GRAPHS OF SMALL RANK-WIDTH ARE PIVOT-MINORS OF GRAPHS OF SMALL TREE-WIDTH

O-JOUNG KWON AND SANG-IL OUM

Abstract. We prove that every graph of rank-width \( k \) is a pivot-minor of a graph of tree-width at most \( 2k \). We also prove that graphs of rank-width at most 1, equivalently distance-hereditary graphs, are exactly vertex-minors of trees, and graphs of linear rank-width at most 1 are precisely vertex-minors of paths. In addition, we show that bipartite graphs of rank-width at most 1 are exactly pivot-minors of trees and bipartite graphs of linear rank-width at most 1 are precisely pivot-minors of paths.

1. Introduction

Rank-width is a width parameter of graphs, introduced by Oum and Seymour [6], measuring how easy it is to decompose a graph into a tree-like structure where the “easiness” is measured in terms of the matrix rank function derived from edges formed by vertex partitions. Rank-width is a generalization of another, more well-known width parameter called tree-width, introduced by Robertson and Seymour [8]. It is well known that every graph of small tree-width also has small rank-width; Oum [7] showed that if a graph has tree-width \( k \), then its rank-width is at most \( k + 1 \). The converse does not hold in general, as complete graphs have rank-width 1 and arbitrary large tree-width.

Pivot-minor and vertex-minor relations are graph containment relations such that rank-width cannot increase when taking pivot-minors or vertex-minors of a graph [6]. Our main result is that for every graph \( G \) with rank-width at most \( k \) and \( |V(G)| \geq 3 \), there exists a graph \( H \) having \( G \) as a pivot-minor such that \( H \) has tree-width at most \( 2k \) and \( |V(H)| \leq (2k+1)|V(G)| - 6k \). Furthermore, we prove that for every graph \( G \) with linear rank-width at most \( k \) and \( |V(G)| \geq 3 \), there exists a graph \( H \) having \( G \) as a pivot-minor such that \( H \) has path-width at most \( k + 1 \) and \( |V(H)| \leq (2k+1)|V(G)| - 6k \).

As a corollary, we give new characterizations of two graph classes: graphs with rank-width at most 1 and graphs with linear rank-width at most 1. We show that a graph has rank-width at most 1 if and only if it is a vertex-minor of a tree. We also prove that a graph has linear rank-width at most 1 if and only if it is a vertex-minor of a path. Moreover, if the graph is bipartite, we prove that a vertex-minor relation can be replaced with a pivot-minor relation in both theorems. Table 1 summarizes our theorems.

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To prove the main theorem, we construct a graph having $G$ as a pivot-minor, called a rank-expansion. Then we prove that a rank-expansion has small tree-width.

The paper is organized as follows. We present the definition of rank-width and related operations in the next section. In Section 3, we define a rank-expansion of a graph and prove the main theorem. In Section 4, using a rank-expansion, we present new characterizations of graphs with rank-width at most 1 and graphs with linear rank-width at most 1.

2. Preliminaries

In this paper, all graphs are simple and undirected. Let $G = (V, E)$ be a graph. For $v \in V$, let $N(v)$ be the set of vertices adjacent to $v$ and $\deg(v) := |N(v)|$. And let $\delta(v)$ be the set of edges incident with $v$. For $S \subseteq V$, $G[S]$ denotes the subgraph of $G$ induced on $S$. For two sets $A$ and $B$, $A \Delta B = (A \cup B) \setminus (A \cap B)$.

A vertex partition of a graph $G$ is a pair $(A, B)$ of subsets of $V$ such that $A \cup B = V$ and $A \cap B = \emptyset$. A vertex $v \in V$ is a leaf if $\deg(v) = 1$; Otherwise we call it an inner vertex. An edge $e \in E$ is an inner edge if $e$ does not have a leaf as an end. Let $V_I(G)$ and $E_I(G)$ be the set of inner vertices of $G$ and inner edges of $G$, respectively.

For an $X \times Y$ matrix $M$ and subsets $A \subseteq X$ and $B \subseteq Y$, $M[X]_A$ denotes the $X \times Y$ matrix derived from $M$ by removing the rows corresponding to $X \setminus A$ and the columns corresponding to $Y \setminus B$. This operation is called a pivot.

**Pivoting matrices.** Let $M = \begin{pmatrix} X & V \setminus X \\ V \setminus X & A & B \\ C & D \end{pmatrix}$ be a symmetric or skew-symmetric $V \times V$ matrix over a field $F$. If $A = M[X]$ is nonsingular, then we define

$$M \ast X = \begin{pmatrix} X & V \setminus X \\ V \setminus X & A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1}B & D - CA^{-1}B \end{pmatrix}.$$

This operation is called a pivot. Tucker showed the following theorem.

**Theorem 2.1** (Tucker [9]). Let $M[X]$ be a nonsingular principal submatrix of a square matrix $M$. Then $M \ast X[Y]$ is nonsingular if and only if $M[X \Delta Y]$ is nonsingular.

