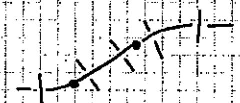
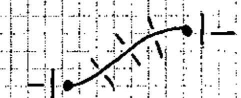
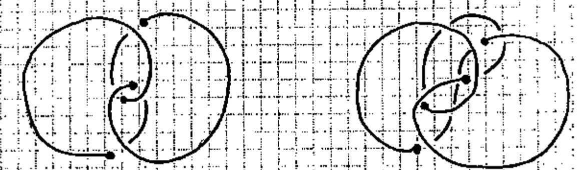
[Colin C. Adams, The knot book] p.64-67

Given a projection of a knot to a plane, define an overpass to be a subarc of the knot that goes over at least one crossing but never goes under a crossing. A maximal overpass is an overpass that cound not be made any longer.


An overpass


A maximal overpass

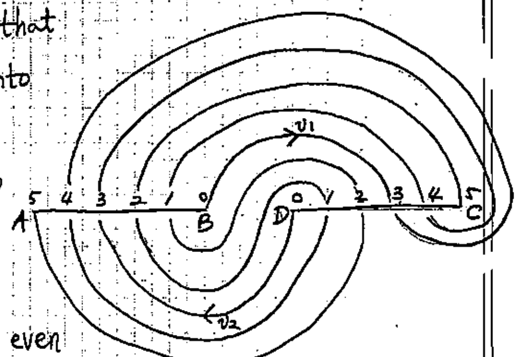
The bridge number of the projection is the number of maximal overpasses in the projection. The bridge number of K is the least bridge number of all of the projections of the knot K.


The trefoil and figure-eight knots

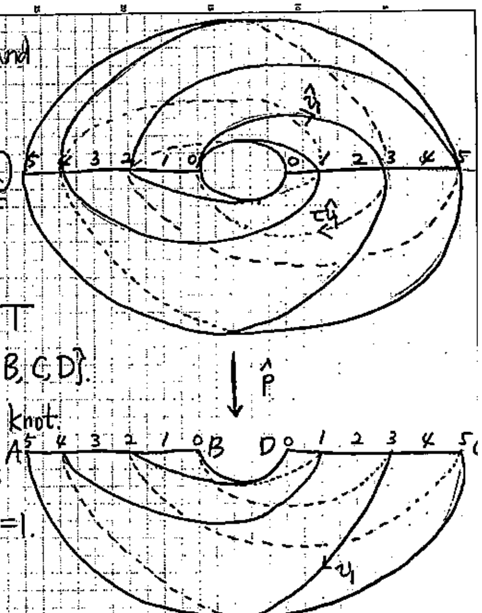
[G. Burde & H. Zieschang, Knots] p.181-185

Schubert's Normal Form of Knots and Links with two Bridges.

H. Schubert [1956] classified knots and links with two bridges. The knot K meets a projection plane $\mathbb{R}^2 \subset \mathbb{R}^3$ in four points: A, B, C, D. We may assume that one pair of arcs is projected onto straight segments $w_1 = AB$, $w_2 = CD$. The other pair is projected onto two disjoint simple curves v_1 ($B \to C$) and v_2 ($D \to A$). The number of double points is even in a reduced diagram with $p-1$ ($p \in \mathbb{N}$) double points on w_1 and w_2.



For a knot p is odd; p even and $\partial v_1 = \{A, B\}$, $\partial v_2 = \{C, D\}$ yields a link. We add a point ∞ at infinity, $S^3 = \mathbb{R}^3 \cup \{\infty\}$, $S^2 = \mathbb{R}^2 \cup \{\infty\}$, and consider the two-fold branched covering T of S^2 with the branch set $\{A, B, C, D\}$. Then $(1-\tau)\hat{v}_i$ is a (q,p)-torus knot. From the construction it follows that $|q| < p$ and that $\gcd(p,q) = 1$.



<u>Proposition</u> For any pair p, q of integers subject to the conditions:

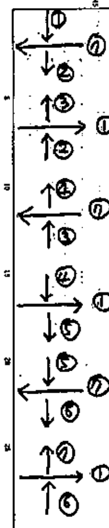
$p>0, \quad -p<q<p, \quad \gcd(p,q)=1, \quad q$ odd,

there is a knot or link with two bridges (p/q) with a reduced diagram with numbers p, q. The 2-fold covering of S^3 branched along (p/q) is the lens space $L(p,q)$.

