A Direct Algorithm for Ultrasound Imaging of Internal Corrosion

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Abstract

We develop a direct (non-iterative) algorithm to address the inverse problem of identifying a collection of disjoint internal corrosive parts of small Hausdorff measures in pipelines from exterior ultrasound boundary measurements. The algorithm is based on an asymptotic expansion of the effect of the corrosion in terms of the size of the corrosive parts. We numerically test the validity of the asymptotic formula at high frequencies. We also propose a simple procedure to remove high-frequency instabilities in our inversion procedure. We illustrate our main findings with a variety of computational examples.

Mathematics subject classification (MSC2000): 35R30
Keywords: direct imaging, corrosion, asymptotic representation formula, reconstruction, ultrasound detection, high frequencies, MUSIC-type algorithm

1 Introduction

Corrosion detection remains a topic of considerable activity in the inverse problems community. Of special interest are algorithms that make use of a priori information or assumptions concerning the nature of target’s internal structure. Such assumptions can usually be exploited to obtain much faster, simpler and more stable algorithms than would otherwise be possible.

In this paper we consider the problem of determining the corrosion damage of an inaccessible part of the surface of a specimen from ultrasound measurements.

The impedance imaging problem for detecting internal corrosion was considered in [5]. In that paper the authors propose an algorithm of MUSIC (multiple signal classification)
type for detecting corrosive parts in pipelines from input fluxes. This algorithm is based on an accurate asymptotic representation formula for the steady state voltage perturbations.

In this paper, we adapt the MUSIC-type algorithm of [5] to the setting in which multiple ultrasound boundary measurements are available and also show how one may quantitatively deduce the number of corrosive parts present. Our approach relies on an asymptotic expansion of the ultrasound reflected wave with respect to the length of the corrosive parts. Yet another method for corrosion detection, the vibration testing, has been considered in a recent paper [4]. Preliminary versions of this paper’s results have been compared with results by electrostatic and vibration tests [3]. The method of this paper (ultrasound measurements) seems to perform as good as that of electrostatic imaging, while the result by vibration testing seems to the worst, which is natural because the modal measurements are limited.

The organization of the paper is as follows. In Section 2 we state the forward and corresponding inverse problem of interest, then review some basic facts on Green’s functions. In Section 3 we state our main theorem that allows us in Section 4 to develop our MUSIC-type algorithm for recovering a collection of corrosive parts. In Section 5 we briefly discuss how to resolve the difficulty in solving our inverse problem that is caused by high oscillations in the measurements data. In Section 6 we provide several computational examples, as well as concluding remarks. We consider only the two-dimensional case, the extension to three dimensions being obvious.

2 Preliminaries and formulation of the inverse problem

Let us first fix notation for this work. Let the annulus $\Omega = \{x : r < |x| < R\}$ represent the specimen to be inspected. Let $\Gamma_e$ and $\Gamma_i$ denote the circles of radius $R$ and $r$ centered at the origin, respectively. Suppose that the inaccessible surface $\Gamma_i$ contains some corrosive parts $I_s$, $s = 1, \ldots, m$. The parts $I_s$ are well-separated and the reciprocal of the surface impedance (the corrosion coefficient) of each $I_s$, $s = 1, \ldots, m$, is $\gamma_s \geq 0$, not identically zero. We assume that each $\gamma_s \in C^1(I_s)$. Let

$$
\gamma(x) = \sum_{s=1}^{m} \gamma_s(x) \chi_s(x), \quad x \in \Gamma_i,
$$

(2.1)

where $\chi_s$ denotes the characteristic function of $I_s$. The annulus $\Omega$ in two dimensions may be considered as a cross section of a pipe inside which there are corrosive parts. We assume that the Hausdorff measures (length) of $I_s$ are small:

$$
|I_s| = O(\epsilon), \quad s = 1, \ldots, m,
$$

(2.2)

where $\epsilon$ is a small parameter representing the common order of magnitude of $I_s$. Here and throughout this paper $|\cdot|$ denotes the one dimensional Hausdorff measure. Then for each $p \geq 1$, we have

$$
\|\gamma\|_{L^p(\Gamma_i)} \leq C\epsilon^{1/p}.
$$

(2.3)

For $\omega > 0$, a fundamental solution $\Phi_\omega(x)$ to the Helmholtz operator $\Delta + \omega^2$ in $\mathbb{R}^2$ is given by

$$
\Phi_\omega(x) = -\frac{i}{4} H^{(1)}_0(\omega|x|),
$$

(2.4)
for \( x \neq 0 \), where \( H_0^{(1)} \) is the Hankel function of the first kind of order 0. The ultrasound wave \( u_\gamma \) generated by a source at \( y \in \Gamma_e \) satisfies
\[
\begin{align*}
(\Delta + \omega^2)u_\gamma &= 0 \quad \text{in } \Omega, \\
\frac{\partial u_\gamma}{\partial \nu} + \gamma u_\gamma &= 0 \quad \text{on } \Gamma_i, \\
u_\gamma &= \Phi_\omega(\cdot - y) \quad \text{on } \Gamma_e,
\end{align*}
\] (2.5)
where \( \nu \) is the outward unit normal to \( \Omega \).

