predicate A.

We now wish to describe how an R-structure is transformed into a Q-structure by a definition scheme. Let S be an R-structure and γ be a W*-assignment in* S, that is a mapping from the variables in W to subsets of the domain D_S of S. The Q-structure T with domain $D_T \subseteq D_S$ is *defined in* (S, γ) by a definition scheme $\Delta = (\varphi, \psi, (\theta_A)_{A \in Q})$ if

(i) $(S, \gamma) \models \varphi$,
(ii) $D_T = \{d \in D_S : (S, \gamma, d) \models \psi\}$,
(iii) for each A in Q, if A is a relation symbol then

$$A_T = \{(d_1, \cdots, d_{\rho(A)}) \in {D_T}^{\rho(A)} : (S, \gamma, d_1, \cdots, d_{\rho(A)}) \models \theta_A\},$$

and if A is a set predicate then

$$A_T = \{(U_1, \cdots, U_{\rho(A)}) \in (\mathcal{P}(D_T))^{\rho(A)} : (S, \gamma, U_1, \cdots, U_{\rho(A)}) \models \theta_A\}.$$

The notation $(S, \gamma, d_1, \cdots, d_{\rho(A)}) \models \theta_A$ means $(S, \gamma') \models \theta_A$, where γ' is the assignment extending γ, such that $\gamma'(x_i) = d_i$ for all $i = 1, \ldots, \rho(A)$; a similar convention is used for $(S, \gamma, d) \models \psi$ and $(S, \gamma, U_1, \cdots, U_{\rho(A)}) \models \theta_A$.

Let us describe the roles of the formulas of a definition scheme Δ. Condition (i) expresses that the values of the parameters specified by the assignment γ satisfy a condition specified by φ. Condition (ii) defines the domain of the output structure T as a subset of that of the input structure S. This restriction is specified by the formula $\psi(x_1)$. Since this formula may also have the parameters as free variables, the domain of T may depend on γ. Condition (iii) defines the relations A of T by means of the formulas θ_A evaluated in S; they also depend on γ. Similarly we define the set predicates of T. An example will be given shortly.

We use the functional notation $def_\Delta(S, \gamma)$ for T because T is associated uniquely with S, γ, and Δ whenever it is defined, in other words, whenever $(S, \gamma) \models \varphi$.

The *transduction defined by* a definition scheme Δ is the mapping $\mathcal{STR}(R) \rightarrow \mathcal{P}(\mathcal{STR}(Q))$ defined as follows:

$$def_\Delta(S) = \{T : T = def_\Delta(S, \gamma) \text{ for some } W\text{-assignment } \gamma \text{ in } S\}.$$

A mapping $\mathcal{STR}(R) \rightarrow \mathcal{P}(\mathcal{STR}(Q))$ is an *MS transduction* if it is equal to

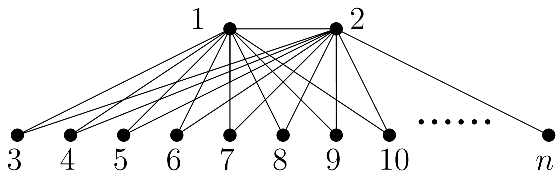


Fig. 3. The graph G_n in Example 5.1

def_Δ for some definition scheme Δ. If the formulas in the considered definition scheme are *C_2MS formulas* or *CMS* formulas, then the associated mapping is called a *C_2MS transduction* or a *CMS transduction* respectively. Hence, like for formulas, we have a hierarchy of classes of transductions: *MS*$\subset$*C_2MS*$\subset$*CMS*.

A mapping $\tau : \mathcal{STR}(R) \to \mathcal{P}(\mathcal{STR}(Q))$ is *isomorphic* to def_Δ if, for each R-structure S, every Q-structure T in $def_\Delta(S)$ is isomorphic to some Q-structure T' in $\tau(S)$ and vice versa.

Example 5.1 (Local complementation). If G is a graph and X is a set of independent vertices, then the local complementations associated with the vertices in X can be performed in any order. We denote by $G * X$ the graph obtained by these local complementations. The mapping $\mathcal{LC}$ that associates with G the set of graphs $G*X$ for all independent sets X of vertices is a $\mathrm{C_2MS}$ transduction defined by the definition scheme $(\varphi, \psi, \theta_{\mathsf{edg}})$ where

(i) φ is $\forall x, y(x \in X \wedge y \in X \Longrightarrow \neg\, \mathsf{edg}(x, y))$ (expressing that X is a set of independent vertices),
(ii) ψ is *true* (because $V(G) = V(G * X)$ and so there is no need to restrict the domain),
(iii) $\theta_{\mathsf{edg}}(x, y)$ is $(x \neq y) \wedge \Big[\mathsf{edg}(x, y) \Leftrightarrow \mathsf{Even}\,(\{z \in X : \mathsf{edg}(x, z) \wedge \mathsf{edg}(y, z)\})\Big]$.

The mapping $\mathcal{LC}$ is thus a $\mathrm{C_2MS}$ transduction with one parameter X. The set predicate Even is necessary, because the mapping $\mathcal{LC}$ is provably not an MS transduction; consider the graphs G_n with vertices $1, 2, \ldots, n$ and edges 1-2, 1-i, 2-i for $i = 3, \ldots, n$ (Fig. 3). Let $X \subseteq \{3, \ldots, n\}$. Then $G * X = G$ if and only if $|X|$ is even. And in the graphs G_n, evenness is not MS expressible (see [13]).

5.3 Fundamental properties of CMS transductions

The following proposition says that if $T = def_\Delta(S, \gamma)$, then the monadic second-order properties of T can be expressed as monadic second-order properties of (S, γ). This is why definable transductions are useful.

Proposition 5.2. (1) Let $\Delta = (\varphi, \psi, (\theta_A)_{A \in Q})$ be a definition scheme, written with a set W of parameters. Let V be a set of variables disjoint from W.

For every formula β in $MS(Q,V)$, there is a formula $\beta^{\#}$ in $MS(R, V \cup W)$ such that, for every R-structure S, every W-assignment γ in S, and every V-assignment η in S, we have the following: $(S, \eta \cup \gamma) \models \beta^{\#}$ if and only if

(i) $def_{\Delta}(S,\gamma)$ is defined,
(ii) η is a V-assignment in $def_{\Delta}(S,\gamma)$, and
(iii) $(def_{\Delta}(S,\gamma),\eta) \models \beta$.

(2) If Δ is a C_2MS (respectively CMS) definition scheme or β is a C_2MS (respectively CMS) formula, then the same holds for some C_2MS (respectively CMS) formula $\beta^{\#}$.

We call $\beta^{\#}$ the *backwards translation* of β relative to the transduction def_{Δ}. Note that, even if $T = def_{\Delta}(S,\gamma)$ is well-defined, the mapping η is not necessarily a V-assignment in T, because the domain of T can be a proper subset of D_S.

Proof sketch. The formula is $\beta^{\#}$ of the form $\varphi_1 \wedge \widehat{\beta}$ where φ_1 is independent of β and $\widehat{\beta}$ is defined inductively from β. We let $V = \{u_1, \ldots, u_m, U_1, \ldots, U_q\}$.

Let φ_1 be

$$\varphi \wedge \psi[u_1] \wedge \cdots \wedge \psi[u_m] \wedge \forall u(u \in U_1 \Longrightarrow \psi[u]) \wedge \cdots \wedge \forall u(u \in U_q \Longrightarrow \psi[u]).$$

This expresses that $def_{\Delta}(S,\gamma)$ is well-defined and η is a V-assignment in $def_{\Delta}(S,\gamma)$. (We denote by $\psi[u]$ the formula resulting from the substitution of u for x_1 in ψ).

We now define $\widehat{\beta}$ recursively. If β is $x = y$ or $x \in X$ or $\mathsf{Even}(X)$ or $\mathsf{Card}_p(X)$, then $\widehat{\beta}$ is β.

If β is $\beta_1 \wedge \beta_2$, or $\beta_1 \vee \beta_2$ or $\neg\beta_1$, then $\widehat{\beta}$ is $\widehat{\beta}_1 \wedge \widehat{\beta}_2$, or $\widehat{\beta}_1 \vee \widehat{\beta}_2$ or $\neg\widehat{\beta}_1$ respectively.

If β is $\exists u.\beta_1$, then $\widehat{\beta}$ is $\exists u.(\psi[u] \wedge \widehat{\beta}_1)$.

If β is $\exists X.\beta_1$, then $\widehat{\beta}$ is $\exists X.[\forall u(u \in X \Longrightarrow \psi[u]) \wedge \widehat{\beta}_1]$.

Universal quantifications are treated as negated existential quantifications.

If β is $A(y_1, \ldots, y_{\rho(A)})$ for some relation symbol A, then $\widehat{\beta}$ is $\theta_A[y_1, \ldots, y_{\rho(A)}]$ (where $\theta_A[y_1, \ldots, y_{\rho(A)}]$ is obtained by substituting $y_1, \ldots, y_{\rho(A)}$ for $x_1, \ldots, x_{\rho(A)}$ in θ_A; the free variables of θ_A are among $x_1, \ldots, x_{\rho(A)}$ and the parameters).

If β is $A(Y_1, \ldots, Y_{\rho(A)})$ for some set predicate A, then $\widehat{\beta}$ is $\theta_A[Y_1, \ldots, Y_{\rho(A)}]$ (where $\theta_A[Y_1, \cdots, Y_{\rho(A)}]$ is obtained as above by substitution of variables).

It is straightforward to verify that $\widehat{\beta}$ has the desired property by induction on the structure of β. $\square$

Proposition 5.3 (Courcelle [11,13]).

(1) If a set of structures has a decidable MS satisfiability problem (respectively C_2MS satisfiability problem), then so has its image under an MS transduction (respectively under a C_2MS transduction).
(2) The composition of two MS transductions (respectively of two C_2MS transductions) is an MS transduction (respectively a C_2MS transduction).

Proof. We only prove (1). Let $\mathcal{C}$ be a set of structures having a decidable MS satisfiability problem, and τ be an MS transduction with parameters $Y_1, \dots, Y_p$. For a given closed MS formula β, we want to know whether $T \models \beta$ for some $T \in \tau(\mathcal{C})$. Consider any $T = def_\Delta(S, \gamma)$ in $\tau(\mathcal{C})$ for S in $\mathcal{C}$. Then, by using Proposition 5.2, $T \models \beta$ if and only if $(S, \gamma) \models \beta^{\#}$ (since β is closed, the set V is empty). Hence $T \models \beta$ for some $T \in \tau(\mathcal{C})$ if and only if $(S, \gamma) \models \beta^{\#}$ for S in $\mathcal{C}$ and some γ. Equivalently we can express it as $S \models \exists Y_1, \dots, Y_p.\beta^{\#}$ for S in $\mathcal{C}$. Since $\mathcal{C}$ has a decidable MS satisfiability problem, we can decide the existence of such a structure S. Therefore we can decide the existence of a structure in $\tau(\mathcal{C})$ satisfying β. $\square$

Since every MS transduction is a C_2MS transduction, the composition of an MS transduction and a C_2MS transduction is a C_2MS transduction.