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**Table 1. Summary of theorems**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Implication</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ has rank-width $\leq k$</td>
<td>$\Rightarrow$ $G$ is a pivot-minor of a graph of tree-width $\leq 2k$</td>
</tr>
<tr>
<td>$G$ has linear rank-width $\leq k$</td>
<td>$\Rightarrow$ $G$ is a pivot-minor of a graph of path-width $\leq k + 1$</td>
</tr>
<tr>
<td>$G$ has rank-width $\leq 1$</td>
<td>$\iff$ $G$ is a vertex-minor of a tree</td>
</tr>
<tr>
<td>$G$ has linear rank-width $\leq 1$</td>
<td>$\iff$ $G$ is a vertex-minor of a path</td>
</tr>
<tr>
<td>$G$ is bipartite and has rank-width $\leq 1$</td>
<td>$\iff$ $G$ is a pivot-minor of a tree</td>
</tr>
<tr>
<td>$G$ is bipartite has linear rank-width $\leq 1$</td>
<td>$\iff$ $G$ is a pivot-minor of a path</td>
</tr>
</tbody>
</table>

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Vertex-minors and pivot-minors. The graph obtained from $G = (V, E)$ by applying local complementation at a vertex $v$ is $G + v = (V, E \Delta \{xy : xv, yv \in E, x \neq y\})$. The graph obtained from $G$ by pivoting an edge $uv$ is defined by $G \setminus uv = G + u \cup v$. To see how we obtain the resulting graph by pivoting an edge $uv$, let $V_1 = N(u) \cap N(v)$, $V_2 = N(u) \setminus N(v) \{v\}$ and $V_3 = N(v) \setminus N(u) \{u\}$. One can easily verify that $G \setminus uv$ is identical to the graph obtained from $G$ by complementing adjacency of vertices between distinct sets $V_i$ and $V_j$ and swapping the vertices $u$ and $v$ [6]. See Figure 1 for example.

In fact, if $uv \in E$, then $A(G \setminus uv) = A(G) \ast \{u,v\}$. Since det $(A(G)(\{u,v\})) = A(G)(u,v)$, Theorem 2.1 is useful for dealing with a sequence of pivoting. In Figure 1, we can easily check that $G \setminus uv \setminus uc = G \setminus vc$. For $X \subseteq V$, if $A(G)(X)$ is nonsingular, then we denote $G \setminus X$ as the graph having the adjacency matrix $A(G) * X$.

A graph $H$ is a vertex-minor of $G$ if $H$ can be obtained from $G$ by applying a sequence of vertex deletions and local complementations. A graph $H$ is a pivot-minor of $G$ if $H$ can be obtained from $G$ by applying a sequence of vertex deletions and pivoting edges. From the definition, every pivot-minor of a graph is a vertex-minor of the graph. Note that every pivot-minor of a bipartite graph is bipartite.

Rank-width and linear rank-width. The cut-rank function $\text{cutrk}_G : 2^V \rightarrow \mathbb{Z}$ of a graph $G$ is defined by

$$\text{cutrk}_G(X) = \text{rank}(A(G)[X, V \setminus X]).$$

A tree is subcubic if it has at least two vertices and every inner vertex has degree 3. A rank-decomposition of a graph $G$ is a pair $(T, L)$, where $T$ is a subcubic tree and $L$ is a bijection from the vertices of $G$ to the leaves of $T$. For an edge $e$ in $T$, $T \setminus e$ induces a partition $(X_e, Y_e)$ of the leaves of $T$. The width of an edge $e$ is defined as $\text{cutrk}_G(L^{-1}(X_e))$. The width of a rank-decomposition $(T, L)$ is the maximum width over all edges of $T$. The rank-width of $G$, denoted by $\text{rw}(G)$, is the minimum width of all rank-decompositions of $G$. If $|V| \leq 1$, then $G$ admits no rank-decomposition and $\text{rw}(G) = 0$.

A subcubic tree is a caterpillar if it contains a path $P$ such that every vertex of a tree has distance at most 1 to some vertex of $P$. A linear rank-decomposition of a graph $G$ is a rank-decomposition $(T, L)$ of $G$, where $T$ is a caterpillar. The linear rank-width of $G$ is defined as the minimum width of all linear rank-decompositions of $G$. If $|V| \leq 1$, then $G$ admits no linear rank-decomposition and $\text{lwr}(G) = 0$. 

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
  $G$ & $G \setminus uv$ & $G \setminus uv \setminus uc$ \\
  \begin{tikzpicture}
  \node[vertex] (u) at (0,0) [label=left:$u$] {};
  \node[vertex] (v) at (1,0) [label=right:$v$] {};
  \node[vertex] (a) at (0,-1) [label=below:$a$] {};
  \node[vertex] (b) at (1,-1) [label=below:$b$] {};
  \node[vertex] (c) at (1,-2) [label=below:$c$] {};
  \draw (u) -- (v) -- (a) -- (b) -- (c) -- (u);
  \end{tikzpicture}
  & \begin{tikzpicture}
  \node[vertex] (v) at (1,0) [label=right:$v$] {};
  \node[vertex] (a) at (0,-1) [label=below:$a$] {};
  \node[vertex] (b) at (1,-1) [label=below:$b$] {};
  \node[vertex] (c) at (1,-2) [label=below:$c$] {};
  \draw (v) -- (a) -- (b) -- (c) -- (v);
  \end{tikzpicture}
  & \begin{tikzpicture}
  \node[vertex] (v) at (1,0) [label=right:$v$] {};
  \node[vertex] (a) at (0,-1) [label=below:$a$] {};
  \node[vertex] (c) at (1,-2) [label=below:$c$] {};
  \node[vertex] (u) at (1,-4) [label=below:$u$] {};
  \draw (v) -- (a) -- (c) -- (u) -- (v);
  \end{tikzpicture}
\end{tabular}
\caption{Pivoting an edge $uv$. Note that $G \setminus uv \setminus uc = G \setminus vc$.}
\end{figure}
Note that if a graph $H$ is a vertex-minor or a pivot-minor of a graph $G$, then $\text{rw}(H) \leq \text{rw}(G)$ and $\text{lrw}(H) \leq \text{lrw}(G)$ [6]. Trivially, $\text{rw}(G) \leq \text{lrw}(G)$.

