A Polynomial Kernel for Block Graph Vertex Deletion

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Joint work with
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Feedback Vertex Set

Input : A graph $G = (V, E)$, an integer k
Parameter : k
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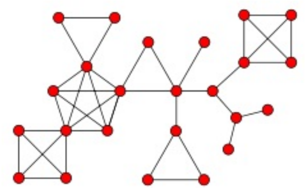
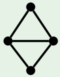
Can we obtain similar results for bigger interesting graph classes?

Block graph

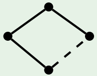
Blocks : maximal connected subgraphs having no cut vertices.
A graph is a **block graph** if every its block is a complete graph.



A graph is a block graph if and only if it has no diamonds and cycles of length at least 4.



diamond



cycles of length $\geqslant 4$

Forests $\subsetneq$ Block graphs $\subsetneq$ (Chordal graphs) $\cap$ (Distance-hereditary graphs)

Block Graph Deletion

Input : A graph $G = (V, E)$, an integer k
Parameter : k
Question : $\exists$? $S \subseteq V$ with $|S| \leqslant k$ such that $G - S$ is a block graph?

Theorem (Kim, K 15)

- Block Graph Deletion can be solved in time $10^k \cdot n^{\mathcal{O}(1)}$.
- Block Graph Deletion admits a kernel of size $\mathcal{O}(k^7)$.

We concentrate on the kernelization algorithm.

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Tools :

(1) Gallai's A-path theorem.
  - finding a (small) hitting set S_v for obstructions containing a fixed vertex v.
  - **complete degree**.

(2) **Sauer-Shelah lemma**.
  - bounding the size of blocks
  $\rightarrow$ existence of some v that $G - (S_v \cup \{v\})$ has many components.

(3) 3-expansion lemma.
  - reducing the graph when $G - (S_v \cup \{v\})$ has many components.

(1) Gallai's A-path theorem

Let $A \subseteq V(G)$. A path in G is called an **A-path** if its two end vertices are on A, and all other vertices are on $V(G) - A$. An edge in $G[A]$ is also an A-path.

Theorem (Gallai 61)

For an integer k, the following are equivalent:

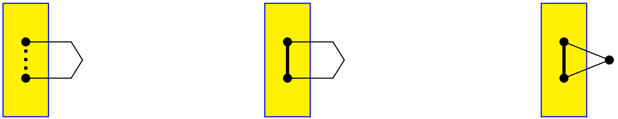
(1) G has no $k+1$ vertex-disjoint A-paths.

(2) There is a vertex set T of size $\leqslant 2k$ such that $G - T$ has no A-paths.

We can find one of them in polynomial time.

Let us fix a vertex v, and let $N_G(v)$ be the set of neighbors of v.
Consider $N_G(v)$-paths in a graph $G - v - E(G[N_G(v)])$.

Three types :



cycle of length $\geqslant 4$ containing v — cycle of length $\geqslant 4$ not containing v — diamond containing v

Proposition

Let G be a graph and let $v \in V(G)$ and let k be a positive integer. Then in $\mathcal{O}(kn^3)$ time, we can find either

(1) $k+1$ obstructions that are pairwise vertex-disjoint, or

(2) $k+1$ obstructions whose pairwise intersections are exactly the vertex v, or

(3) $S_v \subseteq V(G)$ with $|S_v| \leqslant 7k$ such that $G - S_v$ has no obstructions containing v.

Proof :

- We may assume that there is a vertex set T of size $4k$ hitting all obstructions containing at least one vertex of $G - (\{v\} \cup N_G(v))$. All other obstructions are diamonds in $G[\{v\} \cup N_G(v)]$ containing v. We search disjoint P_3's in $N_G(v)$.
  - Output $k+1$ diamonds containing v, or
  - Remove at most $3k$ vertices for hitting all such diamonds.

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Preprocessing : for each vertex v, run the algorithm given in the proposition.
If we find $k+1$ obstructions that are disjoint, then say NO.
If we find $k+1$ obstructions whose pairwise intersections are v, then remove v, and let $k' := k - 1$.
$\Rightarrow$ a reduced graph

For a reduced graph G, $\forall v \in V(G)$, $\exists S_v$ with $|S_v| \leqslant 7k$ where $G - S_v$ has no obstructions containing v.
Especially, the neighborhood of v in $G - S_v$ forms a disjoint union of complete graphs.

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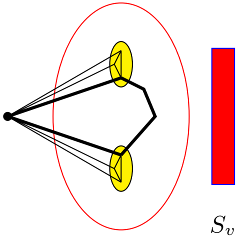
Complete degree

In a reduced graph, **complete degree** of v is

$$\min_{\substack{T \subseteq V(G)-\{v\}, |T| \leqslant 7k \\ G - T \text{ has no obstructions containing } v}} (\text{number of components of } N_{G-T}(v)).$$



You can have 1 complete-neighborhood, but cannot have 2 complete-neighborhoods on a component of $G - (\{v\} \cup S_v)$.

(2) Sauer-Shelah lemma

Suppose it is a $\textsc{Yes}$-instance, and G is sufficiently large.

Is there a vertex of large complete degree?

