A New Optimal Control Approach for the Reconstruction of Extended Inclusions

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Abstract

The aim of this paper is to propose a new optimal control formulation for recovering an extended inclusion from boundary measurements. Our approach provides an optimal representation of the shape of the inclusion. It guarantees local Lipschitz stability for the reconstruction problem. Some numerical experiments are performed to demonstrate the validity and the limitations of the proposed reconstruction method.

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1 Introduction and problem formulation

Let \( \Omega \in \mathbb{R}^d, d = 2, 3, \) be a bounded smooth simply connected domain. Let \( \nu \) denote the outward normal to the boundary \( \partial \Omega. \) Suppose that \( \Omega \) contains an inclusion \( D_\ast \) away from the boundary, with a known material parameter, such as the conductivity, \( 0 < k \neq 1 < +\infty. \) We assume the material parameter of the background to be 1. Throughout this paper \( \gamma_{D_\ast} \) denotes the material parameter distribution, namely,

\[
\gamma_{D_\ast} = 1 + (k - 1)\chi[D_\ast],
\]

where \( \chi \) is the characteristic function.

For \( l = 1, \ldots, N, \) we denote by \( u^{(l)}_\ast \) the solution to

\[
\begin{cases}
\nabla \cdot \gamma_{D_\ast} \nabla u^{(l)}_\ast + \omega^2 u^{(l)}_\ast = 0 & \text{in } \Omega, \\
\frac{\partial u^{(l)}_\ast}{\partial \nu} = g^{(l)} & \text{on } \partial \Omega,
\end{cases}
\]

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where \( g^{(l)} \) are \( N \) given functions and \( \omega \) is the angular frequency. To fix ideas, we restrict ourselves to illuminations by plane and spherical waves:

\[
\frac{\partial}{\partial \nu} e^{i \omega \theta_l \cdot x} = i \omega \theta_l \cdot \nu(x) e^{i \omega \theta_l \cdot x} \quad \text{on} \ \partial \Omega, \quad l = 1, \ldots, N, \tag{1.3}
\]

where \( \{\theta_1, \ldots, \theta_N\} \) is a set of \( N \) unit directions uniformly distributed on the unit sphere, and

\[
g^{(l)}(x) := \frac{i}{4} \frac{\partial}{\partial \nu} H_0^{(1)}(\omega|x - y_l|), \quad l = 1, \ldots, N, \tag{1.4}
\]

where \( H_0^{(1)} \) denotes the zeroth order Hankel function of the first kind and \( y_l \) are \( N \) equidistributed points on \( \partial \Omega \). We emphasize that \( g^{(l)} \) could be of course more general than in (1.3) and (1.4).

We suppose that the size of \( D_* \) as well as the distance between \( D_* \) and \( \partial \Omega \) are large compared to the wavelength \( 2\pi/\omega \). The inverse problem considered in this paper is to reconstruct the extended inclusion \( D_* \) from the boundary measurements \( (u^{(l)}_*|_{l=1}^N) \) on \( \partial \Omega \).

With infinite number of measurements, a uniqueness result for this inverse problem has been proved by Isakov [19].

In this paper we propose optimal control approaches for recovering the inclusion \( D_* \) from the boundary measurements. This inverse problem is known to be severely ill-posed. A first classical approach is to minimize the discrepancy between the measured and computed data using least-squares formulation. One can use total variation regularization to recover sharp discontinuities. Computing the shape derivative of the \( L^2 \) discrepancy shows that a filtering effect is the main difficulty in solving this inverse problem.

The main novelty of the paper is to establish a second optimization approach that reduces filtering of the oscillations. This allows us to determine the optimal inclusion shape with better resolution than by the first optimization approach. More importantly, we provide an optimal representation of the inclusion shape. We also prove a local Lipschitz stability result for reconstructing the inclusion. We also show results of computational experiments to demonstrate efficiency of the proposed algorithms. To handle topology changes such as breaking one component into two, we develop a level set version of our algorithm. Our results in this paper provide mathematical interpretations of important physical notions such as the resolution limit and the filtering effect in wave propagation.

The paper is organized as follows. In Section 2 we present the first quite standard algorithm to solve the reconstruction problem. In Section 3 we propose our new algorithm and provide an optimal shape representation. Local stability results are formulated and proved in Section 4. In Section 5 we discuss a MUSIC-type algorithm to get a good initial guess. Some numerical simulations to test the proposed optimal control approach are presented in Section 6. Section 7 is devoted to a level set version of our algorithm. Our results are generalized to linear elasticity in Section 8. The paper ends with a discussion in Section 9. Some results on the filtering property of the Helmholtz equation are given in the appendices.
2 First algorithm

The first (and quite standard) algorithm to solve the inverse problem is to minimize over $D$ the following cost functional:

$$J[D] := rac{1}{2} \sum_{l=1}^{N} \int_{\partial \Omega} \left| u^{(l)}[D] - u_{*}^{(l)} \right|^2,$$

(2.1)

where $u^{(l)}[D]$ is the solution to

$$\begin{cases}
\nabla \cdot \gamma_D \nabla u^{(l)} + \omega^2 u^{(l)} = 0 & \text{in } \Omega,

\frac{\partial u^{(l)}}{\partial \nu} = g^{(l)} & \text{on } \partial \Omega.
\end{cases}$$

(2.2)

Here, $\gamma_D = 1 + (k - 1)\chi[D]$.

Suppose that $-\omega^2$ is not an eigenvalue of $-\nabla \cdot \gamma_D \nabla$ in $\Omega$ with Neumann boundary conditions. For a given function $h$ on $\partial D$, the shape derivative $d_S J[D]$ of $J[D]$ is given for $h$ of class $C^1$ by

$$(d_S J[D], h) = \sum_{l=1}^{N} \Re \int_{\partial \Omega} (u^{(l)}[D] - u_{*}^{(l)}) \overline{v^{(l)}(h)} d\sigma,$$

(2.3)

where $v^{(l)}(h)$ is the solution to

$$\begin{cases}
\Delta v^{(l)} + \omega^2 v^{(l)} = 0 & \text{in } \Omega \setminus D,

k \Delta v^{(l)} + \omega^2 v^{(l)} = 0 & \text{in } D,

v^{(l)}|_{+} - v^{(l)}|_{-} = (k - 1) h \frac{\partial u^{(l)}[D]}{\partial \nu}|_{-} & \text{on } \partial D,

\frac{\partial v^{(l)}}{\partial \nu}|_{+} - k \frac{\partial v^{(l)}}{\partial \nu}|_{-} = (k - 1) \frac{\partial}{\partial T} \left( h \frac{\partial u^{(l)}[D]}{\partial T} \right) & \text{on } \partial D,

\frac{\partial v^{(l)}}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}$$

(2.4)

Here $T(x)$ denotes the unit tangent vector at $x$ on $\partial D$ and $\partial / \partial T$ stands for the tangential derivative. We note that we restrict the consideration to two dimensions for simplicity (extension to three dimensions is apparent). The proof of (2.3) follows immediately from [10, Theorem 1.2].

