HYPERBOLIC GEOMETRY: ALGORITHMIC, NUMBER THEORETIC, AND NUMERICAL ASPECTS. NIMS SUMMER/WINTER SCHOOL, KIAS, SEOUL, REPUBLIC OF KOREA MARCH 15–19, 2010

ORGANIZERS: KI HYOUNG KO , JA KYUNG KOO (KAIST)

Contents

1. Schedule of Events

- 15th March (Monday)
  - 9:00-5:00: Registration
  - 9:20-10:40: A course on geometry I: Introduction, Inkang Kim (KIAS)
  - 10:50-12:10: A course on orbifolds I : Topology, Suhyoung Choi (KAIST)
  - 12:10-1:30: Lunch
  - 1:30-2:50: Kleinian groups I: General theory, K. Ohshika (Osaka Univ.)
  - 3:00-4:20: A course on orbifolds II: Geometry, Suhyoung Choi (KAIST)
- 16th March (Tuesday)
  - 9:00-5:00: Registration
  - 9:20-10:40: A course in number theory I , Suhhyun Choi (KAIST)
  - 10:50-12:10: Kleinian groups II: Hyperbolizations, K. Oshika
  - 12:10-1:30: Lunch
  - 1:30-2:50: A course on geometry II : Rigidity and arithmeticity, Inkang Kim (KIAS)
  - 3:00-4:20: A course in number theory II, Suhhyun Choi (KAIST)
- 17th March (Wednesday)
  - 9:00-5:00: Registration
  - 9:20-10:40: Lecture 1: Hyperbolic structures on 3-manifolds and 3-orbifolds: Hyperbolic structures from ideal triangulations, Craig Hodgson (Univ. Melbourne)
  - 10:50-12:10: Lecture 1: Basic arithmetic invariants and scissors congruence, Walter Neumann (Columbia Univ.)
  - 12:10-1:30: Lunch
  - 1:30-2:50: Session: Deformation spaces of Kleinian groups: what we can see from geometric limits, Ken'ichi Ohshika (Osaka Univ.)
  - 3:00-4:20: Lecture 1: TBA, Alan Reid (Univ. Texas, Austin)
- 18th March (Thursday)
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  - 9:20-10:40: Lecture 2. Computing hyperbolic structures and invariants, Craig Hodgson (Univ. Melbourne)
  - 10:50-12:10: Lecture 2: Homology-related invariants, Walter Neumann (Columbia Univ.)
  - 12:10-1:30: Lunch
  - 1:30-2:50: Session: Bounds for minimal dilatations of pseudo-Anosovs, Mitsuhiko Takasawa (Titech)
  - 3:00-4:20: Lecture 2: TBA, Alan Reid (Univ. Texas, Austin)
  - 6:00-8:00 Buffet.

- 19th March (Friday)
  - 9:00-5:00: Registration
  - 9:20-10:40: Lecture 3. Hyperbolic structures from angle structures, Craig Hodgson (Univ. Melbourne)
  - 10:50-12:10: Lecture 3: Realizing invariants, Walter Neumann (Columbia Univ.)
  - 12:10-1:30: Lunch
  - 1:30-2:50: Session: Earthquakes and the universal Teichmueller space Hideki Miyachi (Osaka Univ.)
  - 3:00-4:20: Lecture 3: TBA, Alan Reid (Univ. Texas, Austin)

2. Abstracts

2.1. **Suhhyun Choi.**

- Suhhyun Choi, Department of Mathematical Sciences, KAIST, Daejeon, Republic of Korea
- (suhhyun.choi@gmail.com)

- Lecture 1: a course in number theory
  - Abstract: This is the first talk of the preliminary lecture series on number theory. In this talk, I will give definitions of number fields and algebraic integers, class groups, valuations, and completions, and Adeles and Ideles, quadratic forms and give some properties of them. Also, I will give an introduction to quaternion algebras.
- Lecture 2: a course in number theory II
  - Abstract: This is the second talk of the preliminary lecture series on number theory. In this talk, I will continue giving the theorems on quaternion algebras and give an introduction to elliptic curves.

2.2. **Suhyoung Choi.**

- Suhyoung Choi, Department of Mathematical Sciences, KAIST, Daejeon, Republic of Korea
- (schoi@math.kaist.ac.kr)

- Lecture 1: Introduction to orbifolds I: Topology
  - Abstract: In this talk, we define the topological objects called orbifolds, generalizing manifolds. They are useful in some areas of mathematics related to studying discrete group actions. We give some examples, show the existences of the universal covers, and define the fundamental groups. We also study 2-orbifolds by cut and paste methods.
- Lecture 2: Introduction to orbifolds II: Geometry
  - Abstract: In this talk, we define geometric structures on orbifolds and their deformation spaces and prove the fact that the deformation spaces are locally homeomorphic to the representation spaces of the fundamental groups quotient by conjugations. We introduce the Teichmuller spaces of orbifolds, i.e., the deformation spaces of hyperbolic structures on 2-orbifolds, and show that they are homeomorphic to cells.