**Tree-width and path-width.** Let $T$ be a tree, and let $B = \{B_t\}_{t \in V(T)}$ be a family of vertex sets $B_t \subseteq V$ indexed by the vertices $t \in V(T)$, called *bags*. The pair $(T, B)$ is called a *tree-decomposition* of $G$ if it satisfies the following three conditions.

1. $V = \bigcup_{v \in V(T)} B_v$.  
2. For every edge $uv \in E$, there exists a vertex $t$ of $T$ such that $u, v \in B_t$.  
3. For $t_1, t_2$ and $t_3 \in V(T)$, $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever $t_2$ is on the path from $t_1$ to $t_3$.

The *width* of a tree-decomposition $(T, B)$ is $\max(|B_t| - 1 : t \in V(T))$. The *tree-width* of $G$, denoted by $\text{tw}(G)$, is the minimum width of all tree-decompositions of $G$. A *path-decomposition* of a graph $G$ is a tree-decomposition $(T, B)$ where $T$ is a path. The *path-width* of $G$, denoted by $\text{pw}(G)$, is the minimum width of all path-decompositions of $G$.

3. **Rank-expansions and pivot-minors of graphs with small tree-width**

In this section, for a graph $G$ with rank-width $k$, we construct a graph having tree-width at most $2k$ such that it has $G$ as a pivot-minor.

**Theorem 3.1.** Let $k$ be a non-negative integer. Let $G$ be a graph of rank-width at most $k$ and $|V(G)| \geq 3$. Then there exists a graph $H$ having a pivot-minor isomorphic to $G$ such that tree-width of $H$ is at most $2k$ and $|V(H)| \leq (2k + 1)|V(G)| - 6k$.

**Theorem 3.2.** Let $k$ be a non-negative integer. Let $G$ be a graph of linear rank-width at most $k$ and $|V(G)| \geq 3$. Then there exists a graph $H$ having a pivot-minor isomorphic to $G$ such that path-width of $H$ is at most $k + 1$ and $|V(H)| \leq (2k + 1)|V(G)| - 6k$.

We need the following lemma.

**Lemma 3.3.** Let $G$ be a graph and $(A_1, B_1), (A_2, B_2)$ be two vertex partitions such that $A_2 \subseteq A_1$. Let $S \subseteq A_1$ be a set corresponding to a basis of row vectors in $A(G)[A_1, B_1]$. Then there exists a subset of $A_2$ representing a basis of row vectors in $A(G)[A_2, B_2]$ containing $S \cap A_2$.

**Proof.** Because $A_2 \subseteq A_1$, rows in $A(G)[S \cap A_2, B_2]$ are independent. Therefore we can extend $S \cap A_2$ to a basis of rows in $A(G)[A_2, B_2]$. \[\square\]

To prove Theorems 3.1 and 3.2, we construct a *rank-expansion* of a graph. Let $G$ be a connected graph and $(T, L)$ be a rank-decomposition of $G$. We fix a leaf $x \in V(T)$. For $e \in E(T)$, let $T_e$ be the component of $T \setminus e$ which does not contain $x$, and let $A_e = L^{-1}[V(T_e)], B_e = V(G) \setminus A_e$ and $M_e = A(G)[A_e, B_e]$. For each $a \in A_e$, let $R^a_e = M_e[\{a\}, B_e]$ the row vector of $M_e$.

First, for each edge $e = uv \in E(T)$, we orient the edge towards $v$ if $v \in V(T_e)$. We choose a vertex set $U_e \subseteq A_e$ such that $\{R^a_e\}_{a \in U_e}$ forms a basis of row vectors in $M_e$ and $(U_e \cap A_f) \subseteq U_f$ if the tail of an edge $f$ is the head of $e$. Since $R^a_e$ can be uniquely expressed as a linear combination of vectors of $\{R^a_e\}_{a \in U_e}$ for each $a \in A_e$, there exists a unique $A_e \times U_e$ matrix $P_e$ such that $P_e A(G)[U_e, B_e] = A(G)[A_e, B_e]$. If the tail of an edge $f$ is the head of an edge $e$, then let $C_f = P_e[U_f, U_e]$.
Let $H$ be a rank-expansion $R(G, T, L, x, \{U_f\}_{f \in E(T)})$ of a graph $G$ such that

$$V(H) = \bigcup_{v \in V_f(T)} \bigcup_{e \in \delta(v)} (U_e \times \{e\} \times \{v\})$$

$$E(H) = \{(a, e, v) : e = vw \in E_f(T), a \in U_e\}$$

$$\cup \{(a, e, (b, f, v)) : v \in V_f(T), e, f \in E(T), v \text{ is the head of } e \text{ and the tail of } f,$$

$$a \in U_f, b \in U_e \text{ and } C_f(a, b) \neq 0\}$$

$$\cup \{(a, f_1, v), (b, f_2, v) : v \text{ is the tail of both } f_1 \text{ and } f_2 \in E(T),$$

$$a \in U_{f_1}, b \in U_{f_2} \text{ and } ab \in E(G)\}.$$  

For $v \in V_f(T)$, let $S_v = \bigcup_{e \in \delta(v)} U_e \times \{e\} \times \{v\} \subseteq V(H)$. For $e = vw \in E_f(T)$, let 

$$\pi = \{(a, e, v), (a, e, w) : a \in U_e\} \subseteq V(H)$$

and for $W \subseteq \pi$, let $W = \bigcup_{e \in E_T(W)} W \subseteq \pi$. If $e \in E_f(T)$ is directed from $w$ to $v$, let $L_e = S_v \cap \pi$ and $R_e = S_v \cap \pi$. For a vertex $a$ in $V(G)$ and $e = \{L(a), v\} \in E(T)$, let $\pi$ be the unique vertex in $U_e \times \{e\} \times \{v\}$ and let $\pi = \pi$.