<u>Theorem</u> (H. Schubert)

(p/q) and (p'/q') are equivalent, if and only if

$p=p', \quad q^{\pm 1} \equiv q' \mod p.$



$x_1 x_7 = x_7 x_2$

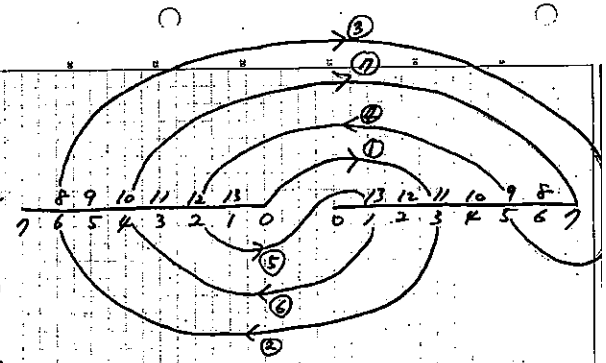
$x_2 x_1 = x_1 x_3$

$x_7 x_3 = x_4 x_7$

$x_1 x_4 = x_5 x_1$

$x_5 x_7 = x_7 x_6$

$x_6 x_1 = x_1 x_7$



$$\Rightarrow \begin{cases} v^{-1}uv = x_2 \\ u^{-1}x_2 u = x_3 \\ v x_3 v^{-1} = x_4 \\ u x_4 u^{-1} = x_5 \\ v^{-1} x_5 v = x_6 \\ u^{-1} x_6 u = v \end{cases} \Rightarrow u^{-1}v^{-1}uvu^{-1}v^{-1}uvuv^{-1}u^{-1}vu = v$$

$\therefore\ uw = vw, \quad w = vuv^{-1}u^{-1}vu$

4.5 Two-Bridge Knots and Links

A two-bridge knot is determined by a pair of relatively prime odd integers (p/q) with $0<q<p$. For $q>1$, the knot complements of these knots have a hyperbolic structure.

Presentations of the knot groups on two meridianal generators:

Let $iq = k_i p + r_i$, $0<r_i<p$ and $e_i = (-1)^{k_i}$

Then $\pi_1(S^3-(p/q)) = \langle u, v \mid uw = wv,\ w = v^{e_1}u^{e_2}\cdots v^{e_{p-2}}u^{e_{p-1}} \rangle$

So the meridians u and v map to parabolics under the complete representation. Thus map u to $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and v to $\begin{pmatrix}1&0\\z&1\end{pmatrix}$, so that if $S^3-(p/q) = \mathbb{H}^3/\Gamma$, then $\mathbb{Q}(tr(\Gamma)) = \mathbb{Q}(z) = k\Gamma$ by <u>Cor 4.2.2</u>.

Let $p-1=2n$ and $w=\begin{pmatrix}a_n&b_n\\c_n&d_n\end{pmatrix}$.

so that $tr(P_1^{\epsilon_1}\cdots P_n^{\epsilon_n}) = \frac{1}{2^n}\, tr\big((P_1^2+I)^{\epsilon_1}\cdots(P_n^2+I)^{\epsilon_n}\big) \in k\Gamma$

From (3.14), $tr(t^2\gamma) = tr(t)\,tr(t\gamma) - tr(\gamma)$.

If $\gamma\in\Gamma^{(2)}$ then $tr(t)\,tr(t\gamma)\in k\Gamma$, so if $tr(t)\in k\Gamma\setminus\{0\}$ then $tr(t\gamma)\in k\Gamma$.

$\Rightarrow \mathbb{Q}(tr\Gamma)\subset k\Gamma \Rightarrow \mathbb{Q}(tr\Gamma) = k\Gamma$ ∎

Cor 4.2.2 If $M=\mathbb{H}^3/\Gamma$ is the complement of a link in a $\mathbb{Z}/2$-homology sphere, then $k\Gamma = \mathbb{Q}(tr\Gamma)$ and $A\Gamma = M_2(\mathbb{Q}(tr\Gamma))$.

