5.1 Discreteness Criteria

if p is a monic irreducible polynomial over $\mathbb{Z}$ of degree n with roots $\alpha_1, \cdots \alpha_n$ then

$$p(z) = \prod_{i=1}^{n}(z-\alpha_i) = z^n - s_1 z^{n-1} + \cdots (-1)^k s_k z^{n-k} + \cdots + (-1)^n s_n$$

If $\deg p < N$ and $|\alpha_i| < M$ $\forall_i$ there are only finitely many choices of s_i's Thus,

Lemma 5.1.1 There are only finitely many algebraic integers of bounded degree such that all of its Galois conjugates are bounded.

Thm 5.1.2.

Γ : finitely generated subgp of $PSL(2,\mathbb{C})$ s.t.

1) $\Gamma^{(2)}$ is irreducible ($\Rightarrow \exists\, g_1, g_2 \in \Gamma^{(2)}$ with no common fixed points)
2) $\mathrm{tr}\,\Gamma$ consists of algebraic integers
3) $\forall\ \sigma : k\Gamma \to \mathbb{C}$ s.t. $\sigma \neq id$ or complex conjugation, $\{\sigma(\mathrm{tr}(f)) \mid f \in \Gamma^{(2)}\}$ is bounded.

Then Γ is discrete

pf) It suffices to prove that $\Gamma^{(2)}$ is discrete because $\Gamma^{(2)}$ is a finite index normal subgp of Γ by 3.3.3

Suppose not then $\exists \{f_n\} \subset \Gamma^{(2)}$ s.t. $f_n \to id$

Let $z_n = \mathrm{tr}\, f_n$, $z_{n,i} = \mathrm{tr}\,[f_n, g_i]$, $\beta(f_n) = z_n^2 - 4$, $\gamma(f_n, g_i) = z_{n,i} - 2$

as $n \to \infty$, $\beta(f_n)$, $\gamma(f_n)$ goes to 0

Thus for large n, $|z_n| < K$ and given $\sigma \neq id$, complex conjugation, $|\sigma(z_n)| < K_\sigma$

Let $R = \max\limits_{\substack{\sigma \neq id \\ \neq cx\ \text{conjugation}}} \{K, K_\sigma\}$ and note that $\deg \underset{\substack{\uparrow \\ \text{monic irredu. poly over } \mathbb{Z} \text{ with root } z_n}}{p_n} \leq [k\Gamma : \mathbb{Q}]$

by 5.1.1 # of $\{z_n\}$ is finite.

So for large n, $\beta(f_n) = 0$ and $\gamma(f_n, g_i) = 0$

$\beta(f_n) = 0 \Rightarrow f_n$ is parabolic $\Rightarrow f_n$ has a single fixed pt $w_n \in \hat{\mathbb{C}}$

$\gamma(f_n, g_i) = 0 \Rightarrow$ g_1, g_2, f_n has a common fixed pt w_n which contradicts to the choice of g_1, g_2. $\square$

Lemma 5.1.3

3) is equivalent to the following 3')

All embeddings $\sigma \neq id$, cx conjugation are real and $A\Gamma$ is ramified at all real places (i.e. when $A\Gamma = \left(\frac{a,b}{k\Gamma}\right)$, $\left(\frac{\sigma(a),\sigma(b)}{\mathbb{R}}\right) \cong \left(\frac{a,b}{k\Gamma}\right) \otimes_{k\Gamma} \mathbb{R} \cong \mathcal{H}$)

pf). if 3') holds, given $\sigma : k\Gamma' \to \mathbb{R}$, $\exists\ \tau : A\Gamma' \to \mathcal{H}$
s.t. $\sigma(\mathrm{tr} f) = \mathrm{tr}(\tau(f))$ for each $f \in \Gamma^{(2)}$
note that $\mathrm{tr} f$ is the usual trace for $f \in SL(2,\mathbb{C})$, $n(f)$ is also $\det(f)$.
and $\mathrm{tr}(\tau(f)) = \tau(f) + \overline{\tau(f)}$ (See the last paragraph in P 114).
Since $\det(f) = 1$, $n(\tau(f)) = \tau(f)\overline{\tau(f)} = 1$ and thus
$\mathrm{tr}(\tau(f)) \in [-2,2]$ ($\because$ $\tau(f) = a + bi + cj + dk$, $n(\tau(f)) = a^2+b^2+c^2+d^2$)