5.4 Seese's Conjecture

Seese [49] asked the following question:

> Is it true that if a set of graphs has a decidable monadic second-order theory, then it is interpretable in a set of trees?

This question concerns infinite as well as finite graphs. We say that a class $\mathcal{C}$ of structures has a *decidable monadic second-order theory* if there exists an algorithm that decides whether an input MS formula φ is valid for all structures in $\mathcal{C}$. We observe that a formula φ is true in every structure in $\mathcal{C}$ if and only if the formula $\neg\varphi$ is not safisfied in any structure of C. Hence, $\mathcal{C}$ has a decidable monadic theory if and only if it has a decidable MS satisfiability problem.

Proposition 5.4. Let $\mathcal{C}$ be a set of graphs. The following are equivalent.

(i) The set $\mathcal{C}$ has bounded clique-width.

(ii) The set $\mathcal{C}$ is the image of a set of trees under an MS transduction.
(iii) The set $\mathcal{C}$ is the image of a set of trees under a C_2MS transduction.

Proof. The first equivalence is proved in [18,24]. One can also replace "trees" by "binary trees" and "is the image" by "is contained in the image". For the last equivalence, let us consider a set $\mathcal{C}$ of graphs that is the image of a set $\mathcal{T}$ of trees under a C_2MS transduction η. There exist a set $\mathcal{B}$ of binary trees and a bijective MS transduction β of $\mathcal{B}$ onto $\mathcal{T}$. Hence $\mathcal{C} = \eta \circ \beta(\mathcal{B})$, and $\eta \circ \beta$ is a C_2MS transduction. But on binary trees a linear order is definable by an MS formula. Hence the atomic formulas $\mathsf{Even}(X)$ in the formulas of the definition scheme of $\eta \circ \beta$ can be replaced by MS formulas, and $\eta \circ \beta$ also has an MS definition scheme. Hence $\mathcal{C}$ is the image of a set of trees under an MS transduction. $\square$

This proof also works for CMS instead of C_2MS. One important consequence of this result and Proposition 5.3.2 is that the image of a set of graphs of bounded clique-width under a CMS transduction has bounded clique-width. This is not immediate from the definitions of clique-width operations on the one hand, and of CMS transductions on the other.

By Proposition 2.1, clique-width can be replaced by rank-width in this statement. Clique-width is also defined for directed graphs [20] and Proposition 5.4 is valid for them.

Using the terminology of the present article, the conjecture by Seese [49] can be stated as follows.

Conjecture. If a set of graphs has a decidable MS satisfiability problem, then it is contained in the image of a set of trees under an MS transduction, equivalently, it has bounded clique-width.

Any two isomorphic graphs satisfy the same formulas, have the same clique-width and one is the image of a set of trees under an MS transduction if and only if the other is. Concrete constructions will handle graphs but this conjecture and the related statements actually concern isomorphism classes of graphs.

This conjecture has been proved for various graph classes: planar graphs [49], graphs of bounded degree, graphs without a fixed graph as a minor, *uniformly k-sparse graphs* (meaning that every subgraph H satisfies that $|E(H)| \leq k|V(H)|$) [14], interval graphs, line graphs, partial orders of dimension 2 [16]. Furthermore, Courcelle [16] proved the following.

Proposition 5.5 (Courcelle [16]). Seese's Conjecture is valid for graphs if and only if it is valid for one of the following classes: bipartite graphs, directed graphs, comparability graphs, and partial orders.

We can ask a similar question for matroids. Hliněný and Seese [34] answered positively for matroids representable over a fixed finite field.

One of the main results of this article is the proof of the following weakening of the conjecture.

Theorem 5.6. If a set of graphs has a decidable C_2MS satisfiability problem, then it is contained in the image of a set of trees under an MS transduction, or equivalently, it has bounded clique-width and bounded rank-width.

The proof of Proposition 5.5 yields the corresponding results for directed graphs, partial orders, etc.

For all particular cases where the conjecture has been proved, the proofs use, *via* some reductions based on MS transductions, the result of Robertson and Seymour [45] saying that excluding a planar graph as a minor implies bounded tree-width. Theorem 5.6 uses the analogous result by Geelen, Gerards, and Whittle [30] which implies that bipartite graphs not containing certain graphs, transformable by MS transductions into grids, as vertex-minors have bounded rank-width. We will also give another proof using binary matroids and results by Geelen, Gerards, and Whittle [30] and Hliněný and Seese [34]. For both proofs, connections between bipartite graphs and binary matroids are essential.

5.5 Evaluation of CMS formulas

We explain why and how CMS formulas can be evaluated in linear time on graphs of clique-width at most k that are given by k-expressions.

The *quantifier-height* $qh(\varphi)$ of a CMS formula is defined as follows.

(i) $qh(\varphi) = 0$ if φ is atomic (of the form $x = y$ or $x \in X$ or $\mathsf{Card}_p(X)$ or $A(u_1, \cdots, u_n)$ or $A(U_1, \cdots, U_n)$).
(ii) $qh(\neg\varphi) = qh(\varphi)$.
(iii) $qh(\varphi_1 \wedge \varphi_2) = qh(\varphi_1 \vee \varphi_2) = \max\{qh(\varphi_1), qh(\varphi_2)\}$.
(iv) $qh(\exists u.\varphi) = qh(\forall u.\varphi) = qh(\exists U.\varphi) = qh(\forall U.\varphi) = 1 + qh(\varphi)$.

We denote by $C_p MS^h(R, \emptyset)$ the set of CMS formulas of quantifier-height at most h, written with the relation symbols in a finite set R and the set predicates Card_q for $q \leq p$. This set is infinite because if it contains a formula φ, then it also contains all the formulas $\varphi \vee \varphi \vee \cdots \vee \varphi$. However all these formulas are equivalent. One can actually replace (by an algorithm) every formula φ in

$\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$ by a canonical formula $Can(\varphi)$ in $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$ which is equivalent to φ (so they has the same truth value in every R-structure). This can be done in such a way that $Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$ is finite. This classical fact is described formally in [21]. The cardinality of $Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$ is however a tower of exponentials of height proportional to h.

For every p, R, h as above, and for every R-structure S, we let

$$Th_{p,R,h}(S) = \{\varphi \in Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)) :\ S \models \varphi\ \}.$$

We call it the (p,R,h)-*theory of* S. Thus, there are finitely many (p,R,h)-theories, and each of them is a finite set of formulas.

A k-graph $G = (V_G, E_G, lab_G)$ is represented by the relational structure

$$\langle V_G, \mathsf{edg}_G, p_{1G}, ..., p_{kG}\rangle,$$

also denoted by G, where edg_G is the edge relation and $p_{iG}(x)$ holds when $lab(x) = i$. The following proposition summarizes well-known results. Similar forms have been published in [10,36].

Proposition 5.7 ([13, Theorem 5.7.5]). Let us fix a positive integer k.

(1) Let $R = \{\mathsf{edg}, p_1, ..., p_k\}$ with edg of arity two and p_i of arity 1. For all positive integers p, h, i, j (where $i, j \in [k]$ and $i \neq j$), there exist mappings $f_{k,\oplus}$, $f_{k,\eta_{i,j}}$, $f_{k,\rho_{i\to j}}$ on subsets of $Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$ such that for all k-graphs G and H,

$$\begin{aligned} Th_{p,R,h}(\eta_{i,j}(G)) &= f_{k,\eta_{i,j}}(Th_{p,R,h}(G)),\\ Th_{p,R,h}(\rho_{i\to j}(G)) &= f_{k,\rho_{i\to j}}(Th_{p,R,h}(G)),\\ Th_{p,R,h}(G \oplus H) &= f_{k,\oplus}(Th_{p,R,h}(G), Th_{p,R,h}(H)). \end{aligned}$$

(2) If a graph G is given as $val(t)$ for some k-expression t, then $Th_{p,R,h}(G)$ can be computed in time proportional to the size of t.

(3) Every CMS graph property can be evaluated on graphs of clique-width at most k, given by a k-expression, in time proportional to the number of vertices.

Proof. (1) Let us observe that the mapping $\eta_{i,j}$ is a *quantifier-free transduction* (a transduction defined by a definition scheme consisting of formulas without quantifiers and without parameters). From the proof of Proposition 5.2, it follows that the backwards translation (denoted by $^{\#}$) associated with $\eta_{i,j}$ does not increase quantifier-height and does not add new counting modulo set predicates. Hence for every formula φ in $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$, $\eta_{i,j}(G) \models \varphi$ if and only if $G \models \varphi^{\#}$. This is equivalent to $G \models Can(\varphi^{\#})$. Furthermore, $\varphi^{\#}$ belongs to

$\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$. Hence, we can take, for every subset Φ of $Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$,

$$f_{k,\eta_{i,j}}(\Phi) = \{\varphi \in Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)) :\ Can(\varphi^{\#}) \in \Phi\}.$$

The proof is similar for $\rho_{i\to j}$.

The case of $\oplus$ is a particular case of a result by Feferman and Vaught [26]. The proof is in [10, Lemma (4.5)]. We also refer the reader to the survey by Makowsky [36] for the history and the numerous consequences of this result.

(2) Consider a graph $G = val(t)$ where t is a k-expression.

Each set $Th_{p,R,h}(val(\mathbf{i}))$ can be computed from the definitions. Then, using (1), we can compute $Th_{p,R,h}(val(t))$ by induction on the structure of t. For example, if $t = t_1 \oplus t_2$, then we get

$$Th_{p,R,h}(val(t)) = f_{k,\oplus}(Th_{p,R,h}(val(t_1)), Th_{p,R,h}(val(t_2))).$$

(3) To know whether $val(t) \models \varphi$, we compute by (2) the set $Th_{p,R,h}(val(t))$ where p and h are the smallest integers such that $\varphi \in \mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$. Then we determine whether $Can(\varphi)$ belongs to $Th_{p,R,h}(val(t))$ and this gives the answer. □

This method applies to optimization and enumeration (counting) problems formalized in monadic second-order logic. We refer the reader to the survey by Makowsky [36].