We discuss the number of vertices in the rank-expansion $H$. We easily observe that $|E_f(T)| = |V(G)| - 3$. So if $rw(G) \leq k$, then $|\pi| \leq 2k$ for each $e \in E_f(T)$, and we deduce that $|V(H)| \leq 2k|E_f(T)| + |V(G)| = 2k(|V(G)| - 3) + |V(G)| = (2k + 1)|V(G)| - 6k$.

First, we prove that every rank-expansion of a graph has the given graph as a pivot-minor. To obtain $G$ as a pivot-minor of $H$, we will pivot $\bigcup_{e \in E_f(T)} \pi$ to $H$.

Lemma 3.4. Let $G$ be a graph and $uv \in E(G)$. If $\deg(u) = 1$, then $G \setminus uv \setminus \{u, v\} = G \setminus \{u, v\}$.

Proof. It is clear from the definition. \qed

For convenience, let $\det(A(H)(\emptyset)) = 1.$

Lemma 3.5. Let $W \subseteq E_f(T)$. Then $A(H)[W]$ is nonsingular.

Proof. We proceed by induction on $|W|$. If $W$ is empty, then it is trivial. If $|W| \geq 1$, then $W$ induces a forest in $T$, and therefore there must be an edge $f \in W$ which has a leaf in $T[W]$. By induction hypothesis, $A(H)[W \setminus \{f\}]$ is nonsingular. Since
every edge in $H[\bar{f}]$ is incident with a leaf in $H[\bar{W}]$, by Lemma 3.4, pivoting all edges in $\bar{f}$ does not change the graph $H[\bar{W}\setminus \{f\}]$. So, $A(H[\bar{W} \setminus \{f\}]) = A(H)[\bar{W}\setminus \{f\}]$ and therefore, by Theorem 2.1, $A(H)[\bar{W}\setminus \{f\}]$ is nonsingular. □

**Lemma 3.6.** Let $a, b \in V(G)$ and let $P$ be a path from $L(a)$ to $L(b)$ in $T$. Then for $E(P) \cap E(T) \subseteq W \subseteq E(T)$, $A(H)[W \setminus \{a, b\}]$ is nonsingular if and only if $A(H)[E(P)]$ is nonsingular.

**Proof.** We use induction on $|W|$. If $W = E(P) \cap E(T)$, then it is trivial, because $\bar{W} \setminus \{a, b\} = E(P)$. So we may assume that $|W| > |E(P) \cap E(T)|$. Since $P$ is a maximal path in $T$, the subgraph of $T$ having the edge set $W \cup E(P)$ must have at least 3 leaves. Thus there is an edge $f$ in $W \cup E(P)$ incident with a leaf in $T[W \cup E(P)]$ other than $L(a)$ and $L(b)$. Since every edge in $\bar{f}$ is incident with a leaf in $H[\bar{W}]$, by Lemma 3.4, $A(H[\bar{W} \cup \{a, b\}] \setminus \bar{f})[\bar{W}\setminus \{f\} \cup \{a, b\}] = A(H)[\bar{W}\setminus \{f\} \cup \{a, b\}]$. By induction hypothesis and Theorem 2.1 we deduce that

$A(H)[E(P)]$ is nonsingular $\iff A(H)[W\setminus \{f\} \cup \{a, b\}]$ is nonsingular

$\iff A(H)[W \setminus \{a, b\}] \setminus \bar{f})[\bar{W}\setminus \{f\} \cup \{a, b\}]$ is nonsingular

$\iff A(H)[W \cup \{a, b\}]$ is nonsingular. □

**Lemma 3.7.** Let $P = (e_{n+1}, e_n, \ldots, e_1)$ be the directed path from $w$ to $v$ in $T$. Then $C_{e_1}C_{e_2}\cdots C_{e_n}A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}]$.

**Proof.** We proceed by induction on $n$. If $n = 1$, then by definition, $C_{e_1}A(G)[U_{e_2}, B_{e_2}] = P_{e_2}[U_{e_1}, U_{e_2}]A(G)[U_{e_1}, B_{e_2}] = A(G)[U_{e_1}, B_{e_2}]$. We may assume that $n \geq 2$. By induction hypothesis, $C_{e_1}C_{e_2}\cdots C_{e_n}A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = A(G)[U_{e_2}, B_{e_{n+1}}]$. Since $C_{e_1}A(G)[U_{e_2}, B_{e_2}] = A(G)[U_{e_1}, B_{e_2}]$ and $B_{e_{n+1}} \subseteq B_{e_2}$, $C_{e_1}A(G)[U_{e_2}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}]$. Therefore, we conclude that $C_{e_1}C_{e_2}\cdots C_{e_n}A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = C_{e_1}A(G)[U_{e_2}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}]$. □
Lemma 3.8.

\[
\begin{vmatrix}
0 & C_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_{n+1} \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \\
\end{vmatrix}
= (-1)^n \det(C_1 C_2 \ldots C_{n+1}).
\]

Proof. By elementary row operation,

\[
\begin{vmatrix}
0 & C_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_{n+1} \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \\
\end{vmatrix}
= \det
\begin{vmatrix}
0 & 0 & -C_1 C_2 & 0 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_{n+1} \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \\
\end{vmatrix}
= \det
\begin{vmatrix}
0 & 0 & 0 & (-1)^2 C_1 C_2 C_3 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_{n+1} \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \\
\end{vmatrix}
= \det
\begin{vmatrix}
(-1)^n C_1 C_2 \cdots C_{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & I & C_{n+1} \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \\
\end{vmatrix}
= (-1)^n \det(C_1 C_2 \ldots C_{n+1}). \qed
Proposition 3.9. Let $k \geq 1$. Let $G$ be a connected graph with rank-width $k$ and $|V(G)| \geq 3$. Then a rank-expansion of $G$ has a pivot-minor isomorphic to $G$.