Conversely, assuming 3), given $\sigma : k\Gamma \to \mathbb{C}$
Let λ, λ^{-1} be eigenvalues of $f \in \Gamma^{(2)}$
and Let μ be an extension of σ to $k\Gamma(\lambda)$
Then $\sigma(\mathrm{tr} f^n) = \sigma(\lambda^n + \lambda^{-n}) = \mu(\lambda)^n + \mu(\lambda)^{-n}$
Thus $|\sigma(\mathrm{tr} f^n)| \geq \big|\, |\mu(\lambda)|^n - |\mu(\lambda)|^{-n} \big|$
note that if $|\mu(\lambda)| \neq 1$ then $|\mu(\lambda)|^n$ or $|\mu(\lambda)|^{-n}$ diverges
which contradicts to the boundedness of $|\sigma(\mathrm{tr} f^n)|$.
So, $|\mu(\lambda)| = 1$ and $\sigma(\mathrm{tr} f) = \mu(\lambda) + \overline{\mu(\lambda)} \in [-2,2]$
by (3.38) $A\Gamma \cong \left(\dfrac{\mathrm{tr}^2 g_1 (\mathrm{tr}^2 g_1 - 4),\ \mathrm{tr}[g_1, g_2] - 2}{k\Gamma} \right)$

for some $\langle g_1, g_2 \rangle \subset \Gamma^{(2)}$
Since $\sigma(\mathrm{tr}^2 g_1) \leq 4$, $\sigma(\mathrm{tr}[g_1,g_2]) \leq 2$, $A\Gamma$ is ramified at real places

5.2 Bass's Thm.

Def. $\overline{\mathbb{Q}}$: algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$
$\Gamma < SL(2, \overline{\mathbb{Q}})$ is said to have integral traces
if $\forall \gamma \in \Gamma$, $\mathrm{tr}(\gamma)$ is an algebraic integer

Thm $M = \mathbb{H}^3/\Gamma$ finite volume hyperbolic 3-mfd
and Γ has non-integral trace
Then M contains a closed embedded essential surface

Coro If $M = \mathbb{H}^3/\Gamma$ is non-Haken, then Γ has integral traces
(An irreducible 3-mfd M is called Haken if
M contains a two-sided incompressible surface)

Serre p 41.

Y : a connected graph T : maximal tree in Y.

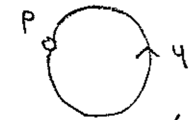
A graph of group (G, Y) consists of Y, G_P's for each $P \in V(Y)$

G_y's for each $y \in E(Y)$ together with

monomorphisms $\underset{a \mapsto a^y}{G_y \xrightarrow{y} G_{t(y)}}$ (note that $G_{\bar{y}} \xrightarrow{\bar{y}} G_{t(\bar{y})} = G_{i(y)}$)

$F(G,Y) := \langle G_P\text{'s}, y\text{'s} \rangle / \langle y\bar{y},\ y a^y y^{-1} (a^{\bar{y}})^{-1} \rangle$

$\pi_1(G, Y, T) := F(G,Y) / \langle y\text{'s in } T \rangle$

example) $Y = \overset{P}{\circ}\xrightarrow{\ y\ }\overset{Q}{\circ}$, $Y =$ 

$\pi_1(G, Y, Y) = G_P \underset{G_y}{*} G_Q$ (= amalgamated free product)

$\pi_1(G, Y, P) = \langle G_P, y \rangle / \langle g a^y g^{-1} a^{\bar{y}} \rangle$ (= HNN extension of G_P)

Thm (Serre p 55, 5.2.7 in p 169)

Let G acts on a tree X without inversion and let $Y = X/G$

Then $\pi_1(G, Y, T) \underset{iso}{\simeq} G$

proof of Bass Thm)

$k = \mathbb{Q}(\mathrm{tr}\Gamma)$ is a finite extension of $\mathbb{Q}$

$A_0\Gamma = k[I, g, h, gh]$ $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ $h = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$ by normalization

$[k(\lambda) : k] \leq 2 \Rightarrow [k(\lambda) : \mathbb{Q}]$ is finite so $k(\lambda) \subset \overline{\mathbb{Q}}$

Coro 3.2.4 says Γ is conjugate into $SL(2, k(\lambda))$

Since Γ has non-integral trace, $\exists$ $k(\lambda)$-prime $\mathcal{P}$ s.t.

$\nu_{\mathcal{P}}(\mathrm{tr}\,\gamma) < 0$ for some $\gamma \in \Gamma$

using injection $i_{\mathcal{P}} : k(\lambda) \longrightarrow k(\lambda)_{\mathcal{P}} := K$

We can inject Γ into $SL(2, K)$

We have seen that $SL(2,K)$ acts on a tree of lattices

and $SL(2,K)_P \underset{\text{Conjugate}}{=} SL(2,R)$ or $\left\{ \begin{pmatrix} a & \pi b \\ \pi^{-1} b & d \end{pmatrix}, a,b,c,d \in R \right\}$, Thus,

If Γ fixes a vertex then $\mathrm{tr}\,\Gamma$ has to be in R

Since $\nu_{\mathcal{P}}(\mathrm{tr}\,\gamma) < 0$, $\mathrm{tr}\,\gamma \notin R$ (= Lemma 5.2.6)

$\Rightarrow$ Γ splits nontrivially as the fundamental group of a graph of group.