To explicitly compute the shape derivative of $J$, we introduce the adjoint state $p^{(l)}[D]$ as the solution to

$$\begin{cases}
\nabla \cdot \gamma_D \nabla p^{(l)}[D] + \omega^2 p^{(l)}[D] = 0 & \text{in } \Omega,

\frac{\partial p^{(l)}[D]}{\partial \nu} = (u^{(l)}[D] - u_{*}^{(l)}) & \text{on } \partial \Omega.
\end{cases}$$

(2.5)

Introduce $N^\omega[D](x, y)$ as the Neumann function for $\nabla \cdot \gamma_D \nabla + \omega^2$ in $\Omega$ corresponding to a Dirac mass at $y$. That is, $N^\omega$ is the unique solution to

$$\begin{cases}
(\nabla_x \cdot \gamma_D \nabla_x + \omega^2) N^\omega[D](x, y) = -\delta_y & \text{in } \Omega,

\frac{\partial N^\omega[D]}{\partial \nu} |_{\partial \Omega} = 0 & \text{on } \partial \Omega.
\end{cases}$$

(2.6)
Note that \( p^{(l)} \) and \( v^{(l)} \) can be written as follows

\[
p^{(l)}[D](x) = \int_{\partial \Omega} (u^{(l)}[D] - u^{*^{(l)}}(y)) N^w[D](x,y) d\sigma(y)
\]

and

\[
v^{(l)}(x) = \int_{\partial D} h(y) M(y) \nabla u^{(l)}[D](y) \cdot \nabla_x N^w[D](x,y) d\sigma(y),
\]

where

\[
M(y) := (k - 1)(k \nu(y) \otimes \nu(y) + T(y) \otimes T(y)) \quad \text{for } y \in \partial D.
\]

Now, using \( p^{(l)} \) to express the integral

\[
\int_{\partial \Omega} (u^{(l)}[D] - u^{*^{(l)}}(y)) v^{(l)}(x) h d\sigma,
\]

we find that \( d_S J[D] \) in the direction of \( h \nu \) is given by

\[
(d_S J[D], h) = (k - 1) \int_{\partial D} h \Re \sum_{l=1}^{N} \left( k \frac{\partial p^{(l)}[D]}{\partial \nu} - \frac{\partial w^{(l)}[D]}{\partial \nu} \right) - \frac{\partial p^{(l)}[D]}{\partial T} \frac{\partial w^{(l)}[D]}{\partial T} \right). \quad (2.7)
\]

Set

\[
w[D] = \Re \sum_{l=1}^{N} \left( k \frac{\partial p^{(l)}[D]}{\partial \nu} - \frac{\partial w^{(l)}[D]}{\partial \nu} \right) - \frac{\partial p^{(l)}[D]}{\partial T} \frac{\partial w^{(l)}[D]}{\partial T} \right). \quad (2.8)
\]

The algorithm consists in replacing, in each step,

\[
\partial D \mapsto \partial D + h \nu,
\]

where

\[
h = -J[D] \frac{w[D]}{\int_{\partial D} (d_S J[D], w[D])} = -J[D] \frac{w[D]}{(k - 1) \int_{\partial D} w[D]^2}. \quad (2.9)
\]

One can regularize the problem by

\[
\min_D J[D] := \frac{1}{2} \sum_{l=1}^{N} \int_{\partial \Omega} |u^{(l)}[D] - u^{*^{(l)}}|^2 + \rho TV(\gamma_D),
\]

where \( \rho \) is the regularization parameter and \( TV \) is the total variation defined for \( q \in L^1 \) by

\[
TV(q) := \sup \left\{ \int_{\Omega} q \nabla \cdot g : g \in C_0^1(\Omega)^d, \ |g(x)| \leq 1 \text{ for all } x \in \Omega \right\}.
\]

Here \( |\cdot| \) is the Euclidean norm of a vector. The regularization by total variation was introduced in [13, 14] for the purpose of obtaining sharp images of the discontinuity.
3 Second algorithm

For the first optimization algorithm, $dSJ[D]$ acts like a filter of the oscillations in the boundary changes. Roughly speaking, write $h = h_{\text{low}} + h_{\text{high}}$, where $h_{\text{low}}$ and $h_{\text{high}}$ are respectively the low- and high-frequency components of $h$.

Fix the threshold to separate the high-frequency component of $h$ from the low-frequency one by

$$\frac{||\partial h_{\text{high}}/\partial T||_{L^2(\partial D)}}{||h_{\text{high}}||_{L^2(\partial D)}} > \beta.$$  \hspace{1cm} (3.1)

One can see that there exists $\beta_0$ such that for $\beta \geq \beta_0$

$$\int_{\partial D} h_{\text{high}} w[D] \approx 0,$$  \hspace{1cm} (3.2)

which shows that $h_{\text{high}}$ can not be reconstructed from boundary measurements. See Appendix A for this filtering property of $w[D]$ defined by (2.8) and a derivation of (3.2).

To reduce the filtering effect, one should allow the adjoint state $p^{(l)}$ to be in a richer space than the one spanned by the solutions of (2.5). Note that $p^{(l)}$ is given by

$$p^{(l)} = \Lambda_D((w^{(l)}[D] - u^{(l)}_*)) \quad \text{for } l = 1, \ldots, N,$$

where the operator $\Lambda_D : L^2(\partial \Omega) \to L^2(\partial D)$ is defined by

$$\Lambda_D(f) = p|_{\partial D},$$

with

$$\begin{cases} 
\nabla \cdot \gamma_D \nabla p + \omega^2 p = 0 & \text{in } \Omega, \\
\frac{\partial p}{\partial \nu} = f & \text{on } \partial \Omega.
\end{cases}$$  \hspace{1cm} (3.3)