6 Logical expression of vertex-minors

6.1 From a graph to locally equivalent graphs

We will represent an isotropic system $S = (V, L)$ by the structure $\langle V, \mathsf{Member}_S\rangle$ (also denoted by S) where the ternary set predicate $\mathsf{Member}_S(X, Y, Z)$ holds if and only if X, Y, Z are pairwise disjoint subsets of V and the vector $a \in K^V$ is in L when

$$a(v) = \begin{cases} \alpha & \text{if } v \in X, \\ \beta & \text{if } v \in Y, \\ \gamma & \text{if } v \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.1. There exists an MS transduction that maps an isotropic system S to the set of isotropic systems strongly isomorphic to S.

Proof. A strong isomorphism of isotropic systems with base set V is defined from a family $\Pi = (\pi_v)_{v \in V}$ of linear permutations of K. Since a linear permutation is nothing but a permutation of $\{\alpha, \beta, \gamma\}$, there are six such permutations, say $\pi^1, \ldots, \pi^6$. Hence a family Π as above can be specified by six set variables $W_1, \ldots, W_6$ forming a partition of V, with the condition that $\pi_v = \pi^i$ if and only if $v \in W_i$. With this assumption, it is then straightforward to write an MS formula expressing $\mathsf{Member}_{\Pi(S)}$ in terms of Member_S and $W_1, \ldots, W_6$. $\square$

We recall a construction from Proposition 4.3. Let $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ be the vectors in K^V such that $\overline{\alpha}(v) = \alpha$, $\overline{\beta}(v) = \beta$, and $\overline{\gamma}(v) = \gamma$ for all $v \in V$. If $G = \langle V, \mathsf{edg} \rangle$ is a graph, then we denote by $S(G)$ the isotropic system $S(G, \overline{\alpha}, \overline{\beta})$. This definition of $S(G)$ corresponds to the particular choice of the pair $(\overline{\alpha}, \overline{\beta})$ of supplementary complete vectors.

Proposition 6.2.

(1) The set predicate $\mathsf{Member}_{S(G)}$ is expressible in $\langle V, \mathsf{edg} \rangle$ by a C_2MS formula.
(2) The mapping from a graph G to the isotropic systems $S(G)$ is a C_2MS transduction.
(3) There is a C_2MS transduction that maps a graph G to the set of isotropic systems strongly isomorphic to $S(G)$, which is the set of isotropic systems having G as a fundamental graph.

Proof. (1) We first show how to define $S(G) = (V, L)$ in logical terms. Let $b_v^G = \overline{\alpha}[n_G(v)] + \overline{\beta}[\{v\}]$. By definition of $S(G)$, the set $\{b_v^G : v \in V(G)\}$ is a basis of L. We represent a vector $c \in K^V$ by a triple (X, Y, Z) of subsets of V such that X, Y, Z are pairwise disjoint and $c = \overline{\alpha}[X] + \overline{\beta}[Y] + \overline{\gamma}[Z]$. The vector $\overline{\alpha}[X] + \overline{\beta}[Y] + \overline{\gamma}[Z]$ is in L if and only if there exists a subset U of V such that $\sum_{x \in U} b_x^G = \overline{\alpha}[X] + \overline{\beta}[Y] + \overline{\gamma}[Z]$.

From the definitions, we have

$$b_x^G(v) = \begin{cases} \alpha & \text{if } x \text{ is adjacent to } v, \\ \beta & \text{if } x = v, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{x \in U} b_x^G(v) = \begin{cases} \beta & \text{if } v \in U \text{ and } |n_G(v) \cap U| \text{ is even (because } \alpha + \alpha = 0), \\ \gamma & \text{if } v \in U \text{ and } |n_G(v) \cap U| \text{ is odd (because } \alpha + \beta = \gamma), \\ 0 & \text{if } v \notin U \text{ and } |n_G(v) \cap U| \text{ is even,} \\ \alpha & \text{if } v \notin U \text{ and } |n_G(v) \cap U| \text{ is odd.} \end{cases}$$

From these observations, it is easy to write a C_2MS formula expressing these conditions.

(2) The mapping S from graphs to isotropic systems is thus a C_2MS transduction.

(3) From $S(G)$, we obtain all strongly isomorphic isotropic systems by applying the MS transduction of Proposition 6.1. The composition of these two transductions is a C_2MS transduction. □

Remark. In the definition of $S(G)$ we have chosen the particular pair $(\overline{\alpha}, \overline{\beta})$ of supplementary vectors so that it is easy to encode $S(G)$ by logical formulas because all components are the same. By taking any other pair, we obtain an isotropic system strongly isomorphic to $S(G)$. The transformation of $S(G)$ into the isotropic systems strongly isomorphic to it is done by using Proposition 6.1. Applying a family Π of permutations to $S(G)$ is exactly the same thing as changing $(\overline{\alpha}, \overline{\beta})$ into another pair of supplementary vectors.

We now consider the inverse transformation.

Proposition 6.3. The mapping from an isotropic system to the set of its fundamental graphs is an MS transduction ν.

Proof. Let $S = (V, L)$ be an isotropic system. Let a be a vector in K^V, described by (X_a, Y_a, Z_a). We can express that the vector a is complete by the condition $V = X_a \cup Y_a \cup Z_a$. (The corresponding logical formula is $\forall x, x \in X_a \vee x \in Y_a \vee x \in Z_a$, but we omit the detailed form.) The vector a is an Eulerian vector of S if a is complete and $a[U] \notin L$ when U is a nonempty subset of V. This is equivalent to the following MS logic formula:

$$(V = X_a \cup Y_a \cup Z_a) \wedge$$
$$\forall X \forall Y \forall Z \Big(X \subseteq X_a, Y \subseteq Y_a, Z \subseteq Z_a, \mathsf{Member}_S(X, Y, Z) \Rightarrow X = Y = Z = \emptyset \Big)$$

So we can thus "select" an Eulerian vector and express by an MS formula that it is actually Eulerian. The parameters of the transduction that we are defining are the variables X_a, Y_a, Z_a representing an Eulerian vector. By Proposition 4.3, for every v in V, there exists a unique vector b_v in L such that

$$b_v(v) \neq 0 \qquad \text{and} \quad b_v(w) \in \{0, a(w)\} \text{ for } w \neq v.$$

The fundamental graph of S with respect to the Eulerian vector a is a graph (V, E) such that two vertices v and w are adjacent if $b_v(w) \neq 0$. (Different graphs are obtained from other Eulerian vectors, but they are all locally equivalent).

The translation in MS logic is easy. We let $\nu_1(X, Y, Z, X_a, Y_a, Z_a, v)$ be the formula:

$$\begin{aligned}&\mathsf{Member}(X,Y,Z)\wedge v\in X\cup Y\cup Z\\ \wedge&\forall w[w\neq v\Rightarrow\{(w\in X\Rightarrow w\in X_a)\wedge(w\in Y\Rightarrow w\in Y_a)\wedge(w\in Z\Rightarrow w\in Z_a)\}].\end{aligned}$$

It expresses that (X, Y, Z) represents b_v. Now two vertices v and w in the fundamental graph G are adjacent if and only if

$$v\neq w\wedge\exists X,Y,Z[\nu_1(X,Y,Z,X_a,Y_a,Z_a,v)\wedge w\in X\cup Y\cup Z].$$

Hence we have constructed an MS transduction ν that transforms an isotropic system given with a triple (X_a, Y_a, Z_a) of sets representing an Eulerian vector into the corresponding fundamental graph. □

Corollary 6.4. There exists a C_2MS transduction that maps a graph G to the set of graphs locally equivalent to G.

Proof. In Proposition 6.2, we constructed a C_2MS transduction S that maps a graph to an isotropic system. In Proposition 6.3, we obtained an MS transduction ν with parameters X_a, Y_a, Z_a that maps an isotropic system to the set of its fundamental graphs. By results recalled in Section 4, the composition $\nu \circ S$ of these transductions is the desired one. It is a C_2MS transduction by Proposition 5.3.2, with parameters X_a, Y_a, Z_a. □

6.2 From a graph to its vertex-minors

Theorem 6.5.

(1) There exists a C_2MS transduction $\overline{\mu}$ that maps a graph to the set of its vertex-minors.
(2) For every graph H, there is a closed C_2MS logic formula expressing that a graph contains a vertex-minor isomorphic to H.

Proof. (1) A graph H is a vertex-minor of a graph G if and only if H is an induced subgraph of a graph G' that is locally equivalent to G. Hence the mapping from a graph to the set of its vertex-minors is the composition $\overline{\mu}$ of two transductions: the C_2MS transduction in Corollary 6.4 and the MS transduction with a parameter U that maps a graph to the set of its induced subgraphs. Hence their composition is a C_2MS transduction with four parameters X_a, Y_a, Z_a, and U.

(2) For every graph H with vertices $1, \ldots, n$, we can construct a closed MS formula $\varkappa_H$ that is valid in a graph if and only if this graph is isomorphic to

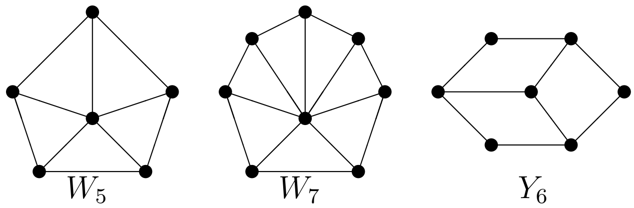


Fig. 4. Obstructions for circle graphs

H. This formula is written as following.

$$\begin{aligned} \exists x_1,\ldots,x_n[&\text{“}x_1,\ldots,x_n \text{ are pairwise distinct”} \\ &\wedge \text{“every vertex is equal to } x_i \text{ for some } i\text{”} \\ &\wedge \text{“for all } i,j,\ \mathsf{edg}(x_i,x_j) \Leftrightarrow i \text{ and } j \text{ are neighbors in } H\text{”}] \end{aligned}$$

This formula is actually a first-order formula, because no set quantification is used. The backwards translation relative to the transduction $\overline{\mu}$ in (1) is a C_2MS formula $\varkappa_H^{\#}$ with free variables X_a, Y_a, Z_a, and U. It is valid on a graph G if and only if its vertex-minor defined by the sets X_a, Y_a, Z_a, and U (“defined” in the sense of the first part of the corollary) is isomorphic to H. Hence G has a vertex-minor isomorphic to H if and only if it satisfies $\exists X_a, Y_a, Z_a, U.\, \varkappa_H^{\#}$. □

Let us discuss one application of Theorem 6.5. A *circle graph* is the intersection graph of a set of chords of a circle so that vertices are chords of a circle and two vertices are adjacent if and only if the corresponding chords intersect.

