Graphs of small rank-width are pivot-minors of graphs of small tree-width

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(Joint work with Sang-il Oum)

June 22, 2012
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Preliminaries

- For a graph $G$, $V(G)$ denote the vertex set of $G$ and $E(G)$ denote the edge set of $G$.
- A tree is **subcubic** if every non-leaf vertex has degree 3.
- A tree is **caterpillar** if there is a path in the tree such that every vertex in the tree is incident with a vertex in that path.
A rank-decomposition \((T, L)\) of \(G\) consists of a subcubic tree \(T\), bijective function \(L\) from \(V(G)\) to leaves of \(T\).
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• **Width of an edge of** $T$: the rank of the matrix with the partition induced by the edge.
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$$\begin{pmatrix}
    a_1 & a_2 & a_3 \\
    1 & 1 & 1 \\
    a_4 \\
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    a_5 \\
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    a_6 \\
    0 & 0 & 1 \\
    a_7
\end{pmatrix} = 3$$
• Width of an edge of $T$: the rank of the matrix with the partition induced by the edge.

$$\text{Width of } e = \text{rank} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & 1 & 1 & 1 \\ a_5 & 1 & 0 & 0 \\ a_6 & 1 & 0 & 1 \\ a_7 & 0 & 0 & 1 \end{pmatrix} = 3$$
• Width of an edge of $T$: the rank of the matrix with the partition induced by the edge.
• Width of $(T, L)$: maximum width of all edges in $T$
• Rank-width of $G$: minimum width of all rank-decompositions of $G$
• If we restrict to use only caterpillar subcubic trees, then we call it the linear rank-width of $G$
Tree-decomposition
(84, Robertson and Seymour)

- A tree-decomposition \((T, \{B_v\}_{v \in V(T)})\) of \(G\) consists of a tree \(T\), mapping from each vertex \(v\) of \(T\) to a subset \(B_v\) of \(V(G)\). and it satisfies following axioms.
  1. Two vertices of an edge must be contained in a bag.
  2. If \(x\) is the path from \(v\) to \(w\) in \(T\), then \(B_v \cap B_w \subseteq B_x\).

\(B_v\) is called a bag.

\[\begin{array}{c}
\text{Tree-decomposition} \\
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\(B_v\) is called a bag.
• Width of \((T, \{B_v\}_{v \in V(T)})\): \(\max\{|B_v| - 1 : v \in V(T)\}\)

• **Tree-width** of \(G\): minimum width of all tree-decompositions of \(G\)

• If we restrict to use only paths, we call it the **path-width** of \(G\)
Local complementation and pivoting

We are interested in two operations.

- **Local complementation** on a vertex $v \in V$
  \[
  G * v = (V, E \Delta \{xy : x, y \in \delta(v)\})
  \]
Local complementation and pivoting

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- **Pivot-operation** on an edge \( uv \in E \)
  \[
  G \land uv = G * u * v * u
  \]
Local complementation and pivoting

- **Local complementation** on a vertex \( v \in V \)
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Why these operations?

- These operations preserve the rank of the matrix induced by a partition of the graph.
- They also preserve the rank-width of the graph.

Definition (Oum, 05)

$H$ : a **vertex-minor** of a graph $G$ 
if $H$ is obtained from $G$ by applying a sequence of local complementations and vertex deletions.

$H$ : a **pivot-minor** of a graph $G$ 
if $H$ is obtained from $G$ by applying a sequence of pivoting edges and vertex deletions.

- If $H$ is a vertex-minor or a pivot-minor of $G$, 
then $\text{rw}(H) \leq \text{rw}(G)$. 

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## Known results

### Rank-width and tree-width of some graph classes

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<tr>
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**Theorem (Oum 08)**

For a graph $G$, $\text{rw}(G) \leq \text{tw}(G) + 1$.

In general, tree-width cannot be bounded by a function of rank-width.
Known results

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Observation

We observe that $K_n$ is a vertex-minor of a path of length $3n$.