It is easy to see that the adjoint $\Lambda^*_D : L^2(\partial D) \to L^2(\partial \Omega)$ is given for $v \in L^2(\partial D)$ by

$$\Lambda^*_D(v) = w|_{\partial \Omega},$$

where $w$ is the solution to

$$\begin{cases} 
\Delta w + \omega^2 w = 0 & \text{in } \Omega \setminus \overline{D}, \\
k \Delta w + \omega^2 w = 0 & \text{in } D, \\
k w|_+ = w|_- & \text{on } \partial D, \\
\frac{\partial w}{\partial \nu}_+ - \frac{\partial w}{\partial \nu}_- = v & \text{on } \partial D, \\
\frac{\partial w}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}$$  \hspace{1cm} (3.4)

Let $\{f_i[D] : i = 1, \ldots, M\}$ be a basis of the image space of $\Lambda^*_D \Lambda_D$ after truncating “small” eigenvalues of $\Lambda^*_D \Lambda_D$. The threshold for such truncation is fixed in terms of the signal-to-noise ratio in the measured data.
It is worthwhile to note that the integral kernel of $\Lambda_D^* \Lambda_D$ is given by

$$
\int_{\partial D} N^\omega[D](x, z) \overline{N^\omega[D](z, y)} \, d\sigma(z), \quad x, y \in \partial \Omega,
$$

(3.5)

because $\Lambda_D$ is given by

$$
\Lambda_D(f)(x) = \int_{\partial \Omega} f(y) N^\omega[D](x, y) \, d\sigma(y), \quad x \in \partial D,
$$

(3.6)

and hence

$$
\Lambda_D^*(v)(x) = \int_{\partial D} v(y) \overline{N^\omega[D](x, y)} \, d\sigma(y), \quad x \in \partial \Omega.
$$

(3.7)

Instead of minimizing the $L^2$-norm of the discrepancy between the measurements and the computed data, $u_l^{(i)}[D] - u_*^{(i)}$ for $l = 1, \ldots, N$, as in the first optimization procedure, we minimize the $L^2$-projection of $u_l^{(i)}[D] - u_*^{(i)}$ onto the space spanned by $\{f_i[D] : i = 1, \ldots, M\}$ in order to generate the highest possible oscillations in the adjoint states and consequently, get optimal reconstruction of the boundary changes $h$. This can only be done by changing the optimization procedure. Indeed, since the basis $\{f_i[D] : i = 1, \ldots, M\}$ depends on $D$, one should change at each step the minimization cost functional.

Our new algorithm is to minimize at each step $n \geq 1$ over all possible boundary changes $\delta D$ the following cost functional:

$$
J_n[\delta D] := \frac{1}{2} \sum_{l=1}^{N} \sum_{i=1}^{M} \left| \int_{\partial \Omega} (u_l^{(i)}[D_{n-1} + \delta D] - u_*^{(i)}f_i[D_{n-1}]) \, d\sigma \right|^2,
$$

(3.8)

where $D_0$ is an initial guess and $u_l^{(i)}[D_{n-1} + \delta D]$ is the solution to (2.2) with $D := D_{n-1} + \delta D$. Note that at each step, $J_n$ is an $L^2$ projection of the discrepancy between the computed and the measured data onto the space spanned by $\{f_i[D_{n-1}] : i = 1, \ldots, M\}$.

Since finding the minimizer $\delta D$ of $J_n$ is a full optimization problem, we update the inclusion by replacing $D_{n-1}$ by $D_{n-1} + \delta D_n$, where $\delta D_n$ is the solution to the linearized problem of the minimization problem (3.8).

Our algorithm can be interpreted as a two-step algorithm. First, one linearizes the problem and then, one projects the data over an optimal basis that insures stability and gives optimal representation of the boundary changes (see Section 4 for details). It is clear that one has to change the cost functional at each step because of the inherent nonlinearity of the inverse problem of reconstructing the inclusion from boundary measurements.

To find $\delta D_n$, one can compute the shape derivative of the (new) cost-functional $\mathcal{J}[D]$ given by

$$
\mathcal{J}[D] := \frac{1}{2} \sum_{l=1}^{N} \sum_{i=1}^{M} \left| \int_{\partial \Omega} (u_l^{(i)}[D] - u_*^{(i)}) f_i \, d\sigma \right|^2,
$$

in the direction of $h\nu$, where the functions $f_i$ are considered independent of $D$. Analogous to (2.7), we obtain that

$$
(d_s \mathcal{J}[D], h) = \sum_{l=1}^{N} \sum_{i=1}^{M} \Re \left[ \int_{\partial \Omega} (u_l^{(i)}[D] - u_*^{(i)}) f_i \, d\sigma \int_{\partial \Omega} \overline{v_l^{(i)}(h)} f_i \, d\sigma \right],
$$

(3.9)
where $v^{(l)}(h)$ is given by (2.4).

Let $p_{i}[D]$ be the solution to

$$
\begin{align*}
\nabla \cdot \gamma_{D} \nabla p_{i} + \omega^{2} p_{i} = 0 \quad \text{in } \Omega, \\
\frac{\partial p_{i}}{\partial \nu} = f_{i} \quad \text{on } \partial \Omega.
\end{align*}
$$

(3.10)

Then, we have from (2.4)

$$
\int_{\partial \Omega} v^{(l)}(\nu) f_{i} \, d\sigma = (k - 1) \int_{\partial D} h \left( k \frac{\partial p_{i}[D]}{\partial \nu} \left| \frac{\partial u^{(l)}[D]}{\partial \nu} \right|_{-} + \frac{\partial p_{i}[D]}{\partial T} \frac{\partial u^{(l)}[D]}{\partial T} \right).
$$

(3.11)

Set

$$
\alpha_{il}[D] := \frac{(k - 1)}{(k - 1)} \int_{\partial \Omega} (u^{(l)}[D] - u_{*}^{(l)}) T_{i}[D] \, d\sigma, \quad i = 1, \ldots, M, \quad l = 1, \ldots, N,
$$

and

$$
\alpha_{il}[D] := k \frac{\partial p_{i}[D]}{\partial \nu} \left| \frac{\partial u^{(l)}[D]}{\partial \nu} \right|_{-} + \frac{\partial p_{i}[D]}{\partial T} \frac{\partial u^{(l)}[D]}{\partial T}.
$$

(3.12)

Then (3.9) can be rewritten as

$$
(d_{S} J[D], h) = \Re \left( \sum_{i=1}^{N} \sum_{l=1}^{M} \alpha_{il}[D_{n-1}] \int_{\partial D} h w_{i}^{(l)}[D_{n-1}] \right).
$$

(3.13)

