A Polynomial Kernel for Block Graph Vertex Deletion

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Joint work with

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IPEC 2015
17th, Sep, 2015
Feedback Vertex Set

Input: A graph $G = (V, E)$, an integer $k$

Parameter: $k$

Question: $\exists S \subseteq V$ with $|S| \leq k$ such that $G - S$ is a forest?
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Can we obtain similar results for bigger interesting graph classes?
Block graph

**Blocks**: maximal connected subgraphs having no cut vertices.

A graph is a **block graph** if every its block is a complete graph.

A graph is a block graph if and only if it has no diamonds and cycles of length at least 4.

Forests $\nsubseteq$ Block graphs $\nsubseteq$ (Chordal graphs) $\cap$ (Distance-hereditary graphs)
**Block Graph Deletion**

**Input:** A graph \( G = (V, E) \), an integer \( k \)

**Parameter:** \( k \)

**Question:** \( \exists S \subseteq V \) with \( |S| \leq k \) such that \( G - S \) is a block graph?
Theorem (Kim, K 15)

- **Block Graph Deletion** can be solved in time $10^k \cdot n^{O(1)}$.
- **Block Graph Deletion** admits a kernel of size $O(k^7)$.

We concentrate on the kernelization algorithm.
Theorem (Kim, K 15)

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Tools:

1. **Gallai’s A-path theorem.**
   - finding a (small) hitting set $S_v$ for obstructions containing a fixed vertex $v$.
   - **complete degree**.

2. **Sauer-Shelah lemma.**
   - bounding the size of blocks
   $\rightarrow$ existence of some $v$ that $G - (S_v \cup \{v\})$ has many components.

3. **3-expansion lemma.**
   - reducing the graph when $G - (S_v \cup \{v\})$ has many components.
(1) Gallai's $A$-path theorem

Let $A \subseteq V(G)$. A path in $G$ is called an $A$-path if its two end vertices are on $A$, and all other vertices are on $V(G) - A$. An edge in $G[A]$ is also an $A$-path.

**Theorem (Gallai 61)**

For an integer $k$, the following are equivalent:

(1) $G$ has no $k + 1$ vertex-disjoint $A$-paths.

(2) There is a vertex set $T$ of size $\leq 2k$ such that $G - T$ has no $A$-paths.

We can find one of them in polynomial time.

Let us fix a vertex $v$, and let $N_G(v)$ be the set of neighbors of $v$. Consider $N_G(v)$-paths in a graph $G - v - E(G[N_G(v)])$.

Three types:

- cycle of length $\geq 4$ containing $v$
- cycle of length $\geq 4$ not containing $v$
- diamond containing $v$
Proposition

Let $G$ be a graph and let $v \in V(G)$ and let $k$ be a positive integer. Then in $O(kn^3)$ time, we can find either

1. $k + 1$ obstructions that are pairwise vertex-disjoint, or
2. $k + 1$ obstructions whose pairwise intersections are exactly the vertex $v$, or
3. $S_v \subseteq V(G)$ with $|S_v| \leq 7k$ such that $G - S_v$ has no obstructions containing $v$.

Proof:

- We may assume that there is a vertex set $T$ of size $4k$ hitting all obstructions containing at least one vertex of $G - (\{v\} \cup N_G(v))$. All other obstructions are diamonds in $G[\{v\} \cup N_G(v)]$ containing $v$. We search disjoint $P_3$’s in $N_G(v)$.
  - Output $k + 1$ diamonds containing $v$, or
  - Remove at most $3k$ vertices for hitting all such diamonds.
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  - Output $k + 1$ diamonds containing $v$, or
  - Remove at most $3k$ vertices for hitting all such diamonds.

**Preprocessing** : for each vertex $v$, run the algorithm given in the proposition.
If we find $k + 1$ obstructions that are disjoint, then say NO.
If we find $k + 1$ obstructions whose pairwise intersections are $v$, then remove $v$, and let $k' := k - 1$.
⇒ a reduced graph
For a reduced graph $G$, $\forall v \in V(G)$, $\exists S_v$ with $|S_v| \leq 7k$ where $G - S_v$ has no obstructions containing $v$.