Corollary 6.6. There exist C_2MS formulas expressing that a graph is a circle graph, a distance-hereditary graph, or a graph locally equivalent to a tree.

Proof. Bouchet [6] proved that a graph is a circle graph if and only if it has no vertex-minor isomorphic to one of W_5, W_7 or Y_6 shown in Fig. 4. The result follows then from Theorem 6.5.

The articles by Bouchet [4,6] show that a graph is distance-hereditary if and only if it does not have a vertex-minor isomorphic to C_5. We obtain thus the result in the same way.

For graphs locally equivalent to trees, the result follows from the definition by Theorem 6.5 and the fact [13] that the class of trees is characterized by an MS formula. □

Remark. (1) The case of distance-hereditary graphs is given as an example of a set of graphs characterized by known excluded vertex-minors. There are not so many yet. This set is also characterized by an infinite set of excluded induced subgraphs, namely the cycles C_n for $n \geq 5$ and three particular graphs

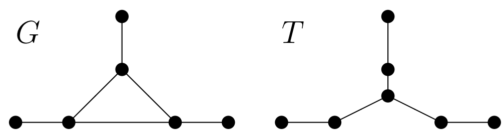


Fig. 5. G is a vertex-minor of a tree T, but not locally equivalent to a tree

(Bandelt and Mulder [1]). A definition of this set by an MS formula is easily derivable from this characterization because the infinitely many cycles C_n for $n \geq 5$ can easily be characterized by a unique MS formula.

(2) The set of graphs locally equivalent to trees is not closed under taking vertex-minors. By using the characterization by Bouchet [4] one can prove that the graph G in Fig. 5 is not locally equivalent to a tree but it is a vertex-minor of the tree T in Fig. 5. One might ask for a characterization of the set of vertex-minors of trees. Since these graphs have rank-width at most 1, they are characterized by a finite set of excluded vertex-minors.

Example 6.7. Let G be the "house" with vertices 1, 2, 3, 4, 5 forming the cycle 1-2-3-4-5-1, augmented with the edge 2-5 (Fig. 6).

Let us illustrate the isotropic system $S(G) = S(G, \overline{\alpha}, \overline{\beta})$. If we use the construction of Proposition 6.2, we obtain the isotropic system $S = (\{1, 2, 3, 4, 5\}, L)$ where L is a subspace of K^V with the following basis:

$$b_1^G = (\beta, \alpha, 0, 0, \alpha), \qquad b_2^G = (\alpha, \beta, \alpha, 0, \alpha), \qquad b_3^G = (0, \alpha, \beta, \alpha, 0),$$
$$b_4^G = (0, 0, \alpha, \beta, \alpha), \qquad b_5^G = (\alpha, \alpha, 0, \alpha, \beta).$$

We note that every nonzero linear combination of them has an entry having β or γ and therefore $\overline{\alpha} = (\alpha, \alpha, \alpha, \alpha, \alpha)$ is an Eulerian vector. And so $\{b_1^G, b_2^G, b_3^G, b_4^G, b_5^G\}$ is a fundamental basis of $S(G)$ with respect to $\overline{\alpha}$.

The subspace L contains 32 vectors spanned by b_1^G, b_2^G, b_3^G, b_4^G, b_5^G. We list some of them here.

$$b_1^G + b_3^G + b_4^G = (\beta, 0, \gamma, \gamma, 0), \qquad b_2^G + b_3^G + b_5^G = (0, \beta, \gamma, 0, \gamma),$$
$$b_1^G + b_2^G = (\gamma, \gamma, \alpha, 0, 0), \qquad b_1^G + b_5^G = (\gamma, 0, 0, \alpha, \gamma),$$
$$b_2^G + b_4^G + b_5^G = (0, \gamma, 0, \gamma, \beta).$$

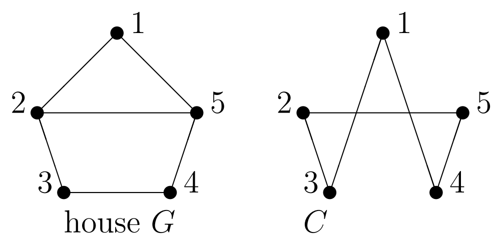


Fig. 6. G and C in Example 6.7

It is straightforward to observe that the above five vectors again form a basis. Moreover, we can see that $\overline{\gamma} = (\gamma, \gamma, \gamma, \gamma, \gamma)$ is an Eulerian vector. In fact, $\{b_1^G + b_3^G + b_4^G, b_2^G + b_3^G + b_5^G, b_1^G + b_2^G, b_1^G + b_5^G, b_2^G + b_4^G + b_5^G\}$ is the fundamental basis of $S(G)$ with respect to $\overline{\gamma}$. The corresponding fundamental graph C with respect to the Eulerian vector $\overline{\gamma}$ is given in Fig. 6.

We can transform C into G by the following sequence of local complementations: 1, 4, 2, 5, 3. The successive Eulerian vectors are

$$\begin{aligned}
&(\gamma, \gamma, \gamma, \gamma, \gamma) \text{ for } C,\\
&(\alpha, \gamma, \gamma, \gamma, \gamma) \text{ for } C * 1,\\
&(\alpha, \gamma, \gamma, \alpha, \gamma) \text{ for } C * 1 * 4,\\
&(\alpha, \alpha, \gamma, \alpha, \gamma) \text{ for } C * 1 * 4 * 2,\\
&(\alpha, \alpha, \gamma, \alpha, \alpha) \text{ for } C * 1 * 4 * 2 * 5,\\
&(\alpha, \alpha, \alpha, \alpha, \alpha) \text{ for } C * 1 * 4 * 2 * 5 * 3 = G.
\end{aligned}$$

6.3 Computing a set of excluded vertex-minors

We recalled in Section 2 that the vertex-minor relation is a well-quasi-ordering of the set of graphs of rank-width at most k. It follows by standard arguments, that if a set L of graphs of bounded clique-width is closed under taking vertex-minors and isomorphisms, then it is characterized by a finite set X of excluded vertex-minors (so that a graph belongs to L if and only if none of its vertex-minors is isomorphic to a graph in X).

How can we compute this finite set? Does there exist an algorithm that would compute this finite set by using input the bound k and a finite formal description of the set L, typically a logical formula?

This question is not trivial. For the graph minor relation, Courcelle, Downey, and Fellows [17] proved that that for a minor-closed set L of graphs a membership algorithm for L is not sufficient to compute the finite set $O_M(L)$ of excluded minors. Formally, there is no algorithm taking as input an MS formula or a Turing Machine characterizing L and producing within a finite time the finite set $O_M(L)$ whenever L is minor-closed.

The following proposition may help in particular cases to compute finite sets of excluded vertex-minors. For every set L of graphs closed under isomorphism, let $O_{VM}(L)$ be the set of graphs not in L whose all proper vertex-minors are in L. *Proper* means that at least one vertex is deleted. For every set K of graphs, let $\text{Forb}_{VM}(K)$ be the set of graphs that have no vertex-minor isomorphic to a graph in K. If L is closed under isomorphism and taking vertex-minors, then

$$L = \text{Forb}_{VM}(O_{VM}(L)). \tag{6.1}$$

We are interested in the computation of $O_{VM}(L)$ when this set is finite up to isomorphism, and in its replacement by a smallest possible set.

Lemma 6.8. Let ξ be a closed CMS formula and let $L = \{G : \ G \models \xi\}$. Then we can algorithmically construct a closed CMS formula ξ' such that $O_{VM}(L) = \{G : \ G \models \xi'\}$.

Proof. We will use the C_2MS-transduction $\overline{\mu}$ of Theorem 6.5 that maps a graph G to the set of its vertex-minors. The parameters of this transduction are X_a, Y_a, Z_a, and U. Let $(\varphi, \psi, \theta_{\mathsf{edg}})$ be the definition scheme of $\overline{\mu}$. Then φ is the MS formula with free variables X_a, Y_a, Z_a, and U, expressing that the parameters are correctly chosen so that a vertex-minor is defined from them by $\overline{\mu}$. The defined vertex-minor is proper if and only if $U \neq V(G)$. We let $\xi^{\#}$ be the backwards translation of ξ with respect to $\overline{\mu}$.

So the desired formula ξ' is $\neg\xi \wedge \forall X_a, Y_a, Z_a, U[\varphi \wedge (\exists x, x \notin U) \Longrightarrow \xi^{\#}]$. $\square$

Note that this construction is correct even if L is not closed under taking vertex-minors. When it is closed under taking vertex-minors, then (6.1) holds. In addition, if L has bounded rank-width, then $O_{VM}(L)$ is finite up to isomorphism by Theorem 2.5. Our objective is to find a "small" finite set K such that $L = \mathrm{Forb}_{VM}(K)$.

It is clear that we do not need two isomorphic graphs in K. Furthermore, we do not need two locally equivalent graphs in K because if a graph H is vertex-minor of a graph locally equivalent to G, then H is isomorphic to a vertex-minor of G; hence we can take K as a subset of $O_{VM}(L)$ such that for each graph G in $O_{VM}(L)$, there is a unique graph in K locally equivalent to a graph isomorphic to G. We call such a set K a *minimal set of vertex-minor obstructions* of L.

We now wish to do this by an algorithm.

Lemma 6.9. For every integer k and every closed CMS formula φ, we can decide whether the set $L = \{G : \mathrm{cwd}(G) \leq k, G \models \varphi\}$ is finite up to isomorphism. Moreover, there exists an algorithm enumerating L when it is finite. In other words, we can compute an integer m from k and φ such that either all graphs in L have at most m vertices or L has arbitrarily large graphs.

Proof sketch. For each k, the graphs of clique-width at most k are the values of the finite terms built with a finite set $\mathcal{F}_k$ of binary operations and nullary symbols where $1, \dots, k$ are the labels (see Section 2). The nullary symbol $\mathbf{i}$ denotes the graph with a single vertex labeled by i, for each $i = 1, \dots, k$. There are only finitely many inequivalent compositions of the unary operations with k labels that relabel vertices (denoted by $\rho_{i \to j}$) and create edges (denoted by

$\eta_{i,j}$). (Two compositions are *equivalent* if they define the same function.) For each equivalence class of these compositions, we select a representative λ and we define a binary operation $\otimes_\lambda$ by $G \otimes_\lambda H = \lambda(G \oplus H)$. Hence we obtain the desired finite signature $\mathcal{F}_k$ consisting of k nullary symbols and the binary operations $\otimes_\lambda$.