Thus in our new algorithm the update $\delta D_{n}$ is chosen to have the form

$$
\delta D_{n} := \left\{ h_{n}(x) \nu(x), \ x \in \partial D_{n-1} \right\},
$$

where

$$
h_{n} = - \frac{J[D_{n-1}]}{\sum_{i,l} \left( (d_{S} J[D_{n-1}], w_{i}^{(l)}[D_{n-1}]) \right)^{2}} \sum_{i,l} \left( d_{S} J[D_{n-1}], w_{i}^{(l)}[D_{n-1}] \right) w_{i}^{(l)}[D_{n-1}].
$$

(3.14)

Here $(d_{S} J[D_{n-1}], w_{i}^{(l)}[D_{n-1}])$ is computed using formula (3.13). Note also that

$$
p^{(l)} = \Lambda_{D}((u^{(l)}[D] - u_{*}^{(l)})) \approx \sum_{i=1}^{M} \sqrt{\lambda_{i}}((u^{(l)}[D] - u_{*}^{(l)}), f_{i}) p_{i}
$$

(3.15)

where $\lambda_{i}$ is the singular value (eigenvalue) of $\Lambda_{D}^{*} \Lambda_{D}$ corresponding to $f_{i}$, and therefore, $h_{n}$ corresponds to back-propagating the projections of the functions $u^{(l)} - u_{*}^{(l)}$ onto the space spanned by the functions $f_{i}$ for $i = 1, \ldots, M$. The algorithm filters $d_{S} J$ given by (2.7) at each step with an optimal filter constructed through a singular value decomposition of the operator $\Lambda_{D}$. In other words, while in the first algorithm the adjoint state corresponds to a back-propagation of the discrepancy, in the new algorithm the discrepancy is projected on an optimal basis at each step and only the vectors of this basis are back-propagated inside...
the domain. The traces of the back-propagated functions on the boundary of the inclusion at step $n$ form a basis to represent the changes in the shape in order to evolve it.

There is a trade off between accuracy and stability. To gain accuracy one has to choose $M$ as high as possible. But if it is too high then it follows from the form of the coefficients $\alpha_{il}$ in (3.11) and the fact that $f_i$ is highly oscillating for large $i$ that the algorithm is unstable in the case of noisy data $(u^{(l)}_*)_{l=1,\ldots,N}$.

Note also that our algorithm extends as well to the case where the inclusion is perfectly insulating or conducting. In those two cases, the analogous of the operator $\Lambda^* D A_D$ are exactly those used in the factorization method. See [23].

Finally, we should note that the basis $\{f_i\}$ can be updated by using the leading-order term in the asymptotic expansion of $f_l[D+\delta D]$ in terms of $\delta D$, which has been obtained in [8]. However, to avoid error accumulation, the basis should be recomputed after few steps.

4 Local stability results

Let $\mathcal{V}[D]$ be the (finite-dimensional) vector space spanned by $\Im w^{(l)}_i$ and $\Re w^{(l)}_i$ for $l = 1, \ldots, N$ and $i = 1, \ldots, M$, where $w^{(l)}_i$ is defined by (3.12).

From Section 3, it follows that an optimal representation of small changes in $D$ is as follows

$$\delta D := \left\{ h(x)\nu(x), \ x \in \partial D \right\}, \ h \in \mathcal{V}[D].$$

This representation is optimal in the sense that any component of the changes $\delta D$ in the space orthogonal to $\mathcal{V}[D]$ in $L^2(\partial D)$ cannot be reconstructed, which can be immediately seen from (3.13). Moreover, as will be shown by Proposition 4.1, the reconstruction of the components of $\delta D$ that belongs to $\mathcal{V}[D]$ is stable. Note that $\mathcal{V}[D]$ is a subset of $H^2(\partial D)$ because of the $H^2$-regularity of $w^{(l)}_i$ on $\partial D$.

A natural representation of $\delta D$ is to expand it on a basis of $L^2(\partial D)$ (constructed for instance by solving the spectral problem for Laplace-Beltrami operator). But, as shown before, high oscillations in $h$ can not be reconstructed which means that only the projection of $\delta D$ on the first basis (not highly oscillating) functions plays a role. However, there is no systematic way to set a threshold. From this point of view, the representation as a linear combination of elements of $\mathcal{V}[D]$ appears to be sparse and the threshold is fixed in a systematic and optimal way.

A few components may be enough to represent all the possible changes that could be reconstructed. Note also that, the larger the distance between $D_*$ and $\partial D$ compared to the wavelength and the size of $D_*$, the smaller the dimension of $\mathcal{V}[D_*]$. When $D_*$ is very small compared to the wavelength $\lambda = 2\pi/\omega$, the dimension of the vector space $\mathcal{V}[D_*]$ could be reduced to 1. In fact, in view of (3.5) we have

$$\Lambda^* D A_D(f)(x) = |\partial D| \int_{\partial \Omega} f(y) N^\omega[D](y, z) d\sigma(y) N^\omega[D](z, x) + o(|\partial D|)$$

and

$$\Lambda^* D A_D(f)(x) = |\partial D| \int_{\partial \Omega} f(y) N^\omega(y, z) d\sigma(y) N^\omega(z, x) + o(|\partial D|), \ x \in \partial \Omega.$$

where $z$ is a point in $D$ and $N^\omega(z, x)$ is the Neumann function for $\Delta + \omega^2$ on $\Omega$ (without inclusion). Here $|\partial D|$ denotes the surface of $\partial D$. It shows that the significant eigenvalue of $\Lambda^* D A_D$ is $|\partial D|$ and corresponding eigenvector is $N^\omega(\cdot, z)$. 

The stability of the reconstruction of the components of the changes that are in $V[D]$ can be stated mathematically by introducing a measure of arbitrarily small changes $\delta D$ in $D$. This can be done by setting

$$|\delta D|^2 = \sum_{i,l} \left| \int_{\partial D} h_w^{(l)} i \right|^2 + \left| \int_{\partial D} \frac{\partial h}{\partial T} \frac{\partial w^{(l)} i}{\partial T} \right|^2. \quad (4.3)$$

We emphasize that since $V[D]$ is finite dimensional, we have

$$|\delta D|^2 \approx \sum_{i,l} \left| \int_{\partial D} h_w^{(l)} i \right|^2.$$

For a general perturbation $\delta D$ (not necessary in $V[D]$) the distance $| |$ corresponds to a truncation of the difference between the total variations of $\gamma_{D+\delta D}$ and $\gamma_D$, or in other words, a truncation of the variation of the total variation. Recall that the total variation of $\gamma_D$ measures the oscillations in $\partial D$. Then the distance $| |$ only measures the “projection” of $\delta D$ on $V[D]$ and neglects oscillations higher than those generated by functions in $V[D]$.