Especially, the neighborhood of $v$ in $G - S_v$ forms a disjoint union of complete graphs.
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**Complete degree**

In a reduced graph, **complete degree** of $v$ is

$$\min_{T \subseteq V(G) - \{v\}, |T| \leq 7k} \text{(number of components of } N_{G - T}(v)),$$

where $G - T$ has no obstructions containing $v$.

You can have 1 complete-neighborhood, but cannot have 2 complete-neighborhoods on a component of $G - (\{v\} \cup S_v)$. 
(2) Sauer-Shelah lemma

Suppose it is a Yes-instance, and $G$ is sufficiently large.

Is there a vertex of large complete degree?
(2) Sauer-Shelah lemma

Suppose it is a \textsc{Yes}-instance, and $G$ is sufficiently large.

Is there a vertex of large complete degree?  \textbf{Currently NO.}

\begin{center}
\begin{tikzpicture}
\node[shape=circle,draw,inner sep=0.5mm,fill=black] (x) at (0,0) {x};
\node[shape=rectangle,fill=yellow] (y) at (0,-3) {y};
\draw (x) -- (y);
\end{tikzpicture}
\end{center}

\textbf{|S|} \leq k \\
almost complete

\textbf{Twin reduction rule :} $G$ has $\geq k + 2$ vertices that are pairwise twins, then we remain $k + 1$ vertices and remove other vertices.

Size looks bounded by $(k + 1)2^k$. 

(2) Sauer-Shelah lemma

First consider $\bigcup_{v \in S} S_v$ on $G - S$.
For the remaining part, there is no obstruction containing $v$ for each $v \in S$.

$|S| \leq k$

$7k^2$
Theorem (Sauer-Shelah, 72)

Let $M$ be a $S \times T$ binary matrix such that no two rows are same. For each $t \geq 2$, if $|S| \geq (|T| + 1)^{t-1}$, then there exists $S' \subseteq S$ and $T' \subseteq T$ with $|S'| = 2^t$, $|T'| = t$ where $M[S', T']$ has all possible row vectors of length $t$. 
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\[ |S| \leq k \]

$\Rightarrow$ a block $U$ in the above part has size $\geq (k + 1)^1$, then $\exists U' \subseteq U, S' \subseteq S$ with $|U'| = 2^2, |S'| = 2$ where all patterns from $U'$ to $S'$ appear.

$\rightarrow \exists$ diamond containing $v$, contradiction.

$\rightarrow$ Each block of $G - S$ has size at most $7k^2 + k$. 
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$\rightarrow$ Each block of $G - S$ has size at most $7k^2 + k$.

$\Rightarrow G - S$ is close to a little 'blow-up' of a tree rather than a complete graph.

$\Rightarrow \exists v \in S$ of large complete degree.
(3) 3-expansion lemma

**Theorem**

Let \((G, k)\) be a reduced instance of Block Graph Deletion that is a YES-instance. If \(G\) has at least \(c \cdot k^7\) vertices then \(G\) has a vertex of complete degree at least \((21 + \epsilon)k\).

\[\exists v\] such that \(G - (S_v \cup \{v\})\) has at least \(21k\) components that are block graphs.
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Reduction:

1. Add a pair of disjoint paths of length 2 between \(v\) and \(S_v\), and remove all edges between \(v\) and selected components.
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**Theorem**

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$|V(G)| + s$ is decreased where $s$ is the number of edges that are not incident with a vertex of degree 2.
Further problems

- Improve the size of kernel.
- Does Chordal Vertex Deletion admit a polynomial kernel?
- Does Distance-hereditary Vertex Deletion admit a polynomial kernel?
- Block graphs are graphs whose blocks are $P_3$-free. What about other vertex deletion problems to graphs whose blocks satisfy a certain property?
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