§ 5.3.1 Manifolds with no geodesic surfaces

Thm 5.3.1 Γ : Kleinian group of finite covolume with the conditions
a) $k\Gamma$ contains no proper subfield other than $\mathbb{Q}$
b) $A\Gamma$ is ramified at at least one infinite place of $k\Gamma$
Then Γ contains no hyperbolic elements
$\Leftrightarrow$ trace is real and has absolute value > 2

pf) Suppose that Γ has a hyperbolic element γ then
$\gamma^2 \in \Gamma^{(2)}$ is also hyperbolic
Let $t := tr(\gamma^2)$. then by def. $t \in k\Gamma \cap \mathbb{R} \underset{\text{by condition a)}}{\overset{}{=}} \mathbb{Q}$ and $|t| > 2$

by condition b), $A\Gamma$ is ramified at an infinite place v of $k\Gamma$
$\exists$ an iso. $\psi : A\Gamma \to \mathcal{H}$ which extends the
Galois embedding $\sigma : k\Gamma \to \mathbb{R}$ associated to v.

Let's recall some definitions
1) A quatemion algebra A over F is said to split over F if $A \simeq M_2(F)$
2) for Archimedian valuation v, on a number field k,
a quaternion algebra A is said to be ramified at v
if $A \otimes_k k_v$ is the unique division algebra $\mathcal{H}$
in this case v, is associated to a real embedding $\sigma : k \to \mathbb{R}$.
3) for non-archimedian valuation v_p on k,
A is said to be ramified at p
if $A \otimes_k k_p$ is the unique division algebra over p $\left(\underset{2.6.4}{=} \left(\frac{\pi, u}{k_p}\right) \right)$

Note $\psi(\Gamma^{(2)}) \underset{\text{norm = det}}{\subseteq} \psi(A\Gamma^1) \subset \mathcal{H}^1$

and $\underset{t \in \mathbb{Q}}{t =} \sigma(t) = \psi(\gamma + \bar{\gamma}) = \psi(\gamma) + \overline{\psi(\gamma)} = tr\,\psi(\gamma)$

but since $tr\mathcal{H}^1 \subset [-2, 2]$, We get a contradiction to $|t| > 2$.

Coro 5.3.2
With the same condition as in 5.3.1
M has no immersed totally geodesic surface

Coro 5.3.3
If $M = \mathbb{H}^3/\Gamma$ is as above then.
Γ contains no non-elementary Fuchsian subgroups

Choose a maximal ideal M_{xy} s.t. $M_{xy} \not\ni xy$
and let $\rho : PSL(2,R) \to PSL(2,R/\mu) \times PSL(2,R/\mu)$ proj.
be the homo. defined by $\rho(\gamma) = (\pi(\gamma), \pi(cx(\gamma)))$ where $\pi : R \to R/\mu$
By the choice of x, y, $\rho(\gamma)$ is a pair of distinct elements in $PSL(2,R/\mu)$
Since $\rho(H)$ lies in the diagonal and R/μ is finite,
We can take K as the ρ^{-1}(diagonal subgp of $(PSL(2,R/\mu))^2$) ∎

Lemma 5.3.7.
Let $M = \mathbb{H}^3/\Gamma$ be a finite volume hyperbolic 3-mfd and $f : S \hookrightarrow M$ be an incompressible immersion of a closed surface
Let $H = f_*(\pi_1(S)) < \Gamma$
If Γ is H-subgp separable then $\exists$ a finite covering M_0 of M s.t. the lifting of f embeds S in M_0.

pf)
$$\begin{tikzcd} \mathbb{H}^3 \arrow[d, "\pi\ \text{covering}"] \\ \mathbb{H}^3/\Gamma = M \end{tikzcd}$$

$\exists$ a cpt fundamental set $D \subset \mathbb{H}^3$ s.t. $\pi(D) = f(S) \subset M$
Since Γ acts on $\mathbb{H}^3$ properly discontinuously,
$\exists$ finite $\gamma_1, \cdots, \gamma_n$ s.t. $\gamma_i D \cap D \neq \phi$
Since H is Fuchsian and thus Γ is H-subgp separable by 5.3.6, $\exists$ finite index subgp $K > H$ s.t. γ_i's $\notin K$ → we may take K as $\bigcap_i K_{\gamma_i}$
so $\mathbb{H}^3/K$ is the required covering.

Thm 5.3.4
M : Closed hyperbolic 3-mfd containing a totally geodesic immersion of a closed surface then $\exists$ a finite sheeted covering of M which contains an embedded closed <u>orientable</u> totally geodesic surface

pf) Let $i : S \hookrightarrow M$ be the totally geodesic immersion
Let $M = \mathbb{H}^3/\Gamma$ and let $H = i_*(\pi_1(S)) < \Gamma$
Since H is Fuchsian, H preserves C as in 5.3.6
and $\mathrm{Stab}(C,\Gamma)$ is separable in Γ by 5.3.6
$\mathrm{Stab}(C,\Gamma)$ contains H and is discrete so $\exists$ finite sheeted covering map $\mathbb{H}^2/H \longrightarrow \mathbb{H}^2/\mathrm{Stab}(C,\Gamma)$
Thus $\mathrm{Stab}(C,\Gamma)$ is also a fundamental group of a closed surface S' (S' may not be orientable)
Since $\mathrm{Stab}(C,\Gamma)$ is separable, $\exists$ a finite covering M_K of M such that S' can be embedded in $M_K = \mathbb{H}^3/K$ by 5.3.7.
If S' is not orientable i.e. if $\mathrm{Stab}(C,\Gamma)$ is not Fuchsian,