Pf. The first part follows from Thm 4.2.1 and the second from the fact that M is non-compact. ∎

$$a_n = 1 + \Big(\hat{\sum_{i_1 \text{ even}}} e_{i_1} e_{i_2}\Big) z + \Big(\hat{\sum_{i_1 \text{ even}}} e_{i_1} e_{i_2} e_{i_3} e_{i_4}\Big) z^2 + \cdots + (e_2 e_3 \cdots e_{2n-1}) z^{n-1}$$

$$b_n = \sum_{i=1}^{n} e_{2i} + \Big(\hat{\sum_{i_1 \text{ even}}} e_{i_1} e_{i_2} e_{i_3}\Big) z + \cdots + (e_2 e_3 \cdots e_{2n}) z^{n-1}$$

$$c_n = \Big(\sum_{i=1}^{n} e_{2i-1}\Big) z + \Big(\hat{\sum_{i_1 \text{ odd}}} e_{i_1} e_{i_2} e_{i_3}\Big) z^2 + \cdots + (e_1 e_2 \cdots e_{2n-1}) z^n$$

$$d_n = 1 + \Big(\hat{\sum_{i_1 \text{ odd}}} e_{i_1} e_{i_2}\Big) z + \cdots + (e_1 e_2 \cdots e_{2n}) z^n \longrightarrow$$

$$\left(\because \quad \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}^{e_{p-2}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{e_{p-1}} = \begin{pmatrix} 1 & 0 \\ e_{p-2} z & 1 \end{pmatrix} \begin{pmatrix} 1 & e_{p-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e_{p-1} \\ e_{p-2} z & e_{p-2} e_{p-1} z + 1 \end{pmatrix}. \right)$$

Then $uw = wv$ iff $d_n = 0$ and $z b_n = c_n$.

$$\left(\because \quad \begin{pmatrix} a_n + c_n & b_n + d_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} a_n + z b_n & b_n \\ c_n + z d_n & d_n \end{pmatrix} \right)$$

Note that $(p-i) q = (q - k_i - 1) p + (p - r_i) \Rightarrow k_{p-i} = q - k_i - 1$

$\Rightarrow e_i = e_{p-i} \Rightarrow z b_n = c_n$ holds in all cases.

Thus z satisfies the integral monic polynomial equation $d_n = 0$.

We use $\hat{\sum}$ to denote a summation over suffixes $i_1, i_2, \dots, i_k$ where $i_1 < i_2 < \cdots < i_k$ and the parity of the suffixes alternates.

A. $5_2 =$  $= (7/3)$

$$\begin{pmatrix} 3 = 0 \cdot 7 + 3 \\ 6 = 0 \cdot 7 + 6 \\ 9 = 1 \cdot 7 + 2 \\ 12 = 1 \cdot 7 + 5 \\ 15 = 2 \cdot 7 + 1 \\ 18 = 2 \cdot 7 + 4 \end{pmatrix}$$

The sequence of e_i is $\{1, 1, -1, -1, 1, 1\}$

$$\begin{aligned} d_3 = 1 &+ (e_1e_2 + e_1e_4 + e_1e_6 + e_3e_4 + e_3e_6 + e_5e_6) z \\ &+ 1(e_1e_2e_3e_4 + e_1e_2e_3e_6 + e_1e_2e_5e_6 + e_1e_4e_5e_6 + e_3e_4e_5e_6) z^2 \\ &+ 1(e_1e_2e_3e_4e_5e_6) z^3 \\ &= 1 + 2z + z^2 + z^3. \end{aligned}$$

If $S^3 - (7/3) = \mathbb{H}^3/\Gamma$ then $k\Gamma = \mathbb{Q}(z)$ and $A\Gamma = M_2(k\Gamma)$.

This field has one complex place and discriminant -23.

B. $6_1 =$  $= (9/5)$

The sequence of $e_i = \{1, -1, -1, 1, 1, -1, -1, 1\}$.

$d_4 = 1 - 2z + 3z^2 - z^3 + z^4$

This polynomial is irreducible and $k\Gamma = \mathbb{Q}(z)$ has two complex places and discriminant 257.

Although there is just one field up to isomorphism with two complex places and discriminant 257, it is not a Galois extension of $\mathbb{Q}$ and so there are two non-real isomorphic subfields of $\mathbb{C}$ with this discriminant. Our approach does not distinguish which of the two isomorphic subfields is the actual invariant trace field. (See §5.5 and §12.7).