2.3. **Inkang Kim.**

- Inkang Kim, KIAS, Republic of Korea
- (inkang@kias.re.kr)
- Lecture 1: Introduction to Geometric structures
  - Abstract: I will review classical geometries like hyperbolic geometry; affine and projective geometries and their inter-relations. In affine geometry, I will discuss about Markus conjecture and its current status. I will discuss about Margulis invariant and related problems. In real projective geometry, I will discuss about strictly convex real projective structures and its related problem about compactification, measure theoretic features like Anosov geodesic flow and critical exponent, Hausdorff measure.
- Lecture 2: Rigidity and arithmeticity
  - Abstract: I will review some rigidity phenomena versus deformation theory in Kleinian groups. I will generally discuss semisimple Lie groups and related topics. For example, local rigidity of hyperbolic lattices in other rank one groups, like

$$SO(3,1) < Sp(1,1) < Sp(n,1), SU(n,1) < Sp(n,1) < SU(2n,2) < SO(4n,4).$$

I will mention about a generalization of Thurston's double limit theorem to function groups.

I will review something about arithmeticity in rank one and higher rank symmetric spaces. If time permits, I will say something about surface groups in general semisimple Lie groups and its relation to Toledo type invariant, its deformability and rigidity.

2.4. **Craig Hodgson.**

- Craig Hodgson, University of Melbourne, Melbourne Australia
- (cdh@ms.unimelb.edu.au)

- Hyperbolic structures on 3-manifolds and 3-orbifolds: Lecture 1. Hyperbolic structures from ideal triangulations
  - Abstract: This talk will give an introduction to ideal triangulations of 3-manifolds, and their application to computation of hyperbolic structures. We will begin with some examples, then discuss Thurston's gluing equations, hyperbolic Dehn surgery, and the computer programs SnapPea and Snap.
- Lecture 2. Computing hyperbolic structures and invariants
  - Abstract: We will begin with a discussion of some geometric and arithmetic invariants for hyperbolic 3-manifolds and 3-orbifolds, canonical triangulations, and applications to isometry and commensurability testing. We will then discuss generalized hyperbolic tetrahedra, the computer program Orb, and some of its applications to the study of knotted graphs and orbifolds.
- Lecture 3. Hyperbolic structures from angle structures
  - Abstract: Let M be the interior of a compact orientable 3-manifold bounded by tori. Given an ideal triangulation T of M, an 'angle structure' on T is a possibly singular hyperbolic structure on M obtained by gluing together hyperbolic ideal tetrahedra so that the sum of dihedral angles around each edge is 2π. The space of all such structures is a convex polytope with compact closure $A(T)$, and the total hyperbolic volume of the tetrahedra gives a continuous, concave function on $A(T)$. If this volume function is maximized at an interior point of $A(T)$, then the corresponding angled structure gives the (unique) complete hyperbolic structure on M. Hyperbolic structures for Dehn fillings on M can be obtained by a similar volume maximization process. We will discuss the proofs of these results (based on work of Rivin, Hodgson, and Chan) and mention some applications.

2.5. **Hideki Miyachi.**

- Hideki Miyachi (joint work with D.Saric), Department of Mathematics, Osaka University
- (miyachi@math.sci.osaka-u.ac.jp)

- Session: Earthquakes and the universal Teichmueller space

  - Abstract: An earthquake deformation is a deformation of hyperbolic surfaces defined by shearing along geodesic laminations (fault movements). In this talk, I will give a natural correspondence between thebounded measured lamination space and the universal Teichmueller space by earthquake deformations.

2.6. **Walter Neumann.**

- Walter Neumann, Department of Mathematics Barnard College, Columbia University
- (wneumann@barnard.edu or neumann@math.columbia.edu)

- Lecture 1: Basic arithmetic invariants and scissors congruence
  - Abstract: This lecture will describe the basic arithmetic invariants of a hyperbolic manifold M: trace field $K(M)$, invariant trace field $k(M)$, invariant quaternion algebra $A(M)$, and their implications for the manifold. This lecture will also make a start on the description of homology-related arithmetic invariants in terms of a scissors congruence and the Bloch group.
- Lecture 2: Homology-related invariants
  - Abstract: Any hyperbolic manifold has a group theoretic fundamental class which lies in the third group homology of $PSL(2, k)$, or even in $H_3(SL(2, k))$ if a spin structure is chosen on the manifold. The lecture will describe this invariant, and its computation through extended versions of the Bloch invariant. This is mostly recent work of Zickert. .
- Lecture 3: Realizing invariants
  - Abstract: The lecture will discuss the question of realization of both arithmetic and homology-related invariants. It will also describe a foliation based approach to trying to prove that every quaternion algebra over a non-real number field arises as the invariant quaternion algebra of a hyperbolic manifold and explain some of the evidence why one might expect this to be true.