The value of each term t in $T(\mathcal{F}_k)$ is a graph $val(t)$ of clique-width at most k, and the number of vertices of $val(t)$ is equal to the number of occurrences of nullary symbols in t. The height of t (the length of a longest branch from the root to a leaf when considering t as a rooted tree) is between $\log_2(|V|)$ and $|V|$, where V is the set of vertices of $val(t)$. Every graph of clique-width at most k is the value of a term in $T(\mathcal{F}_k)$, and there are only finitely many terms denoting a graph.

The set $\{t \in T(\mathcal{F}_k) : val(t) \models \varphi\}$ is the set of terms in $T(\mathcal{F}_k)$ whose values satisfy the closed CMS formula φ and so it is the set of terms having values in $L = \{G : \mathrm{cwd}(G) \leq k, G \models \varphi\}$. This set of terms is defined by a finite tree-automaton $A(k, \varphi)$ that we can construct from k and φ by an algorithm: this is the basic fact underlying the existence of algorithms which verify in linear time the graph properties specified in CMS logic, on graphs of clique-width at most k, given as values of terms in $T(\mathcal{F}_k)$. However, its number of states is a tower of exponentials of height proportional to the quantifier depth of φ (see Section 5.5).

The so-called "Pumping Lemma" for tree-automata states that, if a tree-automaton accepts a term of height more than the number of states, then it accepts infinitely many terms. (Terms are usually called "trees" in automata theory.) It follows that we can decide whether the set of terms accepted by a tree-automaton is finite, and if it is finite, then we can enumerate the accepted terms by an algorithm. For definitions and results on tree-automata, the reader is referred to the book by Comon et al. [7], available on-line.

The set of terms defined by $A(k, \varphi)$ is finite if and only if the set L of graphs is finite up to isomorphism. This can be decided, and if it is finite, then the terms accepted by $A(k, \varphi)$ can be enumerated. By evaluating these terms, we obtain at least one graph isomorphic to each graph in L. It remains to remove graphs which have isomorphic copies in the list (because two different terms may define isomorphic graphs).

Let $m = 2^N$ where N is the number of states of $A(k, \varphi)$. If a graph in L has more than 2^N vertices, it must be defined by a term in $T(\mathcal{F}_k)$ of height more than N and therefore it follows that $A(k, \varphi)$ accepts infinitely many terms. The values of these terms are graphs with an unbounded number of vertices, since the number of vertices of a graph is at least the height of a term $T(\mathcal{F}_k)$. This proves the last assertion. $\square$

Proposition 6.10. There exists an algorithm that takes as input an integer k and a closed CMS formula ξ and produces a minimal set of vertex-minor obstructions for $L = \{G : G \models \xi\}$ if this set is closed under taking vertex-minors and has rank-width at most k. In addition if these conditions are not satisfied, then the algorithm stops but reports a failure or produces irrelevant output.

Proof. Let us assume that $L = \{G : G \models \xi\}$ has rank-width at most k. Then the graphs in $O_{VM}(L)$ have rank-width at most $k+1$. Hence they have clique-width at most $f(k)$, where $f(k) = 2^{k+2} - 1$ by Proposition 2.1. We let ξ' be a closed CMS formula obtained by Lemma 6.8. Then

$$O_{VM}(L) = \{G : G \models \xi'\} = \{G : \mathrm{cwd}(G) \leq f(k), G \models \xi'\}.$$

If L is closed under taking vertex-minors, then $O_{VM}(L)$ is finite up to isomorphism and can be computed by the algorithm of Lemma 6.9, applied to the formula ξ' and the integer $f(k)$. Computed means that one can construct a finite subset K of $O_{VM}(L)$ that contains exactly one graph in each isomorphism class. Then, this set can be reduced into a subset K' of K such that for any graph G in $O_{VM}(L)$, K' contains exactly one graph isomorphic to a graph locally equivalent to G. It is clear that K' is a minimal set of vertex-minor obstructions for L.

If the conditions on L are not satisfied, then the algorithm may report that $\{G : \mathrm{cwd}(G) \leq f(k), G \models \xi'\}$ is infinite or produce a finite set K which does not satisfy $L = \mathrm{Forb}_{VM}(K)$. $\square$

The algorithms of Lemma 6.9 and Proposition 6.10 are clearly not implementable. They are interesting as computability results.

6.4 Recognizing graphs of rank-width at most k

By Corollary 2.6, for every fixed k, there are only finitely many graphs, such that a graph does not contain any of them as a vertex-minor if and only if it has rank-width at most k. By Theorem 6.5, for every fixed graph H, there is a C_2MS formula expressing that H is isomorphic to a vertex-minor of an input graph. In Theorem 2.2, we have an $O(n^9 \log n)$-time algorithm that either confirms that the n-vertex input graph has rank-width at least $k+1$ or confirms that the rank-width is at most $3k+1$ and outputs a rank-decomposition of width at most $3k+1$. Oum and Seymour [42] provided an algorithm that converts the rank-decomposition into a k-expression. In Section 5.5, we recalled that every property specified by a CMS formula can be checked in linear time on graphs given by a k-expression.

By combining all of these, we get the following.

Theorem 6.11. For every fixed k, there is an $O(n^9 \log n)$-time algorithm to check that the n-vertex input graph has rank-width at most k.

Instead of Theorem 2.2, we can use another algorithm by Oum [41] that runs in time $O(n^3)$ and therefore we can produce in this theorem an algorithm running also in time $O(n^3)$.

7 Proof of Seese's Conjecture via vertex-minors

We will prove the following theorem in this section.

Theorem 5.6. If a set of graphs has a decidable C_2MS satisfiability problem, then it has bounded rank-width and bounded clique-width.

To prove this, we will use a family of bipartite graphs S_k for $k > 1$ and build $(2k-2) \times k$ rectangular grids by a fixed MS transduction. The graph S_k has the following property.

Proposition 7.1. If L is a set of bipartite graphs of unbounded rank-width, then for each k there is a graph G in L with a vertex-minor isomorphic to S_k.

Proof. Suppose not. Then, there is an integer k such that no vertex-minors of graphs in L are isomorphic to S_k. By Corollary 3.5, there is an integer l such that every graph in L has rank-width at most $l-1$. Contradiction. □

Proposition 7.2. There exists an MS transduction τ such that the $(2k-2)\times k$ grid belongs to $\tau(S_k)$ for all $k > 1$.

Proof. The idea is to construct the transduction τ as the composition of several transductions. We do not give the explicit formulas but we explain how they can be obtained. We are given S_k as $\langle V, A, B, \mathsf{edg}\rangle$. Our aim is to build a $(2k-2) \times k$ grid in Fig. 7 from S_k.

Step 1 : Ordering A and B.

We first define by MS formulas the orderings of A and B defined by the indices. (The sets A and B are given in $\langle V, A, B, \mathsf{edg}\rangle$ as unordered sets; the indices are used to define S_k shortly, but are not expressed in the relational structure). We assume that $\{b_1\}$ is given by means of a parameter, say Y.

Two elements b and b' of B are *consecutive* if $b = b_i$ and $b' = b_{i+1}$ or vice-versa. It is easy to see that b and b' are consecutive if and only if $|n_G(b)\Delta n_G(b')| = 2$.

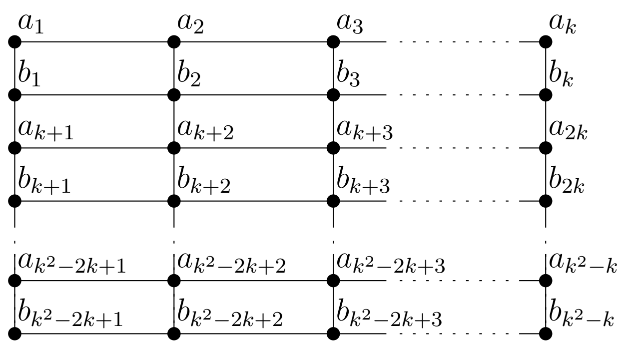


Fig. 7. Grid to be obtained from S_k by an MS transduction

It follows that we can determine the ordering on B such that $b < b'$ if $b = b_i$ and $b' = b_j$ for some $j > i$, because we know b_1 already from Y. To see this, we will say that $b < b'$ if $b \neq b'$ and either $b = b_1$ or there exists a subset X of B containing b_1 and b but not b' such that each of b and b_1 is consecutive to exactly one element of X, and each element of $X \setminus \{b, b_1\}$ is consecutive to exactly two elements of X. This characterization is expressible by an MS formula.

The analogous strict linear order $<$ on A is characterized as follows. We say that $a < a'$ if there exist a neighbor b of a and a neighbor b' of a' such that $b < b'$ and either a is not adjacent to b' or a' is not adjacent to b. This ordering is also expressible by an MS formula. We can thus transform S_k into the structure $S'_k = \langle V, A, B, <, \mathsf{edg}\rangle$ by an MS transduction τ_1.

Step 2 : Some edge modifications.

The edges $b_i a_i$ and $b_i a_{i+k-1}$ are called *minimal* and *maximal* respectively. Each b in B is incident with the unique minimal (respectively maximal) edge, the A-vertex of which is the least (greatest) neighbor of b, where "least" and "greatest" are relative to $<$. On the drawing of S_4 in Fig. 8, the minimal edges are vertical. The maximal edges are oblique and drawn with a thick line. These edges can be identified by MS formulas evaluated in S'_k. We build T_k from S'_k as follows:

1) We add edges between each b_i and a_{i+k} for $i = 1, \ldots, k^2 - 2k$. This is possible because MS formulas can identify b_{k^2-k} (as the maximal element of B), and thus a_{k^2-k} (linked to b_{k^2-k} by a minimal edge), whence also b_{k^2-2k+1} linked to a_{k^2-k} by a maximal edge. Hence an MS formula can identify b_{k^2-2k} as the predecessor of b_{k^2-2k+1}. An MS formula can identify for each b the corresponding a_{i+k-1} where $b = b_i$, $i \in \{1, \ldots, k^2 - 2k\}$. The new edges to be added between b_i and a_{i+k} can thus be defined by an MS formula, since one can determine a_{i+k} as the successor of a_{i+k-1} in A.

2) We delete all edges except the minimal edges and of course, the edges added in 1).