$\Rightarrow$ by Thm 1.5.3 $\exists$ an essential embedded incompressible surface in M.

Lemma 5.2.9

Γ : finitely generated non-ele. subgp of $SL(2,\mathbb{C})$

Γ has integral trace $\Leftrightarrow$ Γ is conjugate to $SL(2,\mathbb{A})$

where $\mathbb{A}$ is the ring of all algebraic integers in $\overline{\mathbb{Q}}$.

pf) $\Leftarrow$) obvious

$\Rightarrow$) Let $k=\mathbb{Q}(\mathrm{tr}\Gamma)$ $A_0\Gamma$ = quaternion algebra of Γ over k

and let $\mathcal{O}\Gamma$ = R_k-module generated by elements of Γ

then $\mathcal{O}\Gamma$ is an order of $A_0\Gamma$

(recall that an order is a complete R_k-lattice which is also a ring with identity)

Recall that when A is a quaternion division algebra over F

$A = F[\mathrm{id}, y, z, yz] \cong \left(\frac{a,b}{F}\right)$ where $y^2=a$, $z^2=b$. (Thm 2.1.8)

Since $A\otimes_F F(y)$ has the element y s.t. $y^2=\mathrm{id}$ in $A\otimes_F F(y)$,

it is not a division algebra ($\because (y-\mathrm{id})(y+\mathrm{id})=0$)

so $A\otimes_F F(y) \cong M_2(F(y))$

when $A\cong M_2(F)$, $A\otimes_F F(y)\cong M_2(F(y))$ also holds.

Thus $A_0(\Gamma)\otimes_k k(y) \cong M_2(k(y))$ Let $K:=k(y)$

by Skolem Noether thm, we may conjugate

so that $A_0\Gamma \subset M_2(K)$ then $\mathcal{O}\Gamma\otimes_{R_k} R_K$ becomes

a suborder of $M_2(R_K;J)$

recall that $M_2(K)$ can be identified with $\mathrm{End}(V)$ ← 2-dim'l vector space over K

and 2.2.8 says that every order of $\mathrm{End}(V)$ can be contained in $\mathrm{End}(L)$ for some complete lattice L.

moreover, thm 2.2.9 says $\exists$ a basis $\{x,y\}$ of V and a fractional ideal J such that $L=Rx+Jy$

In this case, $\mathrm{End}(L)$ is conjugate to

$M_2(R_K;J) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a,d\in R,\ b\in J^{-1}, c\in J \right\}$

fact: If H is a Hilbert class field of K

i.e. maximal abelian unramified extension of K,

then every fractional ideal of K is principal in H

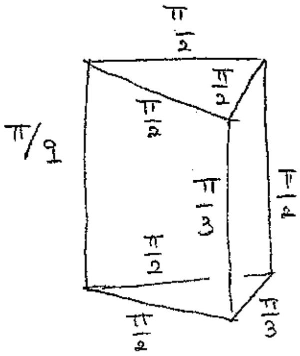
R_H may not be a PID, $[H:K]$ = class number of K

Thus, $J=xR$ in H, $M_2(R_H;J) \underset{\text{conjugate}}{\cong} M_2(R_H)$

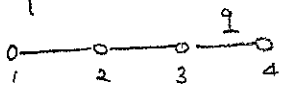
so $\Gamma \underset{\text{conjugate}}{\subset} SL(2,R_H)$

Thm. $M=\mathbb{H}^3/\Gamma$ finite volume hyperbolic 3-mfd not containing any closed embedded essential surface then Γ is conjugate into $PSL(2,\mathbb{A})$

example. In 4.7.3, Γ_q with non integral trace is constructed.



for $q \geq 7$, the triangular prism exists in $\mathbb{H}^3$ and it is the fundamental polyhedron of the coxeter group generated by reflections in the faces by Andreev's Theorem.



$\exists\, Z \in \Gamma_q$ such that $\operatorname{tr}^2 Z - 3 = 1/(2\cos 2\pi/q - 1)$ and it can be shown that $2\cos 2\pi/q - 1$ is not a unit in the ring of integers in $\mathbb{Q}(\cos 2\pi/q)$ if $q = 6p$, $p =$ prime $(\neq 2, 3)$.
So $\operatorname{tr} Z$ is not an algebraic integer for such q