2.7. **Ken'ich Ohshika.**

- Ken'ichi Ohshika, Department of Mathematics, Osaka University
- (ohshika@math.sci.osaka-u.ac.jp)

- Lecture 1: Preliminary lectures on Kleinian groups
  - Abstract: In this two lectures, I shall present some backgrounds on Kleinian group theory, which, I hope, would be useful for understanding more advanced theory, which will be given by the main speakers in the latter half of the workshop. In the first part, I shall talk about the general theory, including definitions of basic notions in this field, and classical results. The basic references for this part is [4], [3] and [5]. In the second part, I shall focus on Thurston's theory of uniformization (or hyperbolization) of Haken manifolds, which is a beautiful but highly complicated theory. The best reference for this part is [2].
- Lecture 2: Preliminary lectures on Kleinian groups II
  - Abstract: In this second lecture, I shall focus on Thurston's uniformization theorem for Haken manifolds. I shall start with some basic notions in 3-manifold topology.
- Session: Deformation spaces of Kleinian groups: what we can see from geometric limits
  - Abstract: I shall talk about the usefulness of considering geometric limits in studying the deformation spaces of Kleinian groups. In particular, I shall show how information on divergence, singularities, and the end invariants of limit points is derived from geometric limits.

References

[1] R. Canary, Marden's tameness conjecture: History and applications, in Geometry, analysis and topology of discrete groups, International Press,

[2] J. W. Morgan, On Thurston's uniformization theorem for three-dimensional manifolds. The Smith conjecture (New York, 1979), 37–125, Pure Appl. Math., 112, Academic Press, Orlando, FL, 1984.

[3] K. Ohshika, Discrete groups. Translated from the 1998 Japanese original by the author. Translations of Mathematical Monographs, 207. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2002.

[4] J. Ratcliffe, Foundations of hyperbolic manifolds. Second edition. Graduate Texts in Mathematics, 149. Springer, New York, 2006.

[5] W. Thurston, Three-dimensional geometry and topology. Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.

2.8. **Mitsuhiko Takasawa.**

- Mitsuhiko Takasawa (joint work with Eiko Kin), Tokyo Institute of Technology, Dept. of Mathematical and Computing Sciences
- (takasawa@is.titech.ac.jp)

- Session: Bounds for minimal dilatations of pseudo-Anosovs
  - Abstract: Let $\Sigma_{g,p}$ be the compact surface of genus g with n boundary component and $\mathrm{Mod}(\Sigma_{g,p})$ be the mapping class group of $\Sigma_{g,p}$. Considering the "complexity" of a mapping class $f \in \mathrm{Mod}(\Sigma_{g,p})$, we can define two numerical invariants for a pseudo-Anosov mapping class. One is the entropy $\mathrm{ent}(f)$ which can be defined as a logarithm of the dilatation $\lambda(f)$ of f, and the other is the hyperbolic volume $\mathrm{vol}(f)$ of the mapping torus T_f of f. The simplest question is the following.

    Question. *Which mapping class has minimal "complexity"? i.e.*
    * *Which mapping class has minimal entropy? and*
    * *Which mapping class has minimal volume?*

    These two questions must be closely related since the entropy and the volume are comparable.

    Thm. [4] *There exists a constant $C = C(\Sigma_{g,n})$ which depends only on the topology of $\Sigma_{g,n}$, such that for any mapping class $f \in \mathrm{Mod}(\Sigma_{g,n})$ the following inequality holds.*

    $$C\ \mathrm{vol}(f) \leq \mathrm{ent}(f)$$

    It is difficult to determine the minimal entropy and we know the minima for only $\Sigma_{0,3}$, $\Sigma_{0,4}$, $\Sigma_{0,5}$, $\Sigma_{1,0}$, $\Sigma_{1,1}$ and $\Sigma_{2,0}$. There are many upper bounds, for example by Penner, Hironaka-Kin, Minakawa, and recently by Hironaka [3]. We construct pseudo-Anosovs with small dilatation from the so-called magic 3-manifold M which is the exterior of the 3-chain link C_3.