C. A similar analysis can be applied to hyperbolic two-bridge link complements (p/q) with $p = 2n$.

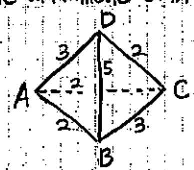
Then $uw = wu \Leftrightarrow c_n = 0$ and $a_n = d_n$

The Whitehead link  $= (8/3)$: $z = -1 + i$, $k\Gamma = \mathbb{Q}(i)$

$A\Gamma = M_2(\mathbb{Q}(i))$

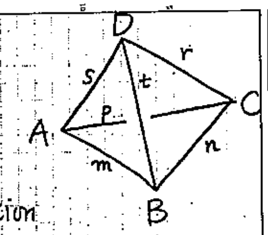
4.7.2 Compact Tetrahedra

The leading candidate for an orientable hyperbolic orbifold of minimal volume is obtained from one of the compact tetrahedra.
(This orbifold is the orientable arithmetic orbifold of minimal volume.)



Let T denote the associated tetrahedral group, so that
$$T = \langle x, y, z \mid x^2 = y^2 = z^3 = (yz)^2 = (zx)^5 = (xy)^3 = 1 \rangle$$
The tetrahedron admits a rotational symmetry of order 2.
Denoting this rotation by w, the extended group
$$\Gamma = \langle x, y, z, w \mid x^2 = y^2 = z^3 = (yz)^2 = (zx)^5 = (xy)^3 = 1,\ w^2 = 1,\ wyw = y,\ wxw = yz,\ wzw = yx \rangle.$$

The edge labelling (e.g. p) indicates the dihedral angle (e.g. π/p) along that edge.



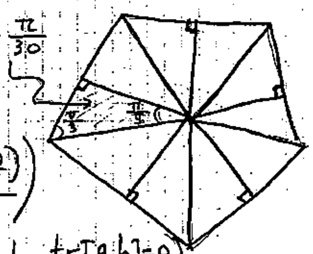
The tetrahedral group then has presentation
$$\langle x, y, z \mid x^m = y^n = z^p = (yz^{-1})^r = (zx^{-1})^s = (xy^{-1})^t = 1 \rangle.$$
These groups may also be described by the Coxeter symbol for the tetrahedron.

The quotient $\mathbb{H}^3/\Gamma$ is the orbifold of minimal volume referred to earlier.
This presentation can be simplified so that Γ is a two-generator group by setting $a = wy$ and $b = z$.
$$\Gamma = \langle a, b \mid a^2 = b^3 = 1,\ c = (ab)^2(ab^{-1})^2,\ c^5 = 1,\ (b^{-1}c^2)^2 = 1 \rangle.$$
$$\begin{aligned}
(\because\ c = (ab)^2(ab^{-1})^2 &= (wyz)^2(wyz^{-1})^2 = wyzw\underline{yz}wyz^{-1}\underline{wy}z^{-1} \\
&= wyzwz^{-1}\underline{yw}yz^{-1}ywz^{-1} = wyzwz^{-1}wz^{-1}\underline{y}wz^{-1} = wyzwz^{-1}wyzwz^{-1} \\
&= (xz^{-1})^2 \\
b^{-1}c^2 &= z^{-1}(xz^{-1})^4 = z^{-1}(xz^{-1})^{-1} = z^{-1}zx^{-1} = x\ )
\end{aligned}$$
Let $A, B \in SL(2, \mathbb{C})$ map onto a and b respectively, chosen so that $\mathrm{tr} A = 0$, $A^2 = -I$, $\mathrm{tr} B = 1$, and $B^3 = -I$.
By Lemma 3.5.8, $k\Gamma = \mathbb{Q}(\mathrm{tr}^2 B, \mathrm{tr}[A,B]) = \mathbb{Q}(\mathrm{tr}[A,B])$.