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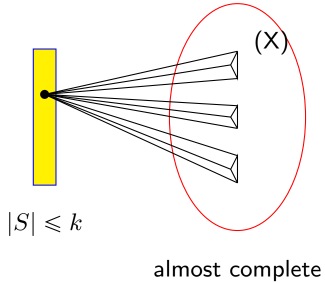
Is there a vertex of large complete degree? Currently NO.

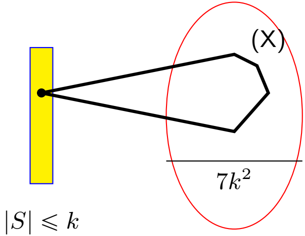


Twin reduction rule : G has $\geqslant k+2$ vertices that are pairwise twins, then we remain $k+1$ vertices and remove other vertices.

Size looks bounded by $(k+1)2^k$.

(2) Sauer-Shelah lemma

First consider $\bigcup_{v \in S} S_v$ on $G - S$.
For the remaining part, there is no obstruction containing v for each $v \in S$.



Theorem (Sauer-Shelah, 72)

Let M be a $S \times T$ binary matrix such that no two rows are same. For each $t \geqslant 2$, if $|S| \geqslant (|T|+1)^{t-1}$, then there exists $S' \subseteq S$ and $T' \subseteq T$ with $|S'| = 2^t, |T'| = t$ where $M[S', T']$ has all possible row vectors of length t.

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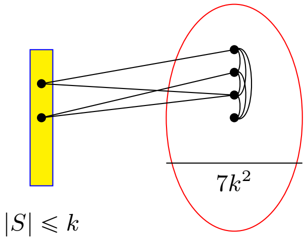

$\Rightarrow$ a block U in the above part has size $\geqslant (k+1)^1$, then $\exists U' \subseteq U$, $S' \subseteq S$ with $|U'| = 2^2, |S'| = 2$ where all patterns from U' to S' appear.

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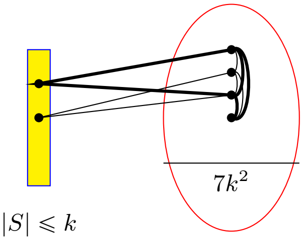

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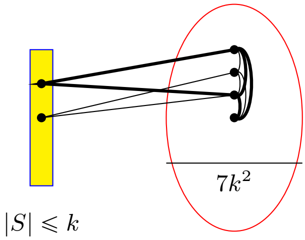

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$\rightarrow$ Each block of $G - S$ has size at most $7k^2 + k$.

$\Rightarrow$ $G - S$ is close to a little 'blow-up' of a tree rather than a complete graph.

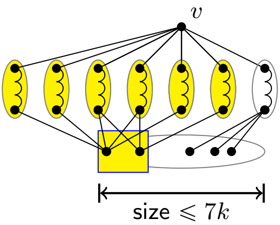
$\Rightarrow$ $\exists v \in S$ of large complete degree.

(3) 3-expansion lemma

Theorem

Let (G, k) be a reduced instance of Block Graph Deletion that is a YES-instance. If G has at least $c \cdot k^7$ vertices then G has a vertex of complete degree at least $(21 + \epsilon)k$.

$\rightarrow \exists v$ such that $G - (S_v \cup \{v\})$ has at least $21k$ components that are block graphs.



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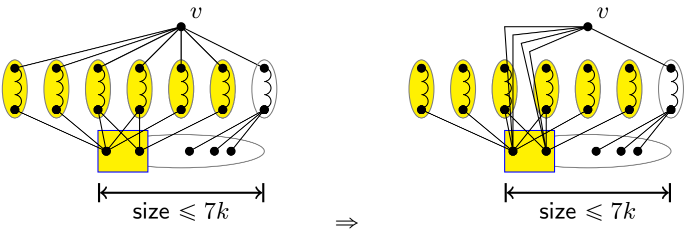

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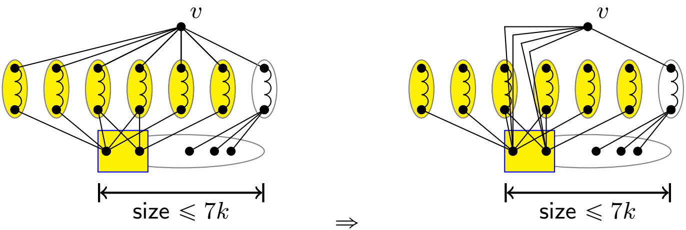

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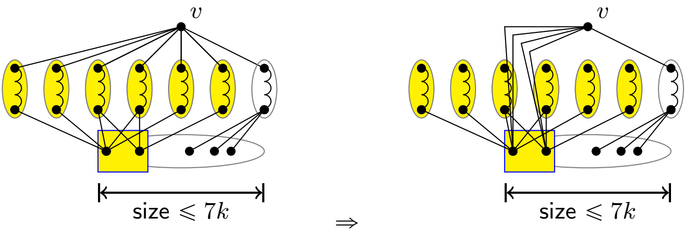

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$|V(G)| + s$ is decreased where s is the number of edges that are not incident with a vertex of degree 2.

Further problems

- Improve the size of kernel.

- Does Chordal Vertex Deletion admit a polynomial kernel?
- Does Distance-hereditary Vertex Deletion admit a polynomial kernel?

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