Let \( u_0 \) denote the solution in absence of the corrosion, i.e., the solution to the problem
\[
\begin{align*}
(\Delta + \omega^2)u_0 &= 0 \quad \text{in } \Omega, \\
\frac{\partial u_0}{\partial \nu} &= 0 \quad \text{on } \Gamma_i, \\
u_0 &= \Phi_\omega(\cdot - y) \quad \text{on } \Gamma_e.
\end{align*}
\] (2.6)
Throughout this paper, we suppose that \( \omega \) is not an eigenvalue of \(-\Delta\) in \( \Omega \) with the Dirichlet boundary condition on \( \Gamma_e \) and the Neumann boundary condition on \( \Gamma_i \). Using the theory of collectively compact operators [2], we can easily prove that (2.5) is uniquely solvable for \( ||\gamma||_{L^p} \) small enough.

Define the Dirichlet function \( G_\omega \) by
\[
\begin{align*}
(\Delta_x + \omega^2)G_\omega(x, y) &= -\delta_y(x) \quad \text{in } \Omega \ (y \in \Omega), \\
\frac{\partial G_\omega}{\partial \nu_x}(x, y) &= 0, \quad x \in \Gamma_i, \\
G_\omega(x, y) &= 0, \quad x \in \Gamma_e,
\end{align*}
\] (2.7)
Then, we have
\[
G_\omega(x, y) = G_\omega(y, x) \quad \text{for } x \neq y \in \Omega,
\] (2.8)
and the solution \( u_0 \) to (2.6) is given by
\[
u_0(x) = -\int_{\Gamma_e} \frac{\partial G_\omega}{\partial \nu_z}(z, x)\Phi_\omega(z - y) \, d\sigma(z), \quad x \in \Omega, \ y \in \Gamma_e,
\] (2.9)
which can be proved using Green’s theorem.

Throughout this paper, we will denote by \( S_{\Gamma_i}^e \) the single layer potential on \( \Gamma_i \) associated with \( G_\omega \), that is,
\[
S_{\Gamma_i}^e[\varphi](x) = \int_{\Gamma_i} G_\omega(x, y)\varphi(y) \, d\sigma(y), \quad x \in \mathbb{R}^2
\] (2.10)
for \( \varphi \in L^2(\Gamma_i) \).

3 Asymptotic formula

We derive in this section an asymptotic expansion of the ultrasound boundary perturbations due to the presence of the corrosive parts.
Let \( u_\gamma \) and \( u_0 \) be the solutions to (2.5) and (2.6), respectively. Let \( v := u_\gamma - u_0 \). Then \( v \) satisfies

\[
\begin{align*}
\Delta v + \omega^2 v &= 0 \quad \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} + \gamma v &= -\gamma u_0 \quad \text{on } \Gamma_i, \\
v &= 0 \quad \text{on } \Gamma_e.
\end{align*}
\]

(3.1)

By integrating the first equation in (3.1) against \( G_\omega(x,y) \) and using the divergence theorem, we get

\[
v + \mathcal{S}_{\Gamma_i}^\omega[\gamma v] = -\mathcal{S}_{\Gamma_i}^\omega[\gamma u_0] \quad \text{in } \Omega,
\]

(3.2)

and hence

\[
\gamma v + \gamma \mathcal{S}_{\Gamma_i}^\omega[\gamma v] = -\gamma \mathcal{S}_{\Gamma_i}^\omega[\gamma u_0] \quad \text{in } \Omega.
\]

(3.3)

The following lemma is of use to us.

**Lemma 3.1** If \( \epsilon \) is sufficiently small, then the operator \( I + \gamma \mathcal{S}_{\Gamma_i}^\omega \) is invertible on \( L^2(\Gamma_i) \).

**Proof.** Since \( \gamma \mathcal{S}_{\Gamma_i}^\omega \) is a compact operator on \( L^2(\Gamma_i) \), it suffices to show the injectivity of \( I + \gamma \mathcal{S}_{\Gamma_i}^\omega \) by the Fredholm alternative. Suppose that \( (I + \gamma \mathcal{S}_{\Gamma_i}^\omega)v = 0 \) for some \( v \in L^2(\Gamma_i) \).

