H.W #5.

1. <Sol> $m_0 = \left| \frac{1}{\varepsilon} \ln \frac{|H|}{\delta} \right|$

$= \left| \frac{1}{0.15} \ln \frac{\left(\frac{100 \cdot 99}{2}\right)^2}{0.05} \right|$

$\approx 133.4$

$\therefore$ If $m \geq 134$, then $P_r[\, \underline{x} \in S(m,\tau) \mid H[\underline{x}] \cap B_\varepsilon = \phi\,] > 0.95$.

2. <Sol>

$P_r[\, err_D(h) > err_S(h) + \varepsilon\,] \leq \exp(-2\ell\varepsilon^2)$

$P_r[\, err_D(h) > err_S(h) + \varepsilon$ for at least one $h \in H\,] \leq |H| \exp(-2\ell\varepsilon^2) \leq \delta$

$|H| \exp(-2\ell\varepsilon^2) \leq \delta$

$\exp(-2\ell\varepsilon^2) \leq \frac{\delta}{|H|}$

$-2\ell\varepsilon^2 \leq \ln \frac{\delta}{|H|}$

$\ell \geq -\frac{1}{2\varepsilon^2} \ln \frac{\delta}{|H|}$

$= \frac{1}{2\varepsilon^2} \ln \frac{|H|}{\delta}$

$\therefore \ell \geq \frac{1}{2\varepsilon^2} \ln \frac{|H|}{\delta}$

3. (a) $VCD(H_R) = 4$

(b) $VCD(H_T) = \infty$.

4. $\Pi_H(m) \underset{①}{\leq} \Phi_d(m) \underset{②}{\leq} \left(\frac{em}{d}\right)^d$

prove ①. i) $m = d$

$\Pi_H(d) = 2^d$

$\Phi_d(d) = \sum_{i=0}^{d} \binom{d}{i} = 2^d$

$\therefore \Pi_H(d) \leq \Phi_d(d)$ holds.

ii) $m = d+1$.

$\Rightarrow \pi_H(d+1) \le 2^{d+1} - 1$. $\quad (\because d+1 \ne VCD(H))$

$= \sum_{i=0}^{d+1} \binom{d+1}{i} - \binom{d+1}{d+1}$

$= \sum_{i=0}^{d} \binom{d+1}{i}$

$= \phi_d(d+1)$

$\therefore \pi_H(d+1) \le \phi_d(d+1)$ holds.

iii) Assume $\pi_H(d+k) \le \phi_d(d+k)$

$\Rightarrow \pi_H(d+k+1) \le 2\pi_H(d+k) - \binom{d+k+1}{d}$

$\le 2\pi_H(d+k) - \binom{d+k}{d}$

$\le 2\phi_d(d+k) - \binom{d+k}{d}$

$= \phi_d(d+k) + \sum_{i=0}^{d} \binom{d+k}{i} - \binom{d+k}{d}$

$= \phi_d(d+k) + \sum_{i=0}^{d-1} \binom{d+k}{i}$

$= \phi_{d-1}(d+k)$

$= \phi_d(d+k+1)$

$\therefore \pi_H(d+k+1) \le \phi_d(d+k+1)$ holds.

By i), ii) and iii)

$\pi_H(m) \le \phi_d(m)$ for all $m \ge d$.

Prove ② $\phi_d(m)$ grows polynomially if $m \ge d$ $\quad (0 \le \frac{d}{m} \le 1)$

$\left(\frac{d}{m}\right)^d \sum_{i=0}^{d} \binom{m}{i} \le \sum_{i=0}^{d} \left(\frac{d}{m}\right)^i \binom{m}{i} \le \sum_{i=0}^{m} \left(\frac{d}{m}\right)^i \binom{m}{i} = \left(1+\frac{d}{m}\right)^m \le e^d$

$\Rightarrow \phi_d(m) = \sum_{i=0}^{d} \binom{m}{i} \le e^d \left(\frac{m}{d}\right)^d = \left(\frac{em}{d}\right)^d$

$\therefore \pi_H(m) \le \phi_d(m) \le \left(\frac{em}{d}\right)^d$ for $\forall m \ge d$.