Note also that the oscillations in $w_i^{(l)}$ are limited and that they are functions of $\omega$ and the distance between $D$ and $\partial \Omega$. As in (3.1), the oscillations in $w_i^{(l)}$ can be measured by the quotient

$$\frac{\int_{\partial D} \frac{|\partial h|}{\partial T}^2}{\int_{\partial D} |w_i^{(l)}|^2}. \quad (4.4)$$

By Shannon’s sampling theorem (see, for instance, [25, page 41]) this yields an estimate of the resolution limit, $\delta_{\text{res}}$, when reconstructing small changes $\partial D$ from boundary measurements that is one over the maximum on $i$ and $l$ of the oscillations in $w_i^{(l)}$:

$$\delta_{\text{res}} = \frac{2\pi \max_{i,l} \int_{\partial D} |w_i^{(l)}|^2}{\int_{\partial D} |\partial w_i^{(l)}|^2}. \quad (4.5)$$

Hence, any detail in the perturbations of size $\delta < \delta_{\text{res}}$ cannot be represented by functions in $V[D]$ since it has oscillations higher than any function in $V[D]$.

Now, from [8], one can prove that if $\delta D := \{h(x)\nu(x), x \in \partial D\}$ with $|h|_{C^1(\partial D)}$ small enough then

$$u^{(l)}[D + \delta D] - u^{(l)}[D] = u^{(l)}(h) + O(||h||_{C^1(\partial D)}^d),$$

uniformly on $\partial \Omega$. Here $d$ is the space dimension. This yields

$$\int_{\partial \Omega} (u^{(l)}[D + \delta D] - u^{(l)}[D]) \bar{f}_i[D] \int_{\partial D} h w_i^{(l)}[D] + O(||h||_{C^1(\partial D)}^d).$$

Thus,

$$|\delta D|^2 \approx \sum_{i=1}^{N} \sum_{l=1}^{M} \left| \int_{\partial \Omega} (u^{(l)}[D + \delta D] - u^{(l)}[D]) \bar{f}_i[D] \right|^2 + O(||h||_{C^1(\partial D)}^{2d-1}).$$

Therefore, the following local stability result holds.
Proposition 4.1 Suppose that \( \delta D := \left\{ h(x)\nu(x), x \in \partial D \right\} \) with \( ||h||_{C^1(\partial D)} < \epsilon \). Then there exist a positive constant \( C \) such that for \( \epsilon < \epsilon_0 \),

\[
|\delta D|^2 \leq C \left( \sum_{l=1}^N \sum_{i=1}^M \left| \int_{\partial \Omega} (u^{(l)}[D + \delta D] - u^{(l)}[D])T_i[D] \right|^2 + \epsilon^{2d-1} \right). \tag{4.6}
\]

Estimate (4.6) shows local Lipschitz stability and uniqueness of the reconstruction (up to \( \epsilon^{2d-1} \)) in the class of perturbations in \( V[D] \). In particular, it clearly indicates that the only information that can be reconstructed from the boundary measurements is the projection of \( \partial D \) onto the space \( V[D] \). Indeed, this reconstruction is stable. To our knowledge, such a result is new. It describes an optimal solution to handle the ill-posedness character of the inverse problem of reconstructing an inclusion from boundary measurements. It gives the exact class where the detectable perturbations should be and ensures their reconstruction in a stable way. By linearization and projection, we have reduced the ill-posed inverse problem of reconstructing an extended inclusion to a family of well-posed ones in much smaller but at each step optimal class (in the sense of detectability) of changes.

5 Initial guess by a MUSIC-type approach

There are many possible ways to get a good initial guess. One of them is to use a standard MUSIC-type projection approach. See, for instance, [18, 9, 7].

Define

\[ V^{(l)}(x) = e^{i\omega \theta_1 \cdot x} \text{ in } \Omega, \quad l = 1, \ldots, N. \]

so that \( g^{(l)} = \frac{\partial V^{(l)}}{\partial \nu} \). We construct the response matrix \( A(A'_{l, l'})_{l, l' = 1}^N \) with

\[ A_{ll'} = \int_{\partial \Omega} u^{(l)}(x) \frac{\partial V^{(l')}}{\partial \nu} - \int_{\partial \Omega} g^{(l)}(x) V^{(l')} \]. \tag{5.1}

Integration by parts shows that

\[ A_{ll'} = (1 - k) \int_D \nabla u^{(l)} \cdot \nabla V^{(l')} \]

Note that the knowledge of \( A_{ll'} \) is equivalent to the knowledge of the far-field of

\[ (1 - k) \int_D \nabla u^{(l)}(y) \cdot \nabla \Gamma(x, y) \]

as \( |x| \to +\infty \) and \( x/|x| \theta_{ll'} \), where \( \Gamma(x, y) \) is the free-space outgoing Green’s function for \( \Delta + \omega^2 \).

Let \( c_1, \ldots, c_d \) be \( d \) unit independent vectors in \( \mathbb{R}^d \). Let for \( z \) in the searching region \( \Omega' \subset \subset \Omega \)

\[ g_j(z) = (c_j \cdot \theta_1 e^{i\omega \theta_1 \cdot z}, \ldots, c_j \cdot \theta_1 e^{i\omega \theta_1 \cdot z})^t \quad \text{for } j = 1, \ldots, d, \]

where \( t \) denotes the transpose. We plot the MUSIC imaging functional

\[ W(z) := \frac{1}{\sum_j ||(I - P)g_j(z)||} \quad \text{for } z \in \Omega', \tag{5.2} \]
where \( P \) is the orthogonal projection onto the range of the response matrix \( A \). The set where \( W(z) \) attains its highest values would be a good initial guess for \( D^* \).

According to the Rayleigh resolution limit, any detail less than one-half of the wavelength cannot be seen [1]. By dividing the search domain \( \Omega' \) into pixels of length order size of half the wavelength, only one point at each pixel will contribute at the image space of the response matrix \( A \). Each of these points can in principle be imaged using the MUSIC imaging functional. The resolution of the image provided by this technique is of order of half a wavelength. Since the measurements are done at the boundary of \( \Omega \), MUSIC-type image can be an initial guess and by using the optimization algorithm described in the last section, higher-resolution in imaging the inclusion can be achieved.