Then, it follows from (2.3) that

\[
\|v\|_{L^2(\Gamma_i)} = \|\gamma \mathcal{S}_{\Gamma_i}^\omega[v]\|_{L^2(\Gamma_i)} \\
\leq \|\gamma\|_{L^2(\Gamma_i)} \|\mathcal{S}_{\Gamma_i}^\omega[v]\|_{L^\infty(\Gamma_i)} \\
\leq C \epsilon^{1/2} \|v\|_{L^2(\Gamma_i)},
\]

and hence \( v = 0 \) if \( \epsilon \) is sufficiently small. This completes the proof. \( \square \)

We now derive an asymptotic formula for \( \frac{\partial(\gamma \gamma - u_0)}{\partial \nu} \) as \( \epsilon \to 0 \), which will be an essential ingredient for designing the reconstruction scheme.

**Theorem 3.2** The following asymptotic formula holds uniformly on \( \Gamma_e \) as \( \epsilon \to 0 \):

\[
\frac{\partial(u_\gamma - u_0)}{\partial \nu}(x) = -\sum_{s=1}^m \langle \gamma \rangle_s u_0(z_s) \frac{\partial G_\omega(x,z_s)}{\partial \nu} + O(\epsilon^{1+1/p}), \quad x \in \Gamma_e,
\]

(3.4)

for all \( p > 1 \), where

\[
\langle \gamma \rangle_s := \int_{I_s} \gamma(x) \, d\sigma(x).
\]

**Proof.** Since there exists a positive constant \( C \) depending only on \( p, \omega, \) and \( r \) such that for \( p > 1 \)

\[
\|\mathcal{S}_{\Gamma_i}^\omega[\gamma v]\|_{L^\infty(\Gamma_i)} \leq C \|\gamma v\|_{L^p(\Gamma_i)} \leq C \epsilon^{\frac{1}{p}} \|v\|_{L^\infty(\Gamma_i)}
\]

and

\[
\|\mathcal{S}_{\Gamma_i}^\omega[\gamma u_0]\|_{L^\infty(\Gamma_i)} \leq C \epsilon^{\frac{1}{p}} \|u_0\|_{L^\infty(\Gamma_i)},
\]

it follows from (3.2) that

\[
\|v\|_{L^\infty(\Gamma_i)} \leq C \epsilon^{\frac{1}{p}}
\]

(3.5)
for some constant $C = C(p, \omega, r, R)$. Since the distance between $\Gamma_e$ and $\Gamma_i$ is $R - r$, we get from (3.2) that

$$
\left\| \frac{\partial}{\partial \nu} S^\epsilon_{\Gamma_i}[\gamma v] \right\|_{L^\infty(\Gamma_i)} \leq C \|\gamma v\|_{L^1(\Gamma_i)} \leq C \|\gamma\|_{L^1(\Gamma_i)} \|v\|_{L^\infty(\Gamma_i)} \leq C \epsilon^{1+\frac{1}{p}},
$$

(3.6)

where $C$ depends on $p, \omega, r, \text{ and } R$. We now obtain from (3.2) and (3.6) that

$$
\frac{\partial v}{\partial \nu}(x) = - \int_{\Gamma_i} (\gamma u_0)(y) \frac{\partial G_\omega}{\partial \nu}(x, y) d\sigma(y) + O(\epsilon^{1+\frac{1}{p}}) \text{ in } L^\infty(\Gamma_e).
$$

(3.7)

Since

$$
\frac{\partial G_\omega}{\partial \nu}(x, y) = \frac{\partial G_\omega}{\partial \nu}(x, z_s) + O(|y - z_s|), \quad y \in I_s
$$

and

$$
\int_{\Gamma_i} \gamma u_0 d\sigma = \sum_s \langle \gamma \rangle_s u_0(z_s) + O(\epsilon^2 \|\nabla u_0\|_{L^\infty(\Gamma_i)}),
$$

we obtain the desired result (3.4).

□

We shall make a remark on the asymptotic formula (3.4). The formula is valid when $\omega$ is fixed and $\epsilon$ goes to zero. But its frequency dependency is not clear in its present form. Finding precisely this frequency dependency is a very challenging problem. In this direction we perform some numerical experiments in the next section.

4 High-frequency asymptotic formula-numerical experiment

In order to investigate the frequency dependency of (3.4), we compute numerically

$$
F(\epsilon, \omega) := \left\| \frac{\partial (u_\gamma - u_0)}{\partial \nu}(x) + \sum_{s=1}^{m} \langle \gamma \rangle_s u_0(z_s) \frac{\partial G_\omega}{\partial \nu}(x, z_s) \right\|_{L^\infty(\Gamma_e)}
$$

(4.1)

for various $\epsilon$ and $\omega$.