Lemma 5.3.12

Let $t = \lambda^{-1} - \lambda$

Then t^2, $rt \in k\Gamma$ and thus $t/r = -[a_1, a_2, b_1, b_2]^{-1} \in k\Gamma$.

pf) $t^2 = tr^2\gamma - 4 = tr^2\eta - 4 \Rightarrow t^2 \in k\Gamma$

note that $tr(\gamma\eta^{-1}) = tr(\gamma\delta\gamma^{-1}\delta^{-1}) \in k\Gamma$

(recall that $k\Gamma = \mathbb{Q}(tr\Gamma^{(2)})$ and
$tr^2 X - 2 = tr X^2$, $tr X^2 Y^2 = tr^2 XY - tr[X, Y]$)

Thus $rt = 2 - tr(\gamma\eta^{-1}) \in k\Gamma$.

$$\gamma\eta^{-1} = \begin{pmatrix} \lambda & 0 \\ r & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \lambda^{-1} & \lambda - \lambda^{-1} \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda^2 - 1 \\ r\lambda^{-1} & r\lambda - r\lambda^{-1} + 1 \end{pmatrix}$$

ω is one of the fixed points of $\gamma = \begin{pmatrix} \lambda & 0 \\ r & \lambda^{-1} \end{pmatrix}$

so $\dfrac{\lambda\omega + 0}{r\omega + \lambda^{-1}} = \omega \Rightarrow r\omega^2 + (\lambda^{-1} - \lambda)\omega = 0 \quad \therefore \omega = \dfrac{-t}{r} = -\dfrac{t^2}{rt} \in k\Gamma$ ☐

Thm 5.3.13

If $M = \mathbb{H}^3/\Gamma$ has a nonsimple closed geodesic, then
$A\Gamma \simeq \left(\dfrac{a, b}{k\Gamma}\right)$ for some $a \in k\Gamma$, $b \in k\Gamma \cap \mathbb{R}$

pf) Suppose M has a nonsimple closed geodesic

Since $\langle \gamma, \eta^{-1} \rangle \subset \Gamma$ is irreducible,

$$A\Gamma = \left(\dfrac{tr^2\gamma - 4,\ tr[\gamma, \eta^{-1}] - 2}{k\Gamma}\right) \text{ by Thm 3.6.1.}$$

Note that $tr^2\gamma - 4 = t^2$, $tr[\gamma, \eta^{-1}] - 2 = tr^2(\gamma\eta^{-1}) - tr(\gamma^2\eta^{-2}) - 2$
$= r^2t^2 + (rt)t^2 = (t^2(r/t))^2(1 + t/r)$

Thus $A\Gamma \simeq \left(\dfrac{t^2,\ 1 + t/r}{k\Gamma}\right)$, $1 + t/r = 1 - \omega \in k\Gamma \cap \mathbb{R}$ ☐

Coro. 5.3.14

If $\exists$ no element $a \in k\Gamma$, $b \in k\Gamma \cap \mathbb{R}$ s.t. $A\Gamma \simeq \left(\dfrac{a, b}{k\Gamma}\right)$
then all closed geodesics of $M = \mathbb{H}^3/\Gamma$ are simple

Note that $\mathrm{Isom}(\mathbb{H}^n) < GL(n+1, \mathbb{R})$

if M is a hyper. n-mfd, $\exists$ a discrete faithful repre.

$\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^n) < GL(n+1, \mathbb{R})$

Let R be the integral domain generated by the entries of a generating set of $\rho(\pi_1(M)) \subset GL(n+1, \mathbb{R})$

then Mal'cev's Thm implies $\pi_1(M)$ is residually finite

because given $g \in \pi_1(M)$, $g \neq id$, $\exists$ maximal ideal $\mathcal{M}_g$ s.t. $\phi \circ \rho(g) \neq id$

$\phi \circ \rho : \pi_1(M) \longrightarrow GL(n+1, R) \longrightarrow GL(n+1, R/\mathcal{M}_g)$

then $\mathrm{Ker}\, \phi \circ \rho \not\ni g$

Question. Let M be a Haken 3-mfd

Does $\pi_1(M)$ admit a faithful repre. into $GL(n, \mathbb{C})$ for some n?

About a normal core

Let G be a finitely generated group

given a finite index subgroup $K < G$

$\mathrm{core}(K) = \bigcap_{g \in G} gKg^{-1}$ is also a finite index subgp

pf) claim: If G is finitely generated, given a fixed index k

$\exists$ only finitely many $H < G$ s.t. $[G : H] = k$.

Let $G = \langle s_1, \cdots s_n \rangle$

and $G = H \cup g_2 H \cup \cdots \cup g_k H$

Each $g \in G$ induces a permutation P_g of cosets $\{g_i H\}$

and $g \in H$ iff P_g fixes the first symbol H

Note that P_g is determined by permutations P_{s_i}'s

there are $(k!)^n$ choices of P_{s_i}'s

So there are at most $(k!)^n$- subgroups of index k.

Thus $\bigcap_{g \in G} gKg^{-1}$ is actually a finite intersection

Since K, gKg^{-1} have the same index, $\mathrm{core}(K)$ has also a finite index in G $\square$