Proof. Let $(T, L)$ be a rank decomposition of a graph $G$ and let $x$ be a leaf in $T$. We orient each edge $f$ away from $x$. For each $f \in E(T)$, if $m$ is the width of $f$, we choose a basis $U_f = \{u_{1f}', u_{2f}', \ldots, u_{mf}'\} \subseteq A_f$ of rows in the matrix $A(G)[A_f, B_f]$ such that $(U_e \cap A_f) \subseteq U_f$ if the head of an edge $e$ is the tail of $f$. Since $G$ is connected, $|U_f| \geq 1$. Let $H$ be a rank-expansion $R(G, T, L, x, \{U_f\}_{f \in E(T)})$ of a graph $G$. By Lemma 3.4, for every $W \subseteq E_I(T)$, $A(H)[W]$ is nonsingular. We will prove that for $a, b \in V(G)$, $\overline{a b} \in E(H \setminus E_I(T))$ if and only if $a b \in E(G)$.

Let $a, b$ be distinct vertices in $V(G)$. We consider the path $P$ from $L(a)$ to $L(b)$ in $T$. By Lemma 3.6, $\overline{a b}$ is adjacent to $\overline{b a}$ in $H \setminus E_I(T)$ if and only if $\overline{a b}$ is adjacent to $\overline{b a}$ in $H[E(P)] \setminus (E(P) \cap E_I(T))$. Therefore, by Theorem 2.1,

$$\overline{a b} \in E(H \setminus E_I(T)) \iff \overline{a b} \in E(H[E(P)] \setminus (E(P) \cap E_I(T)))$$

$$\iff A\bigg((H[E(P)] \setminus (E(P) \cap E_I(T))) \setminus \{(\overline{a b})\}\bigg) \text{ is nonsingular}$$

$$\iff A\bigg((H[E(P)]) \setminus ((E(P) \cap E_I(T)) \Delta \{(\overline{a b})\})\bigg) \text{ is nonsingular}$$

$$\iff A(H[E(P)]) \text{ is nonsingular.}$$

Thus, it is enough to show that $\det(A(H[E(P)])) = A(G)[a, b]$. If $L(b) = x$, then $P = (c_{n+1}, c_n, \ldots, c_1, c_0)$ is a directed path from $L(b)$ to $L(a)$. The submatrix of $A(H)$ induced by $E(P)$ is

$$= \begin{pmatrix} C & 0 \\ 0 & C' \end{pmatrix}.$$
\[ |U_e| \text{ for all edges } e \in E(T), \ A(G)[U_{e+n}, B_{e+n}] = 1. \text{ By Lemma 3.} \]
\[ C_{e_0}C_{e_1} \ldots C_{e_n} = C_{e_0}C_{e_1} \cdots C_{e_n}A(G)[U_{e+n}, B_{e+n}] = A(G)[U_{e_0}, B_{e+n}] = A(G)(a, b). \]
Therefore \( \det(A(H)[E(P)]) = A(G)(a, b) \), as required.

Now we assume that \( L(a) \neq x \) and \( L(b) \neq x \). Then there exists a vertex \( y \) in \( V(P) \) such that it has a shortest distance to \( x \). Let \( P_1 = (e_n, e_{n-1}, \ldots, e_0) \) be the edges of \( P \) from \( y \) to \( L(a) \) and \( P_2 = (f_m, f_{m-1}, \ldots, f_0) \) be the edges of \( P \) from \( y \) to \( L(b) \).

Let \( M = A(H)[R_{e_n}, R_{f_m}] \). By the construction of a rank-expansion, \( M = A(G)[U_{e_n}, U_{f_m}] \). The submatrix of \( A(H) \) induced by \( E(P) \) is
\[
\begin{pmatrix}
\{\pi\} \cup \bigcup_{i=1}^{n} R_{e_i} & \{\pi\} \cup \bigcup_{i=1}^{m} L_{f_i} \\
\{\bar{b}\} \cup \bigcup_{i=1}^{n} L_{e_i} & C
\end{pmatrix}
\]
where \( C \) is
\[
\begin{pmatrix}
0 & L_{e_1} & L_{e_2} & \cdots & L_{e_{n-1}} & L_{e_n} & R_{f_m} & R_{f_{m-1}} & \cdots & R_{f_2} & R_{f_1} \\
0 & I & C_{e_1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I & C_{e_{n-1}} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I & M & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & I & C_{f_{m-1}} & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & I & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I & C_{f_1} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & I
\end{pmatrix}
\]