Let $s = \mathrm{tr} AB$ and $t = \mathrm{tr}[A,B]$. Then $\mathrm{tr} AB^{-1} = -s$ and $t = s^2 - 1$.
$$\begin{aligned}
(\because\ \mathrm{tr} AB &= (\mathrm{tr} A)(\mathrm{tr} B) - \mathrm{tr} AB^{-1} \quad \cdots (3.14) \\
\mathrm{tr}[A,B] &= \mathrm{tr}^2 A + \mathrm{tr}^2 B + \mathrm{tr}^2 AB - \mathrm{tr} A\, \mathrm{tr} B\, \mathrm{tr} AB - 2 \quad \cdots (3.15)\ )
\end{aligned}$$
Again using (3.14), $\mathrm{tr}\, C = (s^2-2)(s^2-1) = t(t-1)$.
$$\begin{aligned}
\mathrm{tr}\, B^{-1}C^2 &= \mathrm{tr}\, B^{-1}C\, \mathrm{tr}\, C - \mathrm{tr}\, B^{-1} = \mathrm{tr} C\ \mathrm{tr}\, B^{-1}(AB)^2(AB^{-1})^2 - 1 \\
&= -\mathrm{tr} C\ \mathrm{tr}(AB)^3 AB^{-1} - 1 \\
&= -\mathrm{tr}\, C\, [\mathrm{tr}(AB)^3\, \mathrm{tr} AB^{-1} - \mathrm{tr}(AB)^3(AB^{-1})^{-1}] - 1 \\
&= -\mathrm{tr}\, C\, [\mathrm{tr}(AB)^3\, \mathrm{tr} AB^{-1} - \mathrm{tr}\, BABAB^2] - 1 \\
&= -\mathrm{tr}\, C\, [-s^4 + 3s^2 - 1] - 1 = \mathrm{tr}\, C\, [\mathrm{tr}\, C - 1] - 1 = 0
\end{aligned}$$
$$\Rightarrow t^4 - 2t^3 + t - 1 = 0.$$
This irreducible polynomial has two real roots and so $\mathbb{Q}(t)$ is a field of degree 4 with one complex place.

Note that the stabiliser of one of the vertices of the tetrahedron contains an irreducible subgroup isomorphic to A_4, and so is generated by two elements g, h of order 3.

Area $= \frac{\pi}{30}$



Using Theorem 3.6.2, we obtain

$$A\Gamma = \left(\frac{tr^2 g(tr^2 g-4),\ tr^2 g\, tr^2 h\,(tr[g,h]-2)}{k\Gamma}\right)$$

$$= \left(\frac{-3,-2}{\mathbb{Q}(t)}\right) \quad (\because\ tr^2 g = tr^2 h = 1,\ tr[g,h]=0)$$

$A\Gamma$ is <u>only</u> ramified at two real places.

($\because$ $-2 \equiv 1 \pmod 3$ and 3 is unramified in the extension $\mathbb{Q}(t)|\mathbb{Q}$

$\Rightarrow$ $A\Gamma$ splits at the primes lying over 3 and also at all other non-dyadic primes by Thm 2.6.6.

<u>Thm 2.6.6</u> Let K be a non-dyadic P-adic field, with integers R and maximal ideal P. Let $A = \left(\frac{a,b}{K}\right)$, where $a, b \in R$.

1. If $a, b \notin P$, then A splits.
2. If $a \notin P$, $b \in P \setminus P^2$, then A splits $\Leftrightarrow$ a is a square mod P

Note that there is only one prime lying over 2 in $k\Gamma$. (See §

By Thm 2.7.3, $A\Gamma$ is only ramified at two real places)

We briefly consider the compact tetrahedron [diagram] whose tetrahedral group Γ has the presentation

$$\langle x, y, z \mid x^2 = y^3 = z^4 = (yz)^2 = (zx)^3 = (xy)^4 = 1 \rangle.$$

Locate the tetrahedron so that the octahedral group $S_4 = \langle y, z \rangle$ fixes the point $(0,0,1)$ in $\mathbb{H}^3$. We have

$$x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad y = \begin{pmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ -\frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}, \quad z = \begin{pmatrix} \frac{1+i}{\sqrt{2}} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{pmatrix}.$$

Taking $\gamma_1 = y$, $\gamma_2 = z$ and $\gamma_3 = z^{-1}x$ as generators of Γ which do not have order 2 and using Lemma 3.5.9,

$$k\Gamma = \mathbb{Q}(\{tr^2\gamma_i,\ 1 \le i \le 3;\ tr\,\gamma_i\gamma_j\, tr\,\gamma_i\, tr\,\gamma_j,\ 1 \le i < j \le 3;\ tr\,\gamma_1\gamma_2\gamma_3\, tr\,\gamma_1\, tr\,\gamma_2\, tr\,\gamma_3\})$$

$$= \mathbb{Q}(\sqrt{-7}).$$

This cocompact group has the same invariant trace field as the once-punctured torus bundle with monodromy R^2L

These groups are <u>not</u> commensurable.