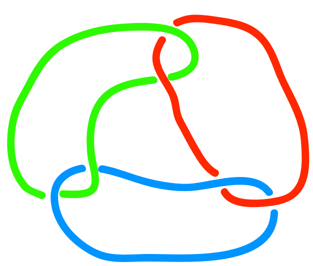
    

    Figure 1: 3-chain link C_3

    Prop. *There exists three series of mapping classes $\{f_{-\frac{1}{2}}(g)\}$, $\{f_{-\frac{3}{2}}(g)\}$, and $\{f_2(g)\}$ such that $f_r(g)$ is a automorphism of $\Sigma_{g,0}$ and the mapping torus $T_{f_r(g)}$ is isomorphic to the Dehn filling $M(r)$.*

    These series gives us sharper upper bound of minimal entropy.

Thm. *Let $t_{k,l}$ be the largest real root of the polynomial $f_{k,l}(t) = t^{2k} - t^{k+l} - t^k - t^{k-l} + 1$.*

$*$ $\mathrm{ent}_{\min}(g, 0) \le \log t_{g+2,1}$ *if* $g \equiv 0, 1, 5, 6 \pmod{10}$ *and* $g \ge 2$

$*$ $\mathrm{ent}_{\min}(g, 0) \le \log t_{g+2,2}$ *if* $g \equiv 7, 9 \pmod{10}$

Conj (weak version)**.** *Our upper bound is best possible for mapping classes whose mapping torus T_f are isomorphic to some Dehn filling of the magic 3-manifold M.*

We will show some evidences for the conjecture from computer experiments.

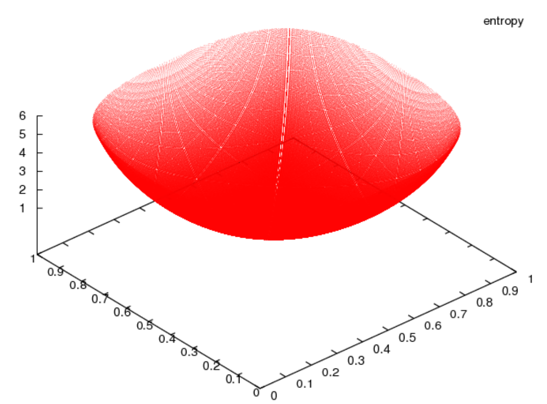
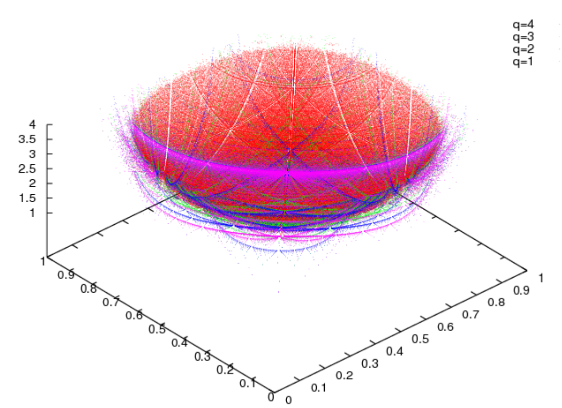


Figure 2: Entropy



function on M (left) and on Dehn fillings of M (right)

Recently, Farb-Leininger-Margalit proved strong theorem in [2] and as a corollary we have the following.

Cor. [2] *There exists a finite set of 3-manifolds $\mathcal{M} = \{M_1, M_2, \cdots, M_k\}$ which have the following property.*
For any $g > 0$, if the mapping class $f \in \mathrm{Mod}(\Sigma_{g,0})$ attains the minimal entropy, there exists $M_i \in \mathcal{M}$ such that T_f is isomorphic to some Dehn filling of M_i.

We can check that the magic 3-manifold belongs to $\mathcal{M}$ and we may have the following conjecture.

Conj (strong version)**.** *Our upper bound is best possible for any mapping classes.*

References

[1] J. W. Aaber and N. M. Dunfield, *Closed surface bundles of least volume*, preprint, arXiv:1002.3423

[2] B. Farb, C. J. Leininger and D. Margalit, *Small dilatation pseudo-Anosovs and* 3*-manifolds*, preprint, arXiv:0905.0219

[3] E. Hironaka, *Small dilatation pseudo-Anosov mapping classes coming from the simplest hyperbolic braid*, preprint, arXiv:0909.4517

[4] E. Kin, S. Kojima and M. Takasawa, *Entropy versus volume for pseudo-Anosovs*, Experimental Mathematics 18 (2009), 397-407.

[5] E. Kin, and M. Takasawa, *Pseudo-Anosovs on closed surfaces having small entropy and the Whitehead sister link exterior*, preprint.

Appendix A. Notes