For computations we take the domain $\Omega \subset \mathbb{R}^2$ to be the annulus centered at $(0,0)$ with radii 0.5 and 0.4. The corrosive part $I$ consists of a single arc with the corrosion coefficient $\gamma = 2$. We choose the mesh points $N = 960$ on each of $\Gamma_e$ and $\Gamma_i$. We then compute the values of $F(\epsilon, \omega)$ for various $\epsilon$ and $\omega$: $\omega = 0.01, 0.1, 1, 3, \cdots, 15$ and 0.0079 $\leq \epsilon \leq 0.2644$. Figure 1 shows the results. The first figure on the right clearly shows that the dependency on $\epsilon$ is $\epsilon^2$. The second one tells us that the dependency on $\omega$ is much more subtle: something of the form $e^{\psi(\omega)}$ for some function $\psi(\omega)$, which is almost constant for low values of $\omega$ (low frequency) and rapidly increasing for high values of $\omega$. The threshold is about 7. These computations suggest that the following formula holds:

$$
\frac{\partial (u_\gamma - u_0)}{\partial \nu}(x) = - \sum_{s=1}^{m} \langle \gamma \rangle_s u_0(z_s) I_s \left| \frac{\partial G_\omega}{\partial \nu}(x, z_s) \right| + O(\epsilon^2 e^{\psi(\omega)}), \quad x \in \Gamma_e.
$$

(4.2)
5 Reconstruction methods

In this section, we adapt the MUSIC-type algorithm of [5] to the setting in which multiple ultrasound boundary measurements are available and also show how one may quantitatively deduce the number of corrosive parts present. Our approach relies on the asymptotic expansion (3.4) of the ultrasound reflected wave with respect to the length of the corrosive parts.

5.1 A MUSIC-type algorithm

Let us first introduce the notion of the Dirichlet-to-Neumann (DtN) map. Let \( u_\gamma \) be the solution to the problem (2.5) and define the DtN map \( \Lambda_\gamma^\omega \) by

\[
\Lambda_\gamma^\omega[y] := \frac{\partial u_\gamma}{\partial \nu} \bigg|_{\Gamma_e}. \tag{5.1}
\]

Let \( \Lambda_0^\omega \) be the DtN map when there is no corrosion. We also define \( T^\omega \) by

\[
T^\omega[y](x) = -\sum_{s=1}^m (\gamma)_s u_0(z_s) \frac{\partial G_\omega}{\partial \nu}(x, z_s), \quad x \in \Gamma_e. \tag{5.2}
\]

Observe that the dependency of \( T^\omega[y] \) on \( y \) is hidden in the term \( u_0(z_s) \) where \( u_0 \) is the solution to (2.6). Then the formula (3.4) now reads

\[
(\Lambda^\omega_\gamma - \Lambda_0^\omega)[y] \approx T^\omega[y]. \tag{5.3}
\]

The MUSIC algorithm of this paper is based on the following simple observation.

**Lemma 5.1** Suppose that \( \omega \) is not a Dirichlet eigenvalue of \( \Delta \) in the disk of radius \( R \). Let \( \Phi_\omega \) and \( G_\omega \) be the functions defined by (2.4) and (2.7), respectively. For \( z \in \Gamma_i \) and \( x \in \Gamma_e \), let

\[
h_z(x) := \int_{\Gamma_e} \frac{\partial G_\omega(y,z)}{\partial \nu} \Phi_\omega(y-x) \, d\sigma(y). \tag{5.4}
\]
If there are complex numbers $a_1, \ldots, a_m$ such that

$$h_z(x) = \sum_{s=1}^{m} a_s h_{z_s}(x) \quad \text{for all } x \in \Gamma_e,$$

then $z \in \{z_1, \ldots, z_m\}$.

**Proof.** We first observe that if $\omega$ is not a Dirichlet eigenvalue of $\Delta$ in the disk of radius $R$ and $\int_{\Gamma_e} f(y) \Phi(\omega)(y - x) \, d\sigma(y) = 0$ for all $x \in \Gamma_e$, then $f \equiv 0$, a proof of which can be found in [8]. The relation (5.5) implies that

$$\int_{\Gamma_e} \left[ \partial G(\omega, y, z) \frac{\partial G(\omega)(y, z_s)}{\partial \nu}(y, z_s) \right] \Phi(\omega)(y - x) \, d\sigma(y) = 0 \quad \text{for all } x \in \Gamma_e,$$

from which it follows that

$$\frac{\partial G(\omega)(y, z)}{\partial \nu} - \sum_{s=1}^{m} a_s \frac{\partial G(\omega)(y, z_s)}{\partial \nu} = 0 \quad \text{for all } y \in \Gamma_e. \quad (5.6)$$