It is enough to show that \( C_{e_0}C_{e_1} \cdots C_{e_{n-1}}MC_{f_{m-1}}^{t}C_{f_{m-2}}^{t} \cdots C_{f_0}^{t} = A(G)(a, b) \).
Since \( M = A(G)[U_{e_n}, U_{f_m}] \leq A(G)[U_{e_n}, B_{e_n}] \), by Lemma 3.7 we have
\[
C_{e_0}C_{e_1} \cdots C_{e_{n-1}}MC_{f_{m-1}}^{t}C_{f_{m-2}}^{t} \cdots C_{f_0}^{t} = A(G)[U_{e_0}, U_{f_m}]C_{f_{m-1}}^{t}C_{f_{m-2}}^{t} \cdots C_{f_0}^{t}
\]
\[
= A(G)[U_{e_0}, U_{f_m}]C_{f_{m-1}}^{t}C_{f_{m-2}}^{t} \cdots C_{f_0}^{t} = (C_{f_0}C_{f_1} \cdots C_{f_{m-1}}C_{e_0}^{t})^{t} \]
\[
= A(G)[U_{f_0}, U_{e_0}]^{t} = A(G)(a, b).
\]

So, \( \det(A(H)[E(P)]) = A(G)(a, b) \), as claimed. Therefore, \( \bar{\pi} \bar{b} \in E(H \wedge E(T)) \) if and only if \( ab \in E(G) \). We conclude that a rank-expansion of \( G \) has a pivot-minor isomorphic to \( G \). \( \square \)

In the next proposition, we show that a rank-expansion has tree-width at most \( 2k \) when \( rw(G) \leq k \).
Proposition 3.10. Let $k \geq 1$. Let $G$ be a connected graph with $|V(G)| \geq 3$. If $G$ has rank-width $k$, then $G$ has a rank-expansion of tree-width at most $2k$. Moreover, if $G$ has linear rank-width $k$, then $G$ has a rank-expansion of path-width at most $k + 1$.

Proof. Let $(T, L)$ be a rank-decomposition of $G$ of width $k$. We fix a leaf $x \in V(T)$ and orient each edge $f$ away from $x$. For each $f \in E(T)$, if $m$ is the width of $f$, we choose a basis $U_f = \{u_1, u_2, \ldots, u_m\} \subseteq A_f$ of rows in the matrix $A(G)[A_f, B_f]$ such that $(U_e \cap A_f) \subseteq U_f$ if the head of an edge $e$ is the tail of $f$. Since $G$ is connected, $|U_f| \geq 1$. Let $H$ be a rank-expansion $R(G, T, L, x, \{U_f\}_{f \in E(T)})$ of a graph $G$.

Let $T'$ be a tree obtained from $T[V_i(T)]$ by replacing each edge from $v$ to $w$ with a path $wz_y^vz_y^w \ldots z_y^{v_{U_j|}}p_y^{v_{U_j|}}p_y^{v_{U_i|}} \ldots p_y^{v_i}$. Let $y$ be the neighbor of $v$ in $T$ and let $B(y) = S_y$. For $v \in V_i(T) \setminus \{y\}$, let $e = vw$ be the edge incoming to $v$ and $f_1, f_2$ be edges outgoing from $v$. Let $R^v = \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a \notin U_e\}$. Since $(U_e \cap A_{f_1}) \subseteq U_{f_1}$ for each $i \in \{1, 2\}$, each vertex in $L_e$ has exactly one neighbor in $R_{f_1} \cup R_{f_2} \setminus R^v$. Let $B(v) = R_{f_1} \cup R_{f_2}$ and $B(z_y^v) = R_e \cup \{\{u_i^v, e, v\}\}$. $B(p_i^v) = R^v \cup L_e \cup \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a = u_i^v\}$. And for each $2 \leq i \leq |U_e|$, we define

\[B(z_y^{u_i^v}) = B(z_y^{u_{i-1}^v}) \setminus \{(u_{i-1}^v, e, w)\} \cup \{(u_i^v, e, v)\}\]
\[B(p_i^v) = B(p_{i-1}^v) \setminus \{(u_{i-1}^v, e, v)\} \cup \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a = u_i^v\}\]

Now we show that the pair $(T', \{B(v)\}_v\in V(T'))$ is a tree-decomposition of $H$. Note that for each $v \in V_i(T) \setminus \{y\}$ with the incoming edge $e$, $E(H[B(z_y^v)]) = E(H[v])$ and $E(H[B(p_i^v)]) = E(H[S_y])$. Therefore all vertices and all edges in $H$ are covered by $B(v)$ for some $v \in V(T')$. So the first and second axioms of a tree-decomposition are satisfied.

For the third axiom, it suffices to show that for every $t \in V(H)$, $T'[\{z : B(z) \ni t\}]$ is a subtree of $T'$. Let $t = \{u_j^v, e, v\} \in V(H)$ for some $e = vw \in E(T)$ and $1 \leq j \leq |U_e|$. If $v$ is the head of $e$, $T'[\{z : B(z) \ni t\}] = T'[\{z_y^v, \ldots, z_y^{u_{j-1}^v}, p_y^{u_j^v}, \ldots, p_y^{v_i}\}]$, and it forms a path. Suppose $v$ is the tail of $e$. Let $f$ be the edge incoming to $v$, and if $a \in U_f$, then let $h$ be the integer such that $a = u_h^v$, if otherwise, let $h = 1$. Then $T'[\{z : B(z) \ni t\}] = T'[\{p_y^{u_0^v}, \ldots, p_y^{u_{j-1}^v}, v, z_y^v, \ldots, z_y^w\}]$. It also forms a path, thus $(T', \{B(v)\}_v\in V(T'))$ is a tree-decomposition of $H$. 

\[\begin{array}{c}
U_e = \{a_4, a_5, a_7\} \\
U_{f_1} = \{a_4, a_5\} \\
U_{f_2} = \{a_6, a_7\}
\end{array}\]
Since $|B(y)| \leq 2k + 1$ and for each $v \in V_1(T) \setminus \{y\}$ with the incoming edge $e$, $|B(z^i_v)| = |B(p^i_v)| = |R_e| + 1 \leq k + 1$, $|B(p^i_v)| = |B(p^i_y)| = |R_e| + |L_e| + 1 \leq (2k - |U_e|) + |U_e| + 1 = 2k + 1$ and $|B(v)| \leq 2k$, the resulting tree-decomposition has width at most $2k$.