6 Numerical experiments

In this section we show results of reconstructions using Method 2 (the second algorithm using (3.14)) which is based on the gradient descent method. We also show the results using Method 1 (the first standard algorithm using (2.9)) without regularization for comparison. Throughout this section, the background domain \( \Omega \) is given by

\[
\frac{x^2}{7^2} + \frac{y^2}{5^2} = 1
\]

and the conductivity inside the inclusion \( D \) is \( k = 4 \) and is assumed to be known. The angular frequency \( \omega \) is 2 except in Example 3 where \( \omega = 1 \) is used as well.

The direct solver uses the boundary integral representation of the solution to (1.2). In the following examples, we use the (outgoing) solution \( u^{(l)}_* \), for \( l = 1, \ldots, N = 10 \), which satisfies

\[
\nabla \cdot \gamma_D \nabla u^{(l)}_* + \omega^2 u^{(l)}_* = -\delta y_l,
\]

where

\[
y_l = 8(\cos((l-1)\pi/5), \sin((l-1)\pi/5)), \quad l = 1, \ldots, 10,
\]

are point sources outside \( \Omega \). Clearly, \( u^{(l)}_* |_{\Omega} \) is a solution to (1.2) with the Neumann datum \( g^{(l)} = \partial u^{(l)}_* / \partial \nu \).

Initial guess. To get an initial guess among disks we use the MUSIC-type algorithm described in Section 5. We collect the grid points where the MUSIC imaging functional \( W \) in (5.2) has a large value, and then find the smallest disk which encircle those grid points. We then modify the radius (among concentric disks) so that the \( L^2 \)-discrepancy of \( u[D] \) and \( u_* \) is minimal. Figure 1 shows results for two different inclusions. It clearly demonstrates that the MUSIC-type algorithm provides quite good initial guesses.

Example 1. This example is for reconstruction of the kite-shape inclusion. We use the disk constructed above (the left figure in Figure 1) as the initial guess. Figure 2 presents the initial guess and shapes for five iterations using Method 2 without noise. It clearly shows how the shapes gradually approach the actual kite-shape. Figure 3 compares the reconstruction results using Method 1 and Method 2 with 0\%, 10\% and 20\% noise (the percentage of noise is measured in terms of \( L^2 \)-norm. It shows that Method 2 catches the detail of the shape better while Method 1 regularizes the shape.
Figure 1: Initial guesses obtained by the MUSIC-type algorithm.

Figure 2: Reconstruction with Method 2 from measurements without error. The gray curve is the actual shape and the black curve is the reconstructed one. Images from top left to bottom right are the initial guess and the results of iterations 1, ..., 5.

**Example 2.** In this example inclusions are unit disks perturbed by $h = 0.2 \cos(m\theta)$ for $m = 3$ and 6. The inclusion is centered at $(-2, -1)$ in Figure 4 and at the origin in Figure 5. Figure 4 and 5 clearly show that Method 2 detects high oscillatory perturbations of the shape better than Method 1. Here we use $\omega = 2$.

**Example 3.** This example is to see the role of the frequency in the reconstruction. Figure 6 shows the reconstruction results with Method 2 using $\omega = 1$ and $\omega = 2$ after 9 iterations. It shows that using low frequencies one cannot detect highly oscillatory part of the shape.

**Example 4.** This example is for reconstruction of non-convex shapes. Figures 7 shows that both Method 1 and 2 work well for reconstruction of a mildly non-convex shape and Method 2 performs better. Figure 8 reveals the limitation of the reconstruction of severely non-convex shapes.
Figure 3: Images after 15 iterations. The first row is result with Method 1 and the second row with Method 2. The first, second and third columns are the reconstruction using the data with 0%, 10%, and 20% noise.

Figure 4: Reconstruction of the unit disk centered at \((-2, -1)\) perturbed by \(h = 0.2 \cos(3\theta)\). The upper row is by Method 1 and the lower row by Method 2 (after 9 iterations). The first, second and third columns are reconstructions from data with 0, 10, and 20% noise.

7 Level set framework

To handle topology changes such as breaking one component into two we develop a level set version of our algorithm.
Figure 5: Reconstruction of the unit disk centered at the origin perturbed by $h = 0.2 \cos(6\theta)$. The upper row is by Method 1 and the lower row by Method 2 (after 9 iterations). The first, second and third columns are reconstructions from data with 0, 10, and 20% noise.

Figure 6: Images obtained after 9 iterations with Method 2 from data without error. The figure on the left is with $\omega = 1$ and the one on the right is with $\omega = 2$. Low frequency data cannot detect high oscillations.

Within the level set framework, we represent $\partial D$ as the zero level set of a continuous function $\phi$ so that $D = \{ \phi < 0 \}$. We convert the minimization problem (3.8) into a level set form by choosing the gradient ascent direction $V(x)$ as

$$V(x) = \Re e \sum_{i=1}^{N} \sum_{l=1}^{M} \alpha_{il} [D] w^{(l)}(D)(x),$$

(7.1)

where $\alpha_{il}$ and $w^{(l)}$ are defined by (3.11) and (3.12), respectively.

Then we evolve $\phi$ by solving the Hamilton-Jacobi equation

$$\frac{\partial \phi}{\partial t} + V |\nabla \phi| = 0,$$

(7.2)
Figure 7: The upper row is the reconstruction with Method 1 and the second one with Method 2 after 21 iterations. The first, second and third columns are the reconstructions from data with 0%, 10%, and 20% noise.

Figure 8: Reconstructed images after 21 iterations. The first and second figures are respectively obtained using Method 1 and 2, from data without noise. The figures reveal the limitation of Methods 1 and 2 for the shape reconstruction of severely non-convex inclusions.

for one time step.

We emphasize that in (7.1), $V$ is only defined on the boundary $\partial D$, even though under the level set framework it has to be defined on the whole domain. We first note that since $\nu = \nabla \phi / |\nabla \phi|$, it follows that

$$w_i^{(1)}[D] = (k - 1) \left( \nabla p_i[D] \cdot \frac{\nabla \phi}{|\nabla \phi|} \right) \left( \nabla u^{(1)}[D] \cdot \frac{\nabla \phi}{|\nabla \phi|} \right) + \nabla p_i[D] \cdot \nabla u^{(1)}[D]. \tag{7.3}$$
Therefore, the equation (7.2) on \( \phi \) can be modified as follows:

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \left( \text{Re} \sum_{i=1}^{N} \sum_{l=1}^{M} \alpha_{il} \left[ (1 - k) \text{sgn}(\phi) + 1 + k - 2 \right] \nabla p_{i}[D] \cdot \frac{\nabla \phi}{|\nabla \phi|} \nabla u^{(i)}[D] \cdot \frac{\nabla \phi}{|\nabla \phi|} \right) |\nabla \phi| + \left( \text{Re} \sum_{i=1}^{N} \sum_{l=1}^{M} \alpha_{il} \nabla p_{i}[D] \cdot \nabla u^{(i)}[D] \right) |\nabla \phi| = 0,
\]

where \( \text{sgn} \) is the sign function. The evolution of the level-set function \( \phi \) then follows from the solution of (7.4) instead of (7.2).