Since $G(\omega)(y, z) = \sum_{s=1}^{m} a_s G(\omega)(y, z_s) = 0$ for all $y \in \Gamma_e$, we have from (5.6) and the unique continuation that $G(\omega)(y, z) = \sum_{s=1}^{m} a_s G(\omega)(y, z_s)$ for all $y \in \Omega$. Since $G(\omega)(y, z)$ has singularity at $y = z$ while $\sum_{s=1}^{m} a_s G(\omega)(y, z_s)$ has at $y = z_s$ for those $s$ such that $a_s \neq 0$, we conclude that $z = z_s$ for some $s$. This completes the proof. \qed

We now discretize the relation (5.5) using uniformly spaced set $\{y_1, \ldots, y_N\}$ on $\Gamma_e$ for a sufficiently large $N$ to conclude that if there are complex numbers $a_1, \ldots, a_m$ such that

$$h_z(y_\ell) = \sum_{s=1}^{m} a_s h_{z_s}(y_\ell) \quad \text{for } \ell = 1, \ldots, N, \quad (5.7)$$

then $z$ approximately equals to $z_s$ for some $s = 1, \ldots, m$. This is the MUSIC characterization of the location of corrosive parts $z_s$. Let us put, for $z \in \Gamma_i$,

$$h_z := (h_z(y_1), \ldots, h_z(y_N))^T.$$

Then the condition (5.7) is equivalent to

$$h_z \text{ is a linear combination of } \{h_{z_1}, \ldots, h_{z_m}\}. \quad (5.8)$$

Thus the main task in determining the location of corrosive parts is to determine those $z$ which satisfy the condition (5.8).

It should be emphasized that we do not have those vectors $h_{z_s}$ in our hand since we do not know the location $z_s$ of the corrosive parts. The key fact in designing the algorithm is that even if we do not know the individual $h_{z_s}$, the space spanned by $h_{z_s}$, $s = 1, \ldots, m$, can be characterized by the boundary measurements, and hence the condition (5.8) can be checked. Let $u_0^\ell$ be the solutions to (2.5) and (2.6) with $y = y_\ell$, respectively, for $\ell = 1, \ldots, N$. Since

$$u_0^\ell(z_s) = - \int_{\Gamma_e} \frac{\partial G(\omega)(y, z_s)}{\partial \nu}(y - y_\ell) \Phi(\omega)(y) \, d\sigma(y) = -h_{z_s}(y_\ell),$$
we have
\[ T^\omega[y_\ell](y) = \sum_{s=1}^{m} \langle \gamma \rangle_s h_{z_s}(y_\ell) \frac{\partial G_\omega}{\partial \nu}(y, z_s). \] (5.9)

In the spirit of the reciprocity gap approach [7, 6], multiplying by \( \Phi_\omega(y - y_{\ell'}) \) and integrating over \( \Gamma_e \), we obtain
\[ \int_{\Gamma_e} T^\omega[y_\ell](y) \Phi_\omega(y - y_{\ell'}) \, d\sigma(y) = \sum_{s=1}^{m} \langle \gamma \rangle_s h_{z_s}(y_\ell) h_{z_s}(y_{\ell'}). \] (5.10)

Define \( T_N = (t_{\ell\ell'})_{\ell,\ell'=1}^N \) by
\[ t_{\ell\ell'} = \int_{\Gamma_e} T^\omega[y_\ell](y) \Phi_\omega(y - y_{\ell'}) \, d\sigma(y). \]

Then (5.10) shows that the condition (5.8) is equivalent to
\[ h_z \in \text{Range}(T_N). \] (5.11)

Define \( N \times N \) matrix \( \Lambda_N \) by
\[ \Lambda_N = \left( \int_{\Gamma_e} (\Lambda^\omega_\gamma - \Lambda^\omega_0)[y_\ell](y) \Phi_\omega(y - y_{\ell'}) \, d\sigma(y) \right)_{\ell,\ell'=1}^N. \] (5.12)

According to (5.3), we have
\[ \Lambda_N \approx T_N, \]
and hence in view of (5.11) the algorithm is to determine those \( z \in \Gamma_i \) satisfying
\[ h_z \in \text{Range}(\Lambda_N). \] (5.13)

It should be emphasized that \( \Lambda_N \) can be computed using the boundary measurements \( (\Lambda^\omega_\gamma - \Lambda^\omega_0)[y] \)

Note that since \( \Lambda^\omega_\gamma - \Lambda^\omega_0 \) is self-adjoint in the sense that
\[ \int_{\Gamma_e} (\Lambda^\omega_\gamma - \Lambda^\omega_0)[x](y) \Phi_\omega(y - z) \, d\sigma(y) = \int_{\Gamma_e} (\Lambda^\omega_\gamma - \Lambda^\omega_0)[z](y) \Phi_\omega(y - x) \, d\sigma(y) \]
for all \( x, z \in \Gamma_e \), which can be proved easily, \( \Lambda_N \) is symmetric and hence admits a (orthogonal) singular value decomposition (SVD). Using eigenvectors of \( \Lambda_N \) we can characterize those \( z \) satisfying (5.13).