Suppose that $G$ has linear rank-width at most $k$. Here, we choose $x \in V(T)$ such that $x$ is an end of a longest path in $T$, and let $y$ be the neighbor of $x$. For $v \in V_1(T)$ with outgoing edges $f_1$ and $f_2$, $|U_{f_1}| = 1$ or $|U_{f_2}| = 1$ because every inner vertex of $T$ is incident with a leaf. Therefore, for each $v \in V_1(T) \setminus \{y\}$ and $1 \leq i \leq |U_e|$, $|B(p^i_v)| \leq (k + 1 - |U_e|) + |U_e| + 1 = k + 2$ and $|B(v)| \leq 2k + 1$, and $|B(y)| \leq k + 2$. Moreover, since $T[V_1(T)]$ is a path, $T'$ is also a path. Therefore $(T', \{B(v)\}_{v \in V(T')})$ is a path-decomposition of $H$ with path-width at most $k + 1$.

**Proof of Theorem 3.1.** If $k = 0$, then it is trivial. We assume that $k \geq 1$. We proceed by induction on the number of vertices.

Suppose $G$ is connected. Since $G$ has rank-width at most $k$ and $|V(G)| \geq 3$, by Proposition 3.10 there is a rank-expansion $H$ of $G$ such that $\text{tw}(H) \leq 2k$, and $|V(H)| \leq (2k + 1)|V(G)| - 6k$. By Proposition 3.9, $H$ has a pivot-minor isomorphic to $G$.

If $G$ is disconnected, then we choose a largest component $Y$ of $G$. Since $k \geq 1$, the component $Y$ has at least 2 vertices. If $|V(Y)| = 2$, then $G$ has rank-width 1 and tree-width 1, and $|V(G)| \leq (2 + 1)|V(G)| - 6$ since $|V(G)| \geq 3$. We assume that $|V(Y)| \geq 3$. Then by induction hypothesis, there is a graph $H_1$ such that $Y$ is isomorphic to a pivot-minor of $H_1$ and $\text{tw}(H_1) \leq 2k$ and $|V(H_1)| \leq (2k + 1)|V(Y)| - 6k$.

If $G \setminus V(Y)$ has tree-width at most 1, then $G$ is isomorphic to a pivot-minor of the disjoint union of two graphs $H_1$ and $G \setminus V(Y)$, and the tree-width of it is equal to the tree-width of $H_1$. Since $|V(H_1)| + |V(G) \setminus V(Y)| \leq (2k + 1)|V(Y)| - 6k + |V(G) \setminus V(Y)| \leq (2k + 1)|V(G)| - 6k$, we obtain the result. If tree-width of $G \setminus V(Y)$ is at least 2, then $|V(G) \setminus V(Y)| \geq 3$. Therefore, by induction hypothesis, there is a graph $H_2$ such that $G \setminus V(Y)$ is isomorphic to a pivot-minor of $H_2$ and $\text{tw}(H_2) \leq 2k$ and $|V(H_2)| \leq (2k + 1)|V(G) \setminus V(Y)| - 6k$. So $G$ is isomorphic to a pivot-minor of the disjoint union of two graphs $H_1$ and $H_2$, and the tree-width of

![Figure 5. Tree-decomposition of a rank-expansion in Figure 3.](image-url)
it is at most $2k$, and $|V(H_1)| + |V(H_2)| \leq (2k + 1)|V(G)| - 6k$. Thus, we conclude the theorem. 

**Proof of Theorem 3.2.** We can easily obtain the proof of Theorem 3.2 from the proof of Theorem 3.1. 

4. Graphs with rank-width or linear rank-width at most 1

Distance-hereditary graphs are introduced by Bandelt and Mulder [2]. A graph $G$ is *distance-hereditary* if for every connected induced subgraph $H$ of $G$ and vertices $a$, $b$ in $H$, the distance between $a$ and $b$ in $H$ is the same as in $G$. Oum [6] showed that distance-hereditary graphs are exactly graphs of rank-width at most 1. Recently, Ganian [5] obtain a similar characterization of graphs of linear rank-width 1. In this section, we obtain another characterization for these classes in terms of vertex-minor relation.

Note that every tree has rank-width at most 1 and every path has linear rank-width at most 1.

**Theorem 4.1.** Let $G$ be a graph. The following are equivalent:

1. $G$ has rank-width at most 1.
2. $G$ is distance-hereditary.
3. $G$ has no vertex-minor isomorphic to $C_5$.
4. $G$ is a vertex-minor of a tree.

**Proof.** ((1) $\iff$ (2)) is proved by Oum [6], and ((2) $\iff$ (3)) follows from the Bouchet’s theorem [3, 4]. Since every tree has rank-width at most 1, ((4) $\implies$ (1)) is trivial.

We want to prove that (1) implies (4).

Let $G$ be a graph of rank-width at most 1. We may assume that $G$ is connected. If $|V(G)| \leq 2$, then $G$ itself is a tree. So we may assume that $|V(G)| \geq 3$. Let $(T, L)$ be a rank-decomposition of $G$ of width 1. From Proposition 3.9, a rank-expansion $H$ with the rank-decomposition $(T, L)$ has $G$ as a pivot-minor.