8 Generalization to linear elasticity

In this section, we extend algorithm for the reconstruction of conductivity inclusions to that of elasticity ones. We restrict ourselves to the two-dimensional case and assume that both \( \Omega \setminus \overline{D} \) and \( D \) are occupied by isotropic and homogeneous elastic materials. Let \( I_{4} \) be the identity 4-tensor and \( I_{2} \) be the identity 2-tensor (the identity 2 \times 2 matrix). The elastic tensor fields \( C_{0} \) and \( C_{1} \) are then of the following form:

\[
C_{m} = \lambda_{m} I_{2} \otimes I_{2} + 2 \mu_{m} I_{4}, \quad m = 0, 1, \tag{8.1}
\]

where \((\lambda_{0}, \mu_{0})\) and \((\lambda_{1}, \mu_{1})\) are the Lamé constants corresponding to \( \Omega \setminus \overline{D} \) and \( D \), respectively, and \((\lambda_{0} - \lambda_{1})^{2} + (\mu_{0} - \mu_{1})^{2} \neq 0 \). We also assume that there are two positive constants \( \alpha_{0} \) and \( \beta_{0} \) such that

\[
\min(\mu_{0}, \mu_{1}) \geq \alpha_{0}, \quad \min(2\lambda_{0} + 2\mu_{0}, 2\lambda_{1} + 2\mu_{1}) \geq \beta_{0}, \tag{8.2}
\]

which guarantees strong convexity of \( C_{0} \) and \( C_{1} \).

Let \( C_{D} = C_{0} \chi[\Omega \setminus D] + C_{1} \chi[D] \) and let \( u_{*}^{(l)}, l = 1, \ldots, N, \) be the solution to

\[
\begin{cases}
\nabla \cdot C_{D} \nabla u_{*}^{(l)} + \omega^{2} u_{*}^{(l)} = 0 & \text{in } \Omega, \\
\frac{\partial u_{*}^{(l)}}{\partial \nu} = g^{(l)} & \text{on } \partial \Omega,
\end{cases}
\]

where \( \nabla u_{*}^{(l)} = \frac{1}{2}(\nabla u_{*}^{(l)} + (\nabla u_{*}^{(l)})^{t}) \) (\( t \) for transpose), \( \partial u/\partial \nu = (\nabla u)\nu \) denotes here the co-normal derivative, and \( g^{(l)}, l = 1, \ldots, N \) are given boundary data. The inverse problem here is to reconstruct the elastic inclusion \( D \) from the boundary measurements \( (u_{*}^{(l)})_{l=1}^{N} \) on \( \partial \Omega \).

As in Section 2, a standard algorithm to solve the inverse problem is to minimize over \( D \) the following cost functional:

\[
J[D] := \frac{1}{2} \sum_{l=1}^{N} \int_{\partial \Omega} \left| u^{(l)}[D] - u_{*}^{(l)} \right|^{2}, \tag{8.4}
\]

where \( u^{(l)}[D] \) is the solution to

\[
\begin{cases}
\nabla \cdot C_{D} \nabla u^{(l)} + \omega^{2} u^{(l)} = 0 & \text{in } \Omega, \\
\frac{\partial u^{(l)}}{\partial \nu} = g^{(l)} & \text{on } \partial \Omega.
\end{cases}
\]

\[ \text{(8.5)} \]
Here, $C_D = C_0 \chi_\Omega - C_1 \chi_D$.

Let

$$p := \frac{\lambda_1 (\lambda_0 + 2\mu_0)}{\lambda_1 + 2\mu_1} \quad \text{and} \quad q := \frac{4(\mu_1 - \mu_0)(\lambda_1 + \mu_1)}{\lambda_1 + 2\mu_1}.$$ 

Define a 4-tensor $K$ by

$$K := pI_2 \otimes I_2 + 2\mu_0 I_4 + qI_2 \otimes (T \otimes T).$$

Given two $2 \times 2$ matrices $A$ and $B$ we denote by $A : B = \sum_{ij} a_{ij} b_{ij}$.

For a given function $h$ on $\partial D$, the shape derivative of $J[D]$ in the direction of $h\nu$ is given by the same formula as (2.3) but with

$$v^{(l)}(h) := \int_{\partial D} h M[\nabla u^{(l)}] : \nabla N^\omega [D],$$

where

$$M[\nabla u^{(l)}] := (C_1 - C_0)C_1^{-1} \left( (\mathcal{K} \nabla u^{(l)}[D]T) \otimes I + (C_0 \nabla u^{(l)}[D]\nu) \otimes \nu \right),$$

and $N^\omega [D]$ is the Neumann function for $\nabla \cdot C_D \nabla + \omega^2$ in $\Omega$ corresponding to $\delta_y I_2$. See [11, 3].

To explicitly compute the shape derivative of $J$, we again introduce the adjoint state $p^{(l)}[D]$ as the solution to

$$\begin{cases}
\nabla \cdot C_D \nabla p^{(l)}[D] + \omega^2 p^{(l)}[D] = 0 & \text{in } \Omega,

\frac{\partial p^{(l)}[D]}{\partial \nu} = (u^{(l)}[D] - u^{(l)}) & \text{on } \partial \Omega,
\end{cases} \quad (8.6)$$

which is given in the elastic case by

$$p^{(l)}[D](x) \int_{\partial \Omega} (u^{(l)}[D] - u^{(l)}(y)) N^\omega [D](x, y) d\sigma(y).$$

Using $p^{(l)}$ to express the integral

$$\int_{\partial \Omega} (u^{(l)}[D] - u^{(l)}(y)) v^{(l)}(h) d\sigma,$$

we arrive at

$$(d_S J[D], h) = \int_{\partial D} h \Re \sum_{l=1}^N \left( M[\nabla u^{(l)}] : \nabla p^{(l)}[D] \right). \quad (8.7)$$