Let us summarize the algorithm which generalizes the one in [5].

[MUSIC-type Algorithm]

**Step 1.** Obtain the Neumann data \( (\Lambda^\omega_\gamma - \Lambda^\omega_0)[y_\ell] \) on \( \Gamma_e \) for \( y_\ell \in \Gamma_e, \ell = 1, \cdots, N; \)

**Step 2.** Compute the matrix \( \Lambda_N \);

**Step 3.** Compute the SVD of \( \Lambda_N \). Let \( \lambda_1 \geq \lambda_2 \geq \cdots \) be the eigenvalues of \( \Lambda_N \) and let \( \{v_p, p = 1, 2, \ldots \} \) be an orthonormal basis of eigenfunctions;
Step 4. For $k = 1, 2, \ldots$, let $P_k$ be the orthogonal projector on to the space spanned by \{v_1, \ldots, v_k\} and define $\theta_k(z)$ by

$$\cot \theta_k(z) = \frac{\|P_k(h_z)\|}{\|(I - P_k)(h_z)\|}.$$  \hspace{1cm} (5.14)

We then compute the minimal values of $\theta_k(z)$ for $k = 1, 2, \ldots$. We repeat this process until the minimal values stabilize. Those $z$ where the minimal value of $\theta_k(z)$ occur are detected locations of the corrosive parts.

Step 5. We then determine the corrosion coefficients $\langle \gamma \rangle_s$ by solving a linear system.

It is worth emphasizing that we may use much less source point $y_i$ in Step 1. In this case the response matrix $\Lambda_N$ is not a square matrix but still admits an SVD.

6 High-frequency instabilities

In many practical situations, $R - r$ is of order $r$ and much larger than the wavelength $\lambda = 2\pi/\omega$ while $\epsilon$ is much smaller than $\lambda$. In this case, formula (4.2) remains valid but does not yield a stable detection procedure because of high oscillations in the computations of $\frac{\partial G}{\partial y}$ and $h_z$ defined by (5.4). In fact, Figure 2 shows the oscillations of $\frac{\partial G}{\partial y}(\cdot, z)$ on $\Gamma_e$ for $\omega = 10, 18, 20, 30, 40, 50$, and Figure 3 shows those of $h_z$ for $\omega = 1, 10, 15, 18, 20$.

To resolve this difficulty, we first introduce the truncation operator $P_n$ defined by

$$P_n(f) := \sum_{|\ell| \leq n} f_\ell e^{i\ell \theta}$$ \hspace{1cm} (6.1)

for $n = 1, 2, \ldots$, when $f$ is given in its Fourier series by $f = \sum_{\ell \in \mathbb{Z}} f_\ell e^{i\ell \theta}$. The operator $P_n$ is a low pass filter operator. We choose $n = \omega(R - r)$ as the threshold of the truncation.

To justify the validity of this filtering, consider the Helmholtz equation $(\Delta + \omega^2)w = 0$ in $|x| > r$ with the boundary condition $w = f$ on $|x| = r$. Then we have

$$w(R, \theta) = \sum_{\ell \in \mathbb{Z}} f_\ell \frac{H_{(1)}^{(1)}(\omega R)}{H_{(1)}^{(1)}(\omega r)} e^{i\ell \theta}, \hspace{1cm} R > r.$$  \hspace{1cm} (6.2)

By making use of Debey’s asymptotic expansion [1, page 366],

$$\frac{H_{(1)}^{(1)}(\omega R)}{H_{(1)}^{(1)}(\omega r)} \approx \frac{e^{l(\alpha - \tanh \alpha)}}{e^{l(\alpha' - \tanh \alpha')}} \approx e^{l(\alpha - \tanh \alpha)} \approx e^{l(\alpha' - \tanh \alpha')}, \hspace{1cm} \text{for } l > \omega R, \hspace{1cm} (6.2)$$

where $\omega R = l \sech \alpha$ and $\omega r = l \sech \alpha'$. Note that $\alpha' > \alpha$. Moreover, from the asymptotic behavior of Hankel functions [1, 9.3.3], i.e.,

$$H_{(1)}^{(1)}(l \sec \beta) = \sqrt{2/(\pi l \tan \beta)} \{ e^{i(l \tan \beta - l\beta - \frac{1}{2} \pi)} + O(1^{-1}) \}, \hspace{0.5cm} 0 < \beta < \frac{1}{2} \pi, \hspace{1cm} (6.3)$$

we have

$$\frac{H_{(1)}^{(1)}(\omega R)}{H_{(1)}^{(1)}(\omega r)} \approx \sqrt{\frac{\tan \beta'}{\tan \beta}} \{ e^{i(l \tan \beta - l \beta - l \tan \beta' + l \beta')} + O(1^{-1}) \}, \hspace{1cm} \text{for } l \leq \omega r, \hspace{1cm} (6.3)$$
where $\omega R = l \sec \beta$ and $\omega r = l \sec \beta'$. From (6.2) we see that the Helmholtz operator acts as a low pass filter. It filters all the
for \( l \) larger than \( \omega r \) and moreover (6.3) would suggest that the angular resolution is of order \( 2\pi/\omega r = \lambda/r \), which is beyond the Rayleigh resolution limit.