The width of each edge in $T$ is 1. Thus for $v \in V_l(T)$, the subgraph $H[S_v]$ is a path of length 2 or a triangle because $G$ is connected. Also for $e \in E_l(T)$, $H[e]$ consists of an edge. Therefore $H$ is connected and does not have cycles of length at least 4.

Let $Q$ be a tree obtained from $H$ by replacing each triangle $abc$ with $K_{1,3}$ by adding a new vertex $d$, making $d$ adjacent to $a$, $b$, $c$ and deleting $ab$, $bc$, $ca$. Clearly $H$ is a vertex-minor of the tree $Q$ because we can obtain the graph $H$ from $Q$ by applying local complementation on those new vertices and deleting them. Therefore $G$ is a vertex-minor of a tree, as required. 

We also obtain a characterization of graphs with linear rank-width at most 1. Obstructions sets for graphs of linear rank-width 1 are $C_5$, $N$ and $Q$ [1], depicted in Figure 6.

**Lemma 4.2.** Every subcubic caterpillar is a pivot-minor of a path.

**Proof.** Let $H$ be a subcubic caterpillar. By the definition of a caterpillar, there is a path $P$ in $H$ such that every vertex in $V(H) \setminus V(P)$ is a leaf. We choose such path $P = p_1p_2 \ldots p_m$ in $H$ with maximum length. We construct a path $Q$ from $P$ by replacing each edge $p_ip_{i+1}$ with a path $p_ia_ib_ip_{i+1}$. We can obtain a pivot-minor of $P$ isomorphic to $Q$ by pivoting each edge $a_ib_i$ and deleting all $a_i$ and deleting $b_i$ if $p_i$ is not adjacent to a leaf in $H$. 

\[ \square \]
Theorem 4.3. Let $G$ be a graph. The following are equivalent:

1. $G$ has linear rank-width at most 1.
2. $G$ has no vertex-minor isomorphic to $C_5$, $N$ or $Q$.
3. $G$ is a vertex-minor of a path.

Proof. $(1 \iff 2)$ is proved by Adler, Farley and Proskurowski [1]. Since every path has linear rank-width at most 1, $(3 \Rightarrow 1)$ is trivial. Let us prove that $(1)$ implies $(3)$.

Let $G$ be a graph of linear rank-width at most 1. We may assume that $G$ is connected and $|V(G)| \geq 3$. Let $H$ be a rank-expansion of $G$ with a linear rank-decomposition $(T, L)$ of width 1. Note that $T$ is a caterpillar.

Since $(T, L)$ is a linear rank-decomposition of width 1, for each triangle in $H$, one of those vertices is of degree 2 in $H$. Let $P$ be a caterpillar obtained from $H$ by replacing each triangle with a path of length 2 whose internal vertex has degree 2 in $H$. We can obtain $H$ from $P$ by applying local complementation on the inner vertex of those paths of length 2, $H$ is a vertex-minor of $P$. And by Lemma 4.2, $P$ is a pivot-minor of a path. Therefore $G$ is a vertex-minor of a path. □

In Theorems 4.1 and 4.2 if a given graph is bipartite, we do not need to apply local complementation at some vertices. To prove it, we need the following lemma.

**Lemma 4.4.** Let $G$ be a connected bipartite graph with rank-width 1 and $|V(G)| \geq 3$. Let $(T, L)$ be a rank-decomposition of width 1. Then a rank-expansion of $G$ with respect to $(T, L)$ is a tree.

Proof. Let $x \in V(T)$ be a leaf and $H$ be a rank-expansion $R(G, T, L, x, \{U_f\}_{f \in E(T)})$ of $G$.

Suppose that $H$ has a triangle. Then there exists a vertex $v \in V_l(T)$ such that $H[S_v]$ is the triangle. Let $e_1, e_2$ and $e_3$ be edges incident with $v$ and assume that $e_1$
is the incoming edge. Let \( U_{e_1} = \{a\} \), \( U_{e_2} = \{b\} \) and \( U_{e_3} = \{c\} \). By the construction of a rank-expansion, \( bc \in E(G) \) and \( R_{a}^{e_1} = R_{b}^{e_1} = R_{c}^{e_1} \). Since \( R_{a}^{e_1} \) is a non-zero vector, there is a vertex \( x \in V(G) \) such that \( x \) is adjacent to all of \( a \), \( b \) and \( c \). Therefore \( xbc \) is a triangle in \( G \), contradiction.

\[ \text{Theorem 4.5. Let } G \text{ be a graph. Then } G \text{ is bipartite and has rank-width at most 1 if and only if } G \text{ is a pivot-minor of a tree.} \]

\[ \text{Proof. We may assume that } G \text{ is connected. Since every tree has rank-width at most 1, backward direction is trivial. If } G \text{ is bipartite and has rank-width at most 1, then by Lemma 4.4, we have a rank-expansion of } G \text{ which is a tree. Hence, } G \text{ is a pivot-minor of a tree.} \]

\[ \text{Theorem 4.6. Let } G \text{ be a graph. Then } G \text{ is bipartite and has linear rank-width 1 if and only if } G \text{ is a pivot-minor of a path.} \]

\[ \text{Proof. We may assume that } G \text{ is connected. Similarly, backward direction is trivial. Suppose } G \text{ is bipartite and has linear rank-width 1. Let } H \text{ be a rank-expansion of } G \text{ with a linear rank-decomposition } (T, L) \text{ of width 1. By Lemma 4.1, the graph } H \text{ is a tree, and since } T \text{ is a caterpillar, } H \text{ is also a caterpillar. By Lemma 4.2, } H \text{ is a pivot-minor of a path, and so is } G. \]

References