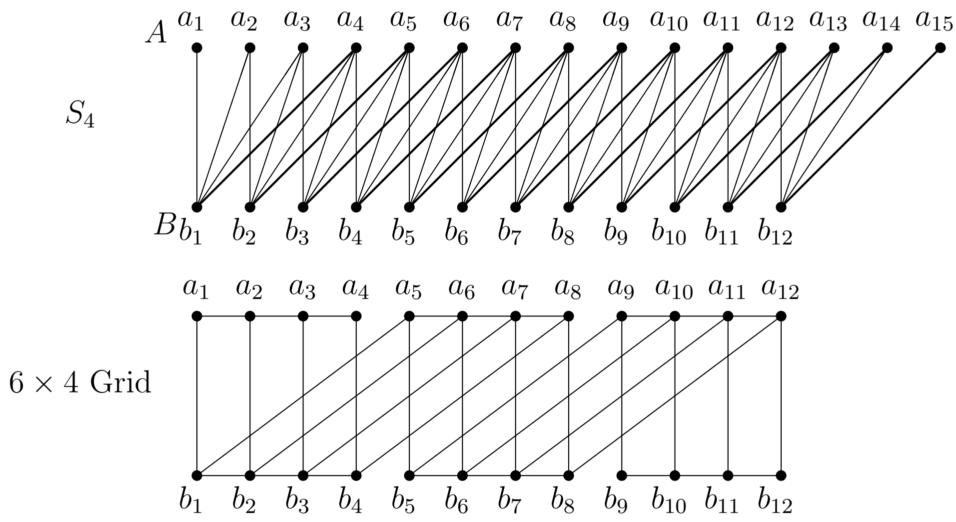


Fig. 8. Getting the grid from S_k

3) We delete the isolated vertices, which are the vertices a_i for $i > k^2 - k$.

We get thus by an MS transduction τ_2 , a graph T_k, equipped with the orderings $<$ of A and B.

Step 3 : Making T_k into a rectangular grid.

The graph T_k consists of k disjoint paths with $2k-2$ vertices. To make T_k into the $(2k-2)\times k$ grid, it suffices to add edges between a_i and a_{i+1}, and between b_i and b_{i+1} for each $i \in I$ defined as $I = \{1, \dots, k^2-k\}\setminus\{pk : p = 1, \dots, k-1\}$. The edges added during this step are the horizontal lines in the 6×4 grid of Fig. 8.

This can be done from the set $U = \{a_i, b_i : i \in I\}$. This set can be "guessed"; it is given as a parameter to the transduction τ_3 we are defining. This transduction also deletes the orderings $<$.

We let τ be the transduction $\tau_3 \circ \tau_2 \circ \tau_1$. It uses actually two parameters, Y intended to specify b_1 (by $Y = \{b_1\}$) and the above set U. Whenever the sets Y and U are "correctly chosen" (so that the above construction works as described) for a graph H isomorphic to S_k, then the structure $\tau(H, Y, U)$ is the $(2k-2)\times k$ grid. If they are not correctly chosen, then a graph that is not a grid may be produced. But we only demand that τ produces grids among other graphs we need not care about. Hence, we are done. $\square$

For a graph $G = (V, E)$, we define $B(G)$ as a bipartite graph on a vertex set $V \times \{1, 2, 3, 4\}$ such that

(i) if $v \in V$ and $i \in \{1, 2, 3\}$ then (v, i) is adjacent to $(v, i + 1)$ in $B(G)$,

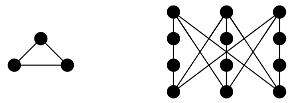


Fig. 9. K_3 and $B(K_3)$

(ii) if $vw \in E$ then $(v, 1)$ is adjacent to $(w, 4)$ in $B(G)$.

Lemma 7.3 (Courcelle [16]). The mapping from a graph G to $\{B(G)\}$ is an MS transduction.

Proof. The transformation of G into $B(G)$ is an MS transduction that duplicates a fixed number of times (here four times) a given structure before defining the new structure inside it. (This technical notion is not defined in this paper. The reader is referred to [11,13,16].) $\square$

For a set $\mathcal{C}$ of graphs, let $B(\mathcal{C})$ be the set $\{B(G) : G \in \mathcal{C}\}$. The above lemma implies that if $\mathcal{C}$ has bounded clique-width, then $B(\mathcal{C})$ has bounded clique-width by Proposition 5.4. For the converse, Courcelle [16] proved that if a set $\mathcal{C}$ of graphs has unbounded clique-width, then $B(\mathcal{C})$ has unbounded clique-width as well by means of several lemmas in his paper using MS transductions. In order to facilitate the reading of the present article, we reproduce the direct proof by Oum [41]. We remark that Oum [38] showed that $\operatorname{rwd}(B(G)) = \max(2\operatorname{rwd}(G), 1)$, but the statement presented here is enough for our result and the proof is conceptually simpler.

Lemma 7.4 (Oum [41]). For a graph $G = (V, E)$, we have $\operatorname{rwd}(G) \leq 4\operatorname{rwd}(B(G))$.

Proof. Let $(T', \mathcal{L}')$ be a rank-decomposition of $B(G)$ of width $k = \operatorname{rwd}(B(G))$. Let T be a minimum subtree of T' containing all leaves in $\mathcal{L}'(V \times \{1\})$ and let $\mathcal{L} : V \to \{t : t \text{ is a leaf of } T\}$ is the bijection defined by $\mathcal{L}(v) = \mathcal{L}'((v, 1))$. We claim that $(T, \mathcal{L})$ is a rank-decomposition of G of width at most $4k$.

For four subsets A_1, A_2, A_3, A_4 of V, we denote $A_1|A_2|A_3|A_4 = (A_1 \times \{1\}) \cup (A_2 \times \{2\}) \cup (A_3 \times \{3\}) \cup (A_4 \times \{4\})$. Let e be an edge of T. Since T is a subtree of T', e is also an edge of T'. Let (X, Y) be a partition of the set of leaves of T' induced by the connected components of $T' \setminus e$. Let $\mathcal{L}'^{-1}(X) = A_1|A_2|A_3|A_4$. Let $\overline{A}_i = V \setminus A_i$ for $i \in \{1, 2, 3, 4\}$. Because the width of $(T', \mathcal{L}')$ is k, we have

$$\operatorname{cutrk}_{B(G)}(A_1|A_2|A_3|A_4) = \operatorname{cutrk}^*_{B(G)}(A_1|A_2|A_3|A_4, \overline{A}_1|\overline{A}_2|\overline{A}_3|\overline{A}_4) \leq k.$$

We now claim that

$$r_1 = \mathrm{cutrk}^*_{B(G)}(A_1|A_2|\emptyset|\emptyset, \overline{A}_1|\overline{A}_2|\emptyset|\emptyset) = |A_1 \Delta A_2|. \tag{7.1}$$
$$r_2 = \mathrm{cutrk}^*_{B(G)}(\emptyset|A_2|A_3|\emptyset, \emptyset|\overline{A}_2|\overline{A}_3|\emptyset) = |A_2 \Delta A_3|. \tag{7.2}$$
$$r_3 = \mathrm{cutrk}^*_{B(G)}(\emptyset|\emptyset|A_3|A_4, \emptyset|\emptyset|\overline{A}_3|\overline{A}_4) = |A_3 \Delta A_4|. \tag{7.3}$$

To see this, we look at the matrix defining the cut-rank functions.

$$\begin{aligned} r_1 &= \mathrm{rank}\left[\begin{array}{c c} & \begin{array}{cc} \overline{A}_1 \times \{1\} & \overline{A}_2 \times \{2\} \end{array} \\ \begin{array}{c} A_1 \times \{1\} \\ A_2 \times \{2\} \end{array} & \begin{pmatrix} 0 & \text{(0-1 submatrix)} \\ \text{(0-1 submatrix)} & 0 \end{pmatrix} \end{array}\right] \\ &= \mathrm{cutrk}^*_{B(G)}(A_1 \times \{1\}, \overline{A}_2 \times \{2\}) + \mathrm{cutrk}^*_{B(G)}(A_2 \times \{2\}, \overline{A}_1 \times \{1\}). \end{aligned}$$

It is easy to observe that $\mathrm{cutrk}^*_{B(G)}(A_1 \times \{1\}, \overline{A}_2 \times \{2\}) = |A_1 \setminus A_2|$ and $\mathrm{cutrk}^*_{B(G)}(A_2 \times \{2\}, \overline{A}_1 \times \{2\}) = |A_2 \setminus A_1|$. Since $|A_1 \setminus A_2| + |A_2 \setminus A_1| = |A_1 \Delta A_2|$, the equation (7.1) is proved. Similarly (7.2) and (7.3) are true.

Since $r_i \leq \mathrm{cutrk}_{B(G)}(A_1|A_2|A_3|A_4)$, we have $|A_i \Delta A_{i+1}| \leq k$ for each $i \in \{1, 2, 3\}$. Adding these inequalities for all i, we obtain that $|A_1 \Delta A_4| \leq 3k$.

Let M be the adjacency matrix of G. We observe that $\mathrm{rank}(M[A_4, \overline{A}_1]) = \mathrm{cutrk}^*_{B(G)}(A_4 \times \{4\}, \overline{A}_1 \times \{1\}) \leq \mathrm{cutrk}_{B(G)}(A_1|A_2|A_3|A_4) \leq k$. Then we have the following bound of $\mathrm{cutrk}_G(A_1)$:

$$\begin{aligned} \mathrm{cutrk}_G(A_1) &= \mathrm{rank}(M[A_1, \overline{A}_1]) \\ &\leq \mathrm{rank}(M[A_4 \cup (A_1 \Delta A_4), \overline{A}_1]) \\ &\leq \mathrm{rank}(M[A_4, \overline{A}_1]) + \mathrm{rank}(M[A_1 \Delta A_4, \overline{A}_1]) \leq 4k. \end{aligned}$$

Therefore the width of $(T, \mathcal{L})$ is at most $4k$. □

Proof of Theorem 5.6. Let $\mathcal{C}$ be a set of graphs having a decidable C_2MS satisfiability problem and unbounded rank-width. We will get a contradiction.

The set $B(\mathcal{C})$ has unbounded rank-width by the above lemma. By applying the C_2MS transduction $\overline{\mu}$ of Theorem 6.5 to $B(\mathcal{C})$, we obtain an infinite set of graphs S_k among the vertex-minors of graphs in $B(\mathcal{C})$ by Corollary 3.5. Then by applying the MS transduction τ of Proposition 7.2, we get an infinite set of $(2k-2) \times k$ grids.