Now it follows from (4.2) that

\[
P_n \left[ \frac{\partial(u_n - u_0)}{\partial \nu} \right] \approx -\sum_{s=1}^{m} \gamma(z_s) u_0(z_s) I_s \left[ P_n \left[ \frac{\partial G_\omega}{\partial \nu} \right] \right](\cdot, z_s) \quad \text{on } \Gamma_e,
\]

since \( P_n \) is linear. The necessary modifications of our MUSIC procedure are obvious. They simply consist in filtering high-frequency oscillations in the function \( h_z \) defined by (5.4). We should replace \( h_z \) by

\[
h_z(x) := P_n \left( \int \frac{\partial G_\omega}{\partial \nu} (y, z) \Phi_\omega(y - x) \ d\sigma(y) \right).
\]

(6.4)

Figure 4 shows the projected measurements \( P_n \left[ \frac{\partial(u_n - u_0)}{\partial \nu} \right] \) for various \( n \) at \( \omega = 20 \). The measurement \( \frac{\partial(u_n - u_0)}{\partial \nu} \) does not seem to yield any useful information on the location of the corrosive part. But the filtering, especially for \( n = 2, 3, 4 \), clearly exhibits the location \( z \).

\section{Numerical results}

This section presents numerical results of finding the internal corrosive parts, using the MUSIC-type algorithm. In the following, \( \Omega \subset \mathbb{R}^2 \) is assumed to be the annulus centered at \((0,0)\) with radii, \( R \) and \( r \). \( \Gamma_e \) and \( \Gamma_i \) are the outer and inner boundary of \( \Omega \), respectively. Let \( I_s, s = 1, \ldots, m \), be the corrosive parts with the corrosion coefficient \( \gamma_s \).

For the computations, we discretize \( \Gamma_e \) by \( \{y_1, \ldots, y_N\} \), where

\[
y_n := r_e (\cos(\theta_n + \pi/N), \sin(\theta_n + \pi/N)) \quad \text{and} \quad x_n := r_i (\cos \theta_n, \sin \theta_n)
\]

with \( \theta_n = 2\pi(n-1)/N \) for \( n = 1, \ldots, N \). Here we take \( N = 256 \).

In order to obtain the measurements \( (\Lambda^\omega_\gamma - \Lambda^\omega_0)[y_n] \) on \( \Gamma_e \) for \( n = 1, \ldots, N \), we solve the direct problems (2.5) and (2.6) using the boundary integral method based on the layer potentials for the Helmholtz equation. For doing this, there is one technical difficulty. If we use the same discrete points \( \{y_n\} \) to evaluate \( (\Lambda^\omega_\gamma - \Lambda^\omega_0)[y_n](y_m) \) numerically, then \( (\Lambda^\omega_\gamma - \Lambda^\omega_0)[y_n](y_m) \) blows up if \( n = m \). To avoid this technical difficulty, we evaluate \( (\Lambda^\omega_\gamma - \Lambda^\omega_0)[y_m] \) at points \( y_m = r_e (\cos \theta_n, \sin \theta_n) \) for \( m = 1, \ldots, N \). In other words the response matrix \( ((\Lambda^\omega_\gamma - \Lambda^\omega_0)[y_n](y_m))^{N}_{n,m=1} \) is our measurements.

In the following examples, the outer radius \( R = 1 \) and the inner one \( r = 0.8 \) and there are four corrosive parts. Figure 5 shows the actual domain with corrosive parts.

Using the response matrix \( ((\Lambda^\omega_\gamma - \Lambda^\omega_0)[y_n](y_m))^{N}_{n,m=1} \), we compute the matrix \( \Lambda_N \) defined by (5.12). In doing this we add \( p\% \) random noise to the measurements for \( p = 0, 1, 5, 10 \). Adding \( p\% \) noise means that our measurement becomes

\[
w + \frac{p}{100} \text{rand}(1) \max\{|w|\},
\]

where \( w \) is the actual measurement (the computed response matrix) on \( \Gamma_e \). Here \( \text{rand}(1) \) is a random number generator in \((-1, 1)\).
Figure 4: Top-left is the actual configuration of the corrosive pipe. The operating frequency \( \omega = 20 \). Top-right are the real and imaginary parts of the measurements without using filtering. The two bottom rows are filtered measurements \( P_n(\Lambda^\omega \gamma - \Lambda^\omega) \) for various \( n \). The data without filtering does not yield much information on the location of the corrosive part. But the filtered data with \( n = 2, 3, 4 \) yield enough information to determine the location. Here the threshold \( \omega(R - r) = 2 \).