Question

If a graph has small rank-width, then it can be a vertex-minor or a pivot-minor of a graph of small tree-width?
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Main result

Theorem (K, Oum 12)

\[ G \text{ has rank-width } \leq k \quad \Rightarrow \quad G \text{ is a pivot-minor of a graph of tree-width } \leq 2k \]

\[ G \text{ has linear rank-width } \leq k \quad \Rightarrow \quad G \text{ is a pivot-minor of a graph of path-width } \leq k + 1 \]

Given a graph \( G \) and a rank-decomposition \((T, L)\) of width \( k \), we explicitly construct a graph \( H \), called a rank-expansion, such that

- tree-width of \( H \) is at most \( 2k \)
- \( G \) is a pivot-minor of \( H \)
- \(|V(H)| \leq (2k + 1)|V(G)| - 6k\)
Main result

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- \( G \) is a pivot-minor of \( H \)
- \( |V(H)| \leq (2k + 1)|V(G)| - 6k \)
First we choose a leaf vertex of $T$ as a root. For an edge $e$ in $T$, let $A_e = \{e_4, e_5, e_6, e_7\}$, $B_e = \{e_1, e_2, e_3\}$. Choose the basis vertices of the row space of the induced matrix. $A(G)[A_e, B_e] = \begin{pmatrix} a_4 & 1 & 1 & 1 \\ a_5 & 1 & 0 & 0 \\ a_6 & 1 & 0 & 1 \\ a_7 & 0 & 0 & 1 \end{pmatrix}$ basis vertices: $\{a_4, a_5, a_7\}$ Note that $R_{a_6} = R_{a_5} + R_{a_7}$. 
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Construction

First we choose a leaf vertex of $T$ as a root.
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basis vertices: \( \{ a_4, a_5, a_7 \} \)
In the matrix corresponding to $a_4$ and $a_5$ are adjacent to $a_6$. 

\[ x = a_2 \]
$R_{a_6} = R_{a_5} + R_{a_7}$
in the matrix corresponding to $e$
$a_4$ and $a_5$ are adjacent to $a_6$
1. After pivoting every blue edges and delete them with vertices, we get the graph $G$.

2. The tree-width of this rank-expansion is at most $2k$. 
Sketch of proof

1. After pivoting every blue edges and delete them with vertices, we get the graph $G$.

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\[
U_d = \{a_4, a_5\} \quad U_e = \{a_4, a_5, a_7\} \\
U_{f_1} = \{a_4, a_5\} \quad U_{f_2} = \{a_6, a_7\}
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Key lemma

Lemma
For a given graph $G$, we make a graph as follows. (left adjacency follows $A(G)$, and right adjacency follows basis) Then we obtain the adjacency between partitions exactly same as the adjancency in the graph $G$ by pivoting matching edges (blue).
Rank-width 1, linear rank-width 1

Theorem (K, Oum 12)

\( G \) has rank-width \( \leq 1 \)
Rank-width 1, linear rank-width 1

Theorem (K, Oum 12)

\[ G \text{ has rank-width } \leq 1 \iff G \text{ is a pivot-minor of a graph of tree-width } \leq 2 \]
Rank-width 1, linear rank-width 1

Theorem (K, Oum 12)

\[ G \text{ has rank-width } \leq 1 \]

\[ \iff G \text{ is a vertex-minor of a tree} \]
Rank-width 1, linear rank-width 1

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Rank-width 1, linear rank-width 1

Theorem (K, Oum 12)

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G \text{ has rank-width } & \leq 1 \\
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\end{align*}

When \( G \) is bipartite,

\begin{align*}
G \text{ has rank-width } & \leq 1 \\
G \text{ has linear rank-width } & \leq 1 \iff G \text{ is a pivot-minor of a tree} \\
G \text{ is a pivot-minor of a path} \\
\end{align*}
To make a rank-expansion of a given graph, we need a rank-decomposition of the graph.

Question. Can we construct a graph satisfying the theorem without rank-decomposition?
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Thank you.