We now observe that these transformations preserve the decidability of C_2MS satisfiability, because B and τ are MS transductions, and $\overline{\mu}$ is a C_2MS transduction. But a set of graphs containing $(2k-2) \times k$ grids for infinitely many k has an undecidable MS satisfiability problem by Seese's Theorem [49]. We have reached a contradiction.

$$
\begin{tikzcd}
\textbf{Graphs} \arrow[r, "\text{MS trsd (1)}"] & \textbf{Bipartite Graphs} \arrow[r, "\text{vertex minor (2)}"] \arrow[d, "\text{C}_2\text{MS trsd (4)}"'] & \mathbf{S_k} \arrow[r, "\text{MS trsd (3)}"] & \textbf{Grids} \\
& \textbf{Isotropic Systems} \arrow[r, "\text{fundamental graphs (MS trsd) (4)}"] & \textbf{Graphs} \arrow[u, "\text{induced subgraphs (4) (MS trsd)}"'] &
\end{tikzcd}
$$

Fig. 10. Sketch of the first proof

Hence if $\mathcal{C}$ has a decidable C_2MS satisfiability problem, it must have bounded rank-width. It has also bounded clique-width. □

The proof is illustrated on Fig. 10: (1) is the MS transduction of Lemma 7.3, (2) is the vertex-minor reduction expressible by C_2MS formulas by means of isotropic systems (Theorem 6.5), and (3) is the MS transduction constructed in Proposition 7.2. The transformation from bipartite graphs to isotropic systems is a C_2MS transduction (Proposition 6.2) and the transformation from isotropic systems to their fundamental graphs is an MS transduction (Proposition 6.3).

Corollary 7.5. There exists a C_2MS transduction θ such that, if $\mathcal{C}$ is a set of graphs of unbounded clique-width or of unbounded rank-width, then $\theta(\mathcal{C})$ contains infinitely many square grids.

Proof. We let $\theta = Ind \circ \tau \circ \overline{\mu} \circ B$ where Ind is the MS transduction that associates with a graph the set of its induced subgraphs. It transforms the $(2k-2) \times k$ grid into a set of graphs containing the $k \times k$ grid. □

By using an MS transduction encoding directed graphs into bipartite graphs defined in [16], we can obtain a similar statement for directed graphs.

We now discuss extensions of Theorem 5.6.

Definition 7.6 (MS orderable classes of graphs (Courcelle [12])). We say that a class $\mathcal{C}$ of graphs is *MS orderable* if there exists a pair

$$(\delta(X_1, \dots, X_n), \sigma(x, y, X_1, \dots, X_n))$$

of MS formulas such that:

1) For every graph G in $\mathcal{C}$, there exist sets of vertices $X_1, \dots, X_n$ such that

$$(G, X_1, \dots, X_n) \models \delta,$$

2) For every n-tuple as above, the binary relation defined by

$$xRy \quad \text{if and only if} \quad (G, x, y, X_1, \ldots, X_n) \models \sigma$$

is a linear ordering of the set of vertices of G.

Theorem 7.7. If a set of graphs (respectively of directed graphs) is MS orderable and has a decidable MS satisfiability problem, then it has bounded clique-width.

Proof. If a set $\mathcal{C}$ of (directed) graphs is MS orderable and has a decidable MS satisfiability problem, then its C_2MS satisfiability problem is decidable (and even the CMS one is), and then we can conclude using Theorem 5.6.

The proof of this claim is as follows. Let φ be a CMS formula for which we ask whether it is satisfied by some graph in $\mathcal{C}$. Then we can rewrite it into an MS formula φ' by expressing the cardinality predicates in term of the linear order defined by σ. The formula φ' has thus free variables $X_1, \ldots, X_n$. Then, for every graph G in $\mathcal{C}$,

$$G \models \varphi \quad \text{if and only if} \quad G \models \exists X_1, \ldots, X_n(\delta(X_1, \ldots, X_n) \wedge \varphi').$$

From the initial hypothesis and since the formula $\exists X_1, \ldots, X_n(\delta(X_1, \ldots, X_n) \wedge \varphi')$ is MS (and not CMS), one can decide whether there exists a graph G in $\mathcal{C}$ such that $G \models \varphi$. □

Example 7.8. Consider the set $\mathcal{D}$ of directed graphs without circuits having a directed Hamiltonian path. The relation "$x = y$ or there exists a directed path from x to y" is a linear ordering and it is definable by an MS formula since MS formulas can express transitive closure. Hence $\mathcal{D}$ satisfies the conditions of Theorem 7.7 and therefore Seese's conjecture is true on $\mathcal{D}$.

The validity of the conjecture for $\mathcal{D}$ cannot be established with the methods of Courcelle [16], by reduction to the result of Robertson and Seymour [45] on excluded planar minors, because these methods apply only to sets of graphs having at most $2^{O(n\log(n))}$ graphs with n vertices. But $\mathcal{D}$ has $2^{(n-2)(n-3)/2}$ directed graphs with n vertices.

Theorem 5.6 extends easily to countable graphs. We first adapt the logical language. The Even predicate is only meaningful for finite sets. Hence, for countable structures, we will use the logical language C_2^fMS containing the following set predicates: $\mathsf{Finite}(X)$ which says that X is finite, and $\mathsf{Even}(X)$ which says that X is finite and has even cardinality. Then we can express that the cardinality of a set is odd by the formula $\mathsf{Finite}(X) \wedge \neg\,\mathsf{Even}(X)$.

The extension of Theorem 5.6 to countable graphs rests on the "compactness" theorem by Courcelle [15] stating that a set of countable graphs has bounded clique-width if and only if the set of all its finite induced subgraphs has bounded clique-width. We refer the reader to this paper for the definition of the clique-width of countable graphs. The above characterization is enough for the following

Theorem 7.9. If a set of finite or countable graphs has a decidable C_2^fMS satisfiability problem, then it has bounded clique-width.

Proof. The mapping associating with a graph the set of its finite induced subgraphs is a C_2^fMS transduction, because the finiteness set predicate makes it possible to restrict graphs to their finite induced subgraphs. Hence the set of finite induced subgraphs of the graphs in the set also has a decidable C_2MS satisfiability problem (by Proposition 5.3.1), hence bounded clique-width. So the set has bounded clique-width by the compactness theorem by Courclle [15]. □

8 Seese's Conjecture proved via matroids

We give another proof of Theorem 5.6 based on binary matroids instead of isotropic systems and using results by Hliněný and Seese [34]. They showed that if a set of matroids representable over a fixed finite field has a decidable monadic second-order theory, then it has bounded branch-width. The result of Geelen, Gerards, and Whittle [30] is essential to both proofs. We assume that matroids are given by their $\{\mathsf{Indep}\}$-structures, described in Section 5.1.

Since binary matroids are closely related to bipartite graphs, it is natural to show the following proposition.

Proposition 8.1. There is a C_2MS transduction with two parameters A and B that maps a bipartite graph G to the set of all binary matroids having G as a fundamental graph.

Proof. Let N be the adjacency matrix of G. Suppose that (A, B) is a bipartition of G and $\mathcal{M} = Bin(G, A, B)$. ($Bin$ is defined in Section 3.2.) The binary matroid $\mathcal{M}$ has a standard representation $P = \begin{pmatrix} I_A & N[A,B] \end{pmatrix}$. It is enough to show that we can express $\mathsf{Indep}(U)$ of $\mathcal{M}$ by a C_2MS logic formula in terms of the edg relation of G.

A subset U of $V(G)$ is independent in $\mathcal{M}$ if and only if columns of P are linearly independent. Thus, it is equivalent to say that there is no subset W of U such that the sum of column vectors of P indexed by elements of W is zero. We claim that we can write a C_2MS logic formula $\mathsf{Zero}(W)$ expressing

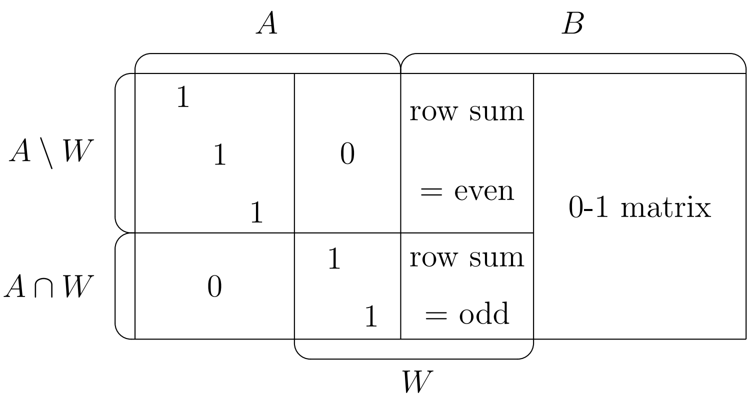


Fig. 11. $\mathsf{Zero}(W)$: the sum of column vectors in W is 0

that the sum of column vectors of P indexed by elements of W is zero. Since each row of P corresponds to an element of A, $\mathsf{Zero}(W)$ is true if and only if for each $x \in A$, the number of neighbors of x in W is odd if $x \in W$, and even otherwise (see Fig. 11). We may easily write this in a $\mathrm{C_2MS}$ logic formula. $\square$

Hliněný and Seese [34] proved the following proposition but stated in a different language.

Proposition 8.2 (Hliněný and Seese [34]). (1) The transduction associating with a matroid the set of its minors is an MS transduction.

(2) There exists an MS transduction ζ from matroids to graphs that maps the $(k-2) \times (k-2)$ grid to the cycle matroid of $k \times k$ grid for k even and at least six.

Proof. Assertion (1) is the content of Lemmas 6.4 and 6.5, and Assertion (2) is that Lemmas 6.6 and 6.7 of [34]. $\square$

Second proof of theorem 5.6. The method is similar to that of the first proof.

By Lemma 7.3 and Lemma 7.4, we need only consider a set $\mathcal{C}$ of bipartite graphs of unbounded rank-width having a decidable $\mathrm{C_2MS}$ satisfiability problem and derive a contradiction.

We will use the Proposition 3.1, which states that for a bipartite graph G with a bipartition $V(G) = A \cup B$, the branch-width of $Bin(G, A, B)$ is exactly one more than the rank-width of G.

Let us apply to $\mathcal{C}$ the transduction $\kappa = \zeta \circ Bin$. Then the set of matroids $Bin(\mathcal{C})$ has unbounded branch-width, hence, by a result of Geelen, Gerards, and Whittle [30], it contains the cycle matroids of $k \times k$ grids for infinitely many k. The transduction κ produces thus from $\mathcal{C}$ infinitely many square grids.

$$
\begin{tikzcd}
\textbf{Graphs} \arrow[d, "\text{MS trsd (1)}"'] & & \textbf{Grids} \\
\textbf{Bipartite Graphs} \arrow[r, "\text{C}_2\text{MS trsd (2)}"] & \textbf{Binary Matroids} \arrow[r, "\text{Matroid Minor (3)}"] & \textbf{Cycle matroids of grids} \arrow[u, "\text{MS trsd (4)}"]
\end{tikzcd}
$$

Fig. 12. Sketch of the second proof

Since we assume that $\mathcal{C}$ has a decidable C_2MS satisfiability problem, and since κ is a C_2MS transduction, then so has $\kappa(\mathcal{C})$. But it cannot contain infinitely many square grids. This is the desired contradiction. □

The schema of the proof is illustrated on Figure 12: (1) is the MS transduction of Lemma 7.3, (2) is the C_2MS transduction *Bin* of Proposition 8.1, the MS transductions of (3) and (4) are from [34].