Figure 6 shows the computed \( \Lambda_N \) under various noise levels. It should be noted that we set the frequency \( \omega = 5 \).

Figure 7 shows the computational results of the MUSIC-type algorithm. The top figures are SVDs of \( \Lambda_N \) and \( T_N \). The SVD of \( T_N \) exhibits a clear drop of the eigenvalues after four significant eigenvalues, which is the number of the corrosive parts. On the other hand, the SVD of \( \Lambda_N \) shows a larger number of significant eigenvalues, which corresponds to the number of nodal points of \( \Gamma_i \) inside the corrosive parts. The figures in the middle are the graphs of \( \theta_k \) defined by (5.14) for \( k = 1, 2, \ldots \), where \( k \) denotes the number of eigenvectors of \( \Lambda_N \) used. We can see that the second and third corrosive parts are detected when there
Figure 5: The actual pipe with four corrosive parts.

Figure 6: $\Lambda_N$ under various noise level. The bottom figures are projections.

is no noise (see (c) of Figure 7). When there is noise, it is interesting to note that since the second and third corrosive parts are close to each other and have relatively low coefficients, they are detected as a single one. Table 1 shows the numerical values of detected quantities. The corrosion coefficients are computed by solving a linear system.

Since the number of significant eigenvalues of the response matrix $\Lambda_N$ contains information not only on the number of corrosive parts but also on the number of nodal points, as was already observed in [5], we are able to identify the length of the corrosive part. Figure 8 is for the detection of extended corrosive parts. It shows that the algorithm of this paper works well for detecting extended corrosion and the length of the corrosive parts can be reconstructed.

Figure 9 shows the performance of the MUSIC-type algorithms at frequencies $\omega = 1, 5, 50, 100$. It clearly exhibits the high-frequency instability. We emphasize that for this computation and the next one we use the discretization $N = 960$ (to enhance the precision of the computation) and the number of source points 10 (to reduce the computation time). So $\Lambda_N$ here is $10 \times 960$ matrix, not $256 \times 256$. The corrosion coefficient is set to be $\gamma = 0.05$.

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<th>$\omega$</th>
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<td>$m^c$</td>
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Table 1: The computational results of the MUSIC-type algorithm with $\omega = 5$.

We then compare these results with those when we use the filtering. The filtering operator $\mathcal{P}_n$ is defined, as before, by

$$\mathcal{P}_n[f] := \sum_{|\ell| \leq \omega (r_e - r_i)} f_\ell e^{i\ell \theta},$$

where $f_\ell$ is the Fourier coefficients of $f$. Figure 10 shows the comparison. It clearly illustrates that by filtering we are able to overcome the high-frequency instability.

### 8 Conclusion

In this paper we have developed a non-iterative algorithm for locating small internal corrosive parts in pipelines from exterior ultrasound boundary measurements. Our algorithm is based on an asymptotic expansion of the effect of the corrosion in terms of the size of the corrosive parts. We have numerically tested the validity of such an asymptotic formula at high-frequencies. We have also proposed a simple procedure to remove high-frequency instabilities in our inversion procedure in the case where the wavelength is small compared to the distance between the accessible and inaccessible parts of the pipeline. Finally, we have presented many computational experiments to show the robustness and the accuracy of our detection algorithm.

### References


(a) The comparison of SVD: \(T_N\) has four significant eigenvalues which correspond to the number of corrosive parts. \(\Lambda_N\) exhibits more than four significant eigenvalues which correspond to the number of nodal points inside the corrosive parts.

(b) The values of \(\theta_k\) increasing the number \(k\) of used eigenvectors under various noise levels.

(c) Zoom of the third figure above.

(d) The detected corrosion parts under various noise levels.

Figure 7: The MUSIC-type algorithm with \(\omega = 5\).
Figure 8: Imaging of an extended corrosive part under 0% noise. The length of the corrosive part can be detected provided that the corrosion coefficient is known.

Figure 9: The MUSIC-type algorithm for various frequencies. The performance of the algorithm degrades as the frequency increases. Here $k$ is the number of eigenvectors used.
Figure 10: A comparison of MUSIC-Type algorithm with and without filtering. The solid lines are results without using the filtering and the dashed lines are results using filtering.