Using the irreducible subgroup $\langle z, y \rangle$ in Thm 3.6.2 gives

$$A\Gamma \cong \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{-7})}\right)$$

The prime 2 splits in the extension $\mathbb{Q}(\sqrt{-7})|\mathbb{Q}$ so that $2\mathcal{O}_7 = PP'$, where P and P' are distinct prime ideals.

The completion k_v of $\mathbb{Q}(\sqrt{-7})$ at the valuation correspondi[ng] to either of these primes is isomorphic to the 2-adic numbe[rs]

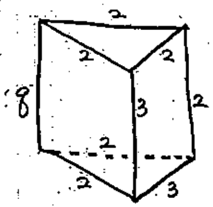
Thus $A\Gamma \otimes_{\mathbb{Q}(\sqrt{-7})} k_v \cong \left(\frac{-1,-1}{\mathbb{Q}_2}\right)$.

The equation $-x^2-y^2=z^2$ has no solution in the ring of 2-adic integers.

Thus by Thm 2.3.1(e) $A\Gamma$ is ramified at ν and so $A\Gamma$ cannot be isomorphic to $M_2(\mathbb{Q}(\sqrt{-7}))$.

We will later see that both of these groups are arithmetic, in which case the invariant trace field and the invariant quaternion algebra are complete commensurability invariants.

4.7.3 Prisms and Non-integral Traces



We construct an infinite family of examples in which there is an infinite subfamily whose members contain elements whose traces are "not" algebraic integers.

For any integer $q \geq 7$, the triangular prisms with dihedral angles which are submultiples of π, satisfy the conditions of Andreev's theorem and so exist in $\mathbb{H}^3$.

These prisms can be obtained from the infinite volume tetrahedron $\underset{1}{\circ}-\underset{2}{\circ}-\underset{3}{\circ}\overset{8}{-}\underset{4}{\circ}$ by truncating the tetrahedron by a face orthogonal to faces numbered 2, 3 and 4.

[E.M. Andreev, On convex polytopes in Lobachevskii spaces and On convex polytopes of finite volume in Lobachevskii space]

$C :=$ an abstract 3-dimensional polyhedron

$C^* :=$ its dual

A simple closed curve γ is a k-circuit if it consists of k edges of C^*.

A circuit γ is prismatic if all of the endpoints of the edges of C which γ meets are different.

Suppose that C is not a tetrahedron and non-obtuse angles $\theta_{ij} \in (0, \frac{\pi}{2}]$ are given corresponding to each edges $F_{ij} = F_i \cap F_j$ of C, where F_i are the faces of C.

Then the following conditions (A1)-(A4) are necessary and sufficient for the existence of a compact 3-dimensional hyperbolic polyhedron P which realizes C with dihedral angle θ_{ij} at each edge F_{ij}.

(A1) If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of C then
$$\theta_{ij} + \theta_{jk} + \theta_{ki} > \pi.$$

(A2) If F_i, F_j, F_k form a prismatic 3-circuit, then
$$\theta_{ij} + \theta_{jk} + \theta_{ki} < \pi.$$

(A3) If F_i, F_j, F_k, F_ℓ form a prismatic 4-circuit, then
$$\theta_{ij} + \theta_{jk} + \theta_{k\ell} + \theta_{\ell i} < 2\pi.$$

(A4) If C is a triangular prism with triangular faces F_1 and F_2, then

$\theta_{13}+\theta_{14}+\theta_{15}+\theta_{23}+\theta_{24}+\theta_{25} < 3\pi$.

Furthermore, this polyhedron is unique up to hyperbolic isometries.

The following conditions $(\tilde{A}1)-(\tilde{A}6)$ are necessary and sufficient for the existence of a 3-dimensional hyperbolic polyhedron P of finite volume which realizes C with dihedral angle $\theta_{ij} \in (0, \frac{\pi}{2}]$ at each edge F_{ij}.