9 Conclusion

We have shown how isotropic systems can be handled in C_2MS logic. Together with other results, we could prove a slight weakening of Seese's Conjecture and obtain polynomial-time algorithms for recognizing graphs of rank-width at most k, for each k. Some questions remain open.

Question 1. Is the original conjecture valid?

Question 2. Is it true that if a set of relational structures without set predicates has a decidable MS (or C_2MS) satisfiability problem, then it is contained in the image of a set of trees under an MS transduction (or a C_2MS transduction).

Even though the graphs of rank-width at most k are recognizable in polynomial time and $\mathrm{rwd}(G) \leq \mathrm{cwd}(G) \leq 2^{\mathrm{rwd}(G)+1} - 1$, this does not answer the following question for $k > 3$.

Question 3. For fixed $k > 3$, is there a polynomial-time algorithm recognizing graphs of clique-width at most k?

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References

[1] H.-J. Bandelt and H. M. Mulder. Distance-hereditary graphs. *J. Combin. Theory Ser. B*, 41(2):182–208, 1986.

[2] A. Bouchet. Isotropic systems. *European J. Combin.*, 8(3):231–244, 1987.

[3] A. Bouchet. Graphic presentations of isotropic systems. *J. Combin. Theory Ser. B*, 45(1):58–76, 1988.

[4] A. Bouchet. Transforming trees by successive local complementations. *J. Graph Theory*, 12(2):195–207, 1988.

[5] A. Bouchet. κ-transformations, local complementations and switching. In *Cycles and rays (Montreal, PQ, 1987)*, volume 301 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 41–50. Kluwer Acad. Publ., Dordrecht, 1990.

[6] A. Bouchet. Circle graph obstructions. *J. Combin. Theory Ser. B*, 60(1):107–144, 1994.

[7] H. Comon, M. Dauchet, R. Gilleron, F. Jacquemard, D. Lugiez, S. Tison, and M. Tommasi. Tree automata techniques and applications. Available on: `http://www.grappa.univ-lille3.fr/tata`, 1997. release October, 1rst 2002.

[8] D. G. Corneil, M. Habib, J.-M. Lanlignel, B. Reed, and U. Rotics. Polynomial time recognition of clique-width ≤ 3 graphs (extended abstract). In *Gonnet, Gastón H. (ed.) et al., LATIN 2000: Theoretical informatics. 4th Latin American symposium, Punta del Este, Uruguay, April 10-14, 2000.*, volume 1776 of *Lecture Notes in Comput. Sci.*, pages 126–134. Springer, Berlin, 2000.

[9] D. G. Corneil, Y. Perl, and L. K. Stewart. A linear recognition algorithm for cographs. *SIAM J. Comput.*, 14(4):926–934, 1985.

[10] B. Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Inform. and Comput.*, 85(1):12–75, 1990.

[11] B. Courcelle. Monadic second-order definable graph transductions: a survey. *Theoret. Comput. Sci.*, 126(1):53–75, 1994.

[12] B. Courcelle. The monadic second-order logic of graphs. X. Linear orderings. *Theoret. Comput. Sci.*, 160(1-2):87–143, 1996.

[13] B. Courcelle. The expression of graph properties and graph transformations in monadic second-order logic. In *Handbook of graph grammars and computing by graph transformation, Vol. 1*, pages 313–400. World Sci. Publishing, River Edge, NJ, 1997.

[14] B. Courcelle. The monadic second-order logic of graphs. XIV. Uniformly sparse graphs and edge set quantifications. *Theoret. Comput. Sci.*, 299(1-3):1–36, 2003.

[15] B. Courcelle. Clique-width of countable graphs: a compactness property. *Discrete Math.*, 266(1-3):127–148, 2004.

[16] B. Courcelle. The monadic second-order logic of graphs XV: On a conjecture by D. Seese. *J. Appl. Log.*, 1:79–114, 2006.

[17] B. Courcelle, R. G. Downey, and M. R. Fellows. A note on the computability of graph minor obstruction sets for monadic second order ideals. In *Proceedings of the First Japan-New Zealand Workshop on Logic in Computer Science (Auckland, 1997)*, volume 3, pages 1194–1198 (electronic), 1997.

[18] B. Courcelle and J. Engelfriet. A logical characterization of the sets of hypergraphs defined by hyperedge replacement grammars. *Math. Systems Theory*, 28(6):515–552, 1995.

[19] B. Courcelle, J. Engelfriet, and G. Rozenberg. Handle-rewriting hypergraph grammars. *J. Comput. System Sci.*, 46(2):218–270, 1993.

[20] B. Courcelle and S. Olariu. Upper bounds to the clique width of graphs. *Discrete Appl. Math.*, 101(1-3):77–114, 2000.

[21] B. Courcelle and P. Weil. The recognizability of sets of graphs is a robust property. *Theoret. Comput. Sci.*, 342(2-3):173–228, 2005.

[22] J. Doner. Tree acceptors and some of their applications. *J. Comput. System Sci.*, 4:406–451, 1970.

[23] R. G. Downey and M. R. Fellows. *Parameterized complexity.* Monographs in Computer Science. Springer-Verlag, New York, 1999.

[24] J. Engelfriet and V. van Oostrom. Logical description of context-free graph languages. *J. Comput. System Sci.*, 55(3):489–503, 1997.

[25] W. Espelage, F. Gurski, and E. Wanke. How to solve NP-hard graph problems on clique-width bounded graphs in polynomial time. In *Graph-theoretic concepts in computer science (Boltenhagen, 2001)*, volume 2204 of *Lecture Notes in Comput. Sci.*, pages 117–128, Berlin, 2001. Springer.

[26] S. Feferman and R. L. Vaught. The first order properties of products of algebraic systems. *Fund. Math.*, 47:57–103, 1959.

[27] M. R. Fellows, F. A. Rosamond, U. Rotics, and S. Szeider. Proving NP-hardness for clique-width II: non-approximability of clique-width. Report TR05-081, Revision 01, Electronic Colloquium on Computational Complexity, 2005.

[28] M. R. Fellows, F. A. Rosamond, U. Rotics, and S. Szeider. Clique-width minimization is NP-hard. In *Proceedings of the 38th ACM Symposium on Theory of Computing*, 2006. to appear.

[29] J. F. Geelen, A. M. H. Gerards, and G. Whittle. Branch-width and well-quasi-ordering in matroids and graphs. *J. Combin. Theory Ser. B*, 84(2):270–290, 2002.

[30] J. F. Geelen, A. M. H. Gerards, and G. Whittle. Excluding a planar graph from GF(q)-representable matroids. Research Report 03-4, School of Mathematical and Computing Sciences, Victoria University of Wellington, 2003.

[31] M. C. Golumbic and U. Rotics. On the clique-width of some perfect graph classes. *Internat. J. Found. Comput. Sci.*, 11(3):423–443, 2000. Selected papers from the Workshop on Theoretical Aspects of Computer Science (WG 99), Part 1 (Ascona).

[32] P. Hliněný. On matroid properties definable in the MSO logic. In *Mathematical foundations of computer science 2003*, volume 2747 of *Lecture Notes in Comput. Sci.*, pages 470–479. Springer, Berlin, 2003.

[33] P. Hliněný. Branch-width, parse trees, and monadic second-order logic for matroids. *J. Combin. Theory Ser. B*, 2006. to appear.

[34] P. Hliněný and D. Seese. Trees, grids, and MSO decidability: From graphs to matroids. *Theoret. Comput. Sci.*, 351(3):372–393, 2006.

[35] D. Kobler and U. Rotics. Edge dominating set and colorings on graphs with fixed clique-width. *Discrete Appl. Math.*, 126(2-3):197–221, 2003.

[36] J. A. Makowsky. Algorithmic uses of the Feferman-Vaught theorem. *Ann. Pure Appl. Logic*, 126(1-3):159–213, 2004.

[37] S. Oum. Approximating rank-width and clique-width quickly. In *Graph-theoretic concepts in computer science (Metz, 2005)*, volume 3787 of *Lecture Notes in Comput. Sci.*, pages 49–58. Springer, 2005.

[38] S. Oum. *Graphs of bounded rank-width.* PhD thesis, Princeton University, 2005.

[39] S. Oum. Rank-width and vertex-minors. *J. Combin. Theory Ser. B*, 95(1):79–100, 2005.

[40] S. Oum. Rank-width and well-quasi-ordering. submitted, 2005.

[41] S. Oum. Approximating rank-width and clique-width quickly. submitted, an extended abstract appeared in [37], 2006.

[42] S. Oum and P. Seymour. Approximating clique-width and branch-width. *J. Combin. Theory Ser. B*, 2006. to appear.

[43] J. G. Oxley. *Matroid theory.* Oxford University Press, New York, 1992.

[44] N. Robertson and P. Seymour. Graph minors. II. Algorithmic aspects of tree-width. *J. Algorithms*, 7(3):309–322, 1986.

[45] N. Robertson and P. Seymour. Graph minors. V. Excluding a planar graph. *J. Combin. Theory Ser. B*, 41(1):92–114, 1986.

[46] N. Robertson and P. Seymour. Graph minors. IV. Tree-width and well-quasi-ordering. *J. Combin. Theory Ser. B*, 48(2):227–254, 1990.

[47] N. Robertson and P. Seymour. Graph minors. X. Obstructions to tree-decomposition. *J. Combin. Theory Ser. B*, 52(2):153–190, 1991.

[48] N. Robertson and P. Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 92(2):325–357, 2004.

[49] D. Seese. The structure of the models of decidable monadic theories of graphs. *Ann. Pure Appl. Logic*, 53(2):169–195, 1991.

[50] J. W. Thatcher and J. B. Wright. Generalized finite automata theory with an application to a decision problem of second-order logic. *Math. Systems Theory*, 2:57–81, 1968.

[51] W. Thomas. Automata on infinite objects. In *Handbook of theoretical computer science, Vol. B: Formal models and semantics*, pages 133–191. Elsevier, Amsterdam, 1990.

[52] E. Wanke. k-NLC graphs and polynomial algorithms. *Discrete Appl. Math.*, 54(2-3):251–266, 1994.