Set

$$w_n = \Re \sum_{l=1}^N \left( M[\nabla u^{(l)}[D_{n-1}]] : \nabla p^{(l)}[D_{n-1}] \right).$$

A first algorithm consists then in replacing, at each step $n$,

$$\partial D_{n-1} \mapsto \partial D_n := \partial D_{n-1} + h_n \nu,$$
where
\[ h_n = -J[D_{n-1}] \frac{w_n}{\int_{\partial D} w^2_n}. \] (8.8)

On the other hand, if we define \( \{ f_i[D] : i = 1, \ldots, M \} \) as the significant singular vectors associated with the operator \( \Lambda_D : L^2(\partial \Omega) \rightarrow L^2(\partial D) \) given by
\[ \Lambda_D(f) = p|_{\partial D}, \]
with
\[
\begin{align*}
\nabla \cdot \nabla_D \nabla p + \omega^2 p &= 0 \quad \text{in } \Omega, \\
\frac{\partial p}{\partial \nu} &= f \quad \text{on } \partial \Omega,
\end{align*}
\]
then in our new algorithm the update \( h_n \) is chosen as in (3.14) with
\[ w_i^{(l)}[D_{n-1}] = M[\nabla u^{(l)}[D_{n-1}]] : \nabla p_i[D_{n-1}], \]
\[ p_i[D_{n-1}] = \Lambda_{D_{n-1}}[f_i[D_{n-1}]], \]
and
\[ \alpha_{il}[D_{n-1}] := \int_{\partial \Omega} (u^{(l)}[D_{n-1}] - u^{(l)}_*) : \nabla f_i[D_{n-1}] \, d\sigma, \quad i = 1, \ldots, M, \quad l = 1, \ldots, N. \]

### 9 Concluding remarks

In this paper we have presented a new optimal control approach for the reconstruction of inclusions from boundary measurements. For doing so, we have constructed an optimal representation (in the sense of detectability) of small changes in the shape of the inclusion and proved a local stability result. We have performed some numerical experiments to demonstrate the validity and the limitations of the proposed optimal control method. The results clearly show that our approach is promising in recovering fine shape details.

To handle topology changes we have developed a level set version of our approach. Our approach also extends to the reconstruction of elastic inclusions from boundary measurements.

Reconstruction of electromagnetic inclusions will be discussed in a forthcoming work. We also intend to generalize our inversion procedure to the case where only a part of the boundary is accessible. A discrete version of our new algorithm in the response matrix framework will be provided. Numerical implementation of the level set version of our reconstruction algorithm is in progress and will be reported elsewhere.

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A High-frequency truncation

For simplicity, we restrict ourselves to the two-dimensional case. We parametrize ∂D by a 2π-periodic function s proportional to the arclength. For a function h write the Fourier expansion of h:

\[ h(s) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \hat{h}_n e^{ins}, \]

where

\[ \hat{h}_n := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} h(s)e^{-ins} ds. \]

Introduce the \( H^1 \)-inner product

\[ (u, v)_1 = \bar{u}_0 v_0 + \sum_{m \in \mathbb{Z}\{0\}} m^2 \bar{u}_m v_m \]

and define \( H^1 \) to be the Sobolev space that consists of all 2π-periodic and complex functions for which \( |v|_1 = \sqrt{(v, v)_1} \) is finite.

Now, because of the filtering properties of the Helmholtz equation (see [26, 27] and also Appendix B), for \( w[D] \) defined by (2.8), there exists \( n_0 \) such that

\[ w[D](s) \approx \sum_{|n| \leq n_0} \hat{w}_n e^{ins}. \]

If one defines the high-frequency component of h by

\[ h_{\text{high}} = \sum_{|n| > n_0} \hat{h}_n e^{ins}, \]

then one can easily see that

\[ \int_{\partial D} h_{\text{high}} w[D] \approx 0. \]

Moreover, we have

\[ \frac{\|\partial h_{\text{high}}/\partial s\|_{L^2(\partial D)}}{\|h_{\text{high}}\|_{L^2(\partial D)}} \geq n_0, \]

and hence \( n_0 \) is the threshold given in (3.1).

B Singular value decomposition of the operator \( \Lambda_D \)

Suppose that \( \Omega \) and \( D \) are respectively the disks of center 0 and radii \( R \) and \( R' \) with \( R' < R \). Write \( f \sum_n f_n e^{in\theta} \), where \( \theta \) is the angular variable. The explicit solution to (3.3) is given in the polar coordinates by

\[ p(r, \theta) = \begin{cases} \sum_n (a_n J_n(\omega r) + b_n Y_n(\omega r))e^{in\theta} & \text{in } \Omega \setminus D, \\ \sum_n c_n J_n(\sqrt{\omega^2 k^2} r)e^{in\theta} & \text{in } D, \end{cases} \]

(B.1)
where the coefficients $a_n, b_n, c_n$ can be explicitly computed in terms of the $f_n$. Here $J_n$ and $Y_n$ are the Bessel functions of the first and second kinds. The singular values of the operator $\Lambda_D : L^2(\partial\Omega) \to L^2(\partial D)$ are given by

$$\frac{c_n J_n\left(\frac{\omega R'}{\sqrt{k}}\right)}{f_n} \frac{A_n J_n(\omega R') - B_n Y_n(\omega R')}{\omega [A_n J'_n(\omega R) - B_n Y'_n(\omega R)]}, \quad n \in \mathbb{Z}, \quad (B.2)$$

where

$$A_n = \frac{1}{\sqrt{k}} Y_n(\omega R') J'_n\left(\frac{\omega}{\sqrt{k}} R'\right) - Y'_n(\frac{\omega}{\sqrt{k}} R) J_n\left(\frac{\omega}{\sqrt{k}} R'\right),$$

$$B_n = \frac{1}{\sqrt{k}} J_n(\omega R') J'_n\left(\frac{\omega}{\sqrt{k}} R'\right) - J'_n(\frac{\omega}{\sqrt{k}} R) J_n\left(\frac{\omega}{\sqrt{k}} R'\right).$$

Since $J_n(x) \sim \frac{1}{(n+1)!} \left(\frac{x}{2}\right)^n$ and $Y_n(x) \sim \frac{n!}{\pi} \left(\frac{x}{2}\right)^n$ for $n > 0$, one can see from (B.2) that singular values decay like $(R'/R)^{|n|}$ as the index $|n| \to +\infty$. In short, the singular values of $\Lambda_D$ decay exponentially.

References