$(\tilde{A}1)$ If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of C, then

$$\theta_{ij}+\theta_{jk}+\theta_{ki} \geq \pi.$$

$(\tilde{A}2)$ is the same as $(A2)$

$(\tilde{A}3)$ is the same as $(A3)$

$(\tilde{A}4)$ is same as $(A4)$

$(\tilde{A}5)$ If $F_{ijkl} = F_i \cap F_j \cap F_k \cap F_l$ is a vertex of C, then

$$\theta_{ij}+\theta_{jk}+\theta_{kl}+\theta_{li} = 2\pi.$$

$(\tilde{A}6)$ If F_i, F_j, F_k are faces with F_i and F_j adjacent, F_j and F_k adjacent, and F_i and F_k are not adjacent but meet in a vertex not in F_j, then

$$\theta_{ij}+\theta_{jk} < \pi.$$

Note that if the vertices of C are all trivalent, then conditions $(\tilde{A}5)$ and $(\tilde{A}6)$ are not needed.

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Of the five faces of the prism, two will be planes P_1 and P_2 orthogonal to $\mathbb{C}$, two will be hemispheres S_1 and S_2 centred at the origin and the last a hemisphere S_3 with centre on the x-axis. P_1, P_2, and S_3 meet S_1 orthogonally and bound a hyperbolic triangle on S_1 with angles $\frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{\pi}{8}$.

Thus $P_2 = \{(x,y,z) \mid y\cos\frac{\pi}{8} = x\sin\frac{\pi}{8}\}$ and

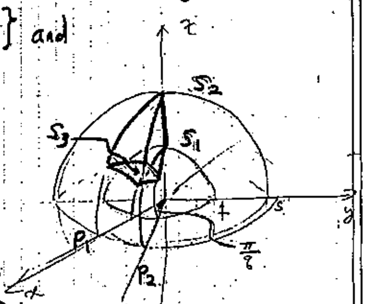
$S_3 = \{(x,y,z) \mid (x-a)^2+y^2+z^2 = t^2\}$,

where $a^2 = t^2+1$.

Choosing $t = 2a\sin\frac{\pi}{8}$ ensures that S_3 meets P_2 at $\frac{\pi}{3}$

$(\because\ (-\sin\frac{\pi}{8}, \cos\frac{\pi}{8}, 0)\cdot(x-a, y, z) = t\cos\frac{\pi}{3}$

$\Rightarrow a\sin\frac{\pi}{8} + (y\cos\frac{\pi}{8} - x\sin\frac{\pi}{8}) = \frac{t}{2})$



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Finally we truncate the region lying outside S_1 and S_3 and bounded by P_1 and P_2 by the hemisphere $S_2 = \{(x,y,z) \mid x^2+y^2+z^2 = s^2\}$ where $s^2+ts-1=0$, which guarantees that S_2 meets S_3 at $\frac{\pi}{3}$.

The polyhedral group T_8 is generated by the three elements $X = \rho_{S_2}\rho_{S_3}$, $Y = \rho_{P_1}\rho_{P_2}$ and $Z = \rho_{S_2}\rho_{S_1}$, where ρ denotes a reflection. We obtain

$$X = \begin{pmatrix} -\frac{1}{ts} & \frac{as}{t} \\ -\frac{a}{ts} & \frac{s}{t} \end{pmatrix} \quad Y = \begin{pmatrix} \exp(\frac{\pi i}{8}) & 0 \\ 0 & \exp(-\frac{\pi i}{8}) \end{pmatrix}, \quad Z = \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix}$$

Now $\mathrm{tr}^2 Z - 3 = (s+\frac{1}{s})^2 - 3 = t^2+1 = a^2 = \frac{1}{2\cos\frac{3\pi}{8}-1}$

Thus Z will have integral trace precisely when $2\cos\frac{2\pi}{8}-1$ is a unit in the ring of integers in $\mathbb{Q}(\cos\frac{2\pi}{8})$.

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For any q of the form $q=6p$, where p is a prime $\neq 2,3$, then $2\cos\frac{2\pi}{q}-1$ fails to be a unit.
Thus for these values of q, the groups Γ_q have elements whose traces are not algebraic integers.

4.8 Dehn Surgery Examples

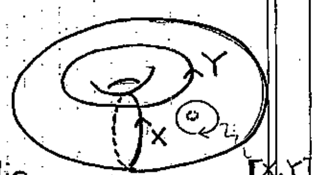
4.8.1 Jørgensen's Compact Fibre Bundles

Jørgensen first showed that there are compact hyperbolic manifolds which fibre over the circle.
The manifolds, M_n, were obtained as finite covers of orbifold bundles over the circle with fibre the 2-orbifold which is a torus with one cone point of order n, $n \geq 2$.
These orbifolds can be obtained by surgery on M, the figure 8 knot complement

$$\Gamma = \pi_1(M) = \langle X, Y, T \mid TXT^{-1} = XYX,\ TYT^{-1} = YX \rangle \rightarrow$$

$$\Gamma_n = \pi_1(M_n) = \langle X, Y, T \mid TXT^{-1} = XYX,\ TYT^{-1} = YX,\ [X,Y]^n = 1 \rangle$$

This group is isomorphic to the orientation-preserving subgroup of the outer automorphism group of $\pi_1(T_0) = F = \langle X, Y \rangle$, the free group on two generators, and so is isomorphic to $SL(2,\mathbb{Z})$.
Then $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is induced by the automorphism ρ where $\rho(X) = X$, $\rho(Y) = YX$ and $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ by λ, where $\lambda(X) = XY$, $\lambda(Y) = Y$.
The commutator $[X,Y]$ is represented by a simple closed loop round the puncture of T_0 so that $[X,Y]$ is parabolic.



A presentation of Γ is obtained as

$$\Gamma = \langle X, Y, T \mid TXT^{-1} = XYX,\ TYT^{-1} = YX \rangle.$$

($\because$ $\rho(\lambda(X)) = \rho(XY) = XYX$ and $\rho(\lambda(Y)) = \rho(Y) = YX$)

Now $\Gamma^{(2)} = \langle X, Y, T^2 \rangle$. Let $a = \operatorname{tr} X$, $b = \operatorname{tr} Y$, $c = \operatorname{tr} XY$
From the defining relations for Γ,

$$b = \operatorname{tr} Y = \operatorname{tr} TYT^{-1} = \operatorname{tr} YX = \operatorname{tr} XY = c \quad \text{and}$$

$$a = \operatorname{tr} X = \operatorname{tr} TXT^{-1} = \operatorname{tr} XYX \underset{(3.14)}{=} (\operatorname{tr} XY)(\operatorname{tr} X) - \operatorname{tr} XYX^{-1} = ac-$$

Since $[X,Y]$ is parabolic,

$$-2 = \operatorname{tr}[X,Y] \underset{(3.15)}{=} \operatorname{tr}^2 X + \operatorname{tr}^2 Y + \operatorname{tr}^2 XY - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} XY - 2$$

$$= a^2 + b^2 + c^2 - abc - 2$$

From these three equations, $a = (3 \pm \sqrt{-3})/2$ and $b = (3 \mp \sqrt{-3})/2$
From (3.25), $\mathbb{Q}(\operatorname{tr} F) = \mathbb{Q}(\operatorname{tr} X, \operatorname{tr} Y, \operatorname{tr} XY) = \mathbb{Q}(\sqrt{-3})$.
Since F has parabolic elements, $A_0F \cong M_2(\mathbb{Q}(\sqrt{-3}))$ by Thm 3
By Cor 4.3.3, $k\Gamma = \mathbb{Q}(\operatorname{tr} F) = \mathbb{Q}(\sqrt{-3})$ and

$$A\Gamma = A_0F = M_2(\mathbb{Q}(\sqrt{-3})).$$

Let F_n be the fundamental group of the orbifold fibre so that

$$F_n = \langle X, Y \mid [X,Y]^n = 1 \rangle$$

As in §4.4.3, let $a = \mathrm{tr}\,X$, $b = \mathrm{tr}\,Y$, $c = \mathrm{tr}\,XY$.

$\Rightarrow$ $b = c$, $a = ac - b$, $a^2 + b^2 + c^2 - abc - 2 = -2\cos\frac{\pi}{n}$

$\Rightarrow$ $(ab)^2 - 3(ab) - (2 - 2\cos\frac{\pi}{n}) = 0$, $a^2 - (ab)a + (ab) = 0$

$\Rightarrow$ $ab = \frac{3 + \sqrt{17 - 8\cos\frac{\pi}{n}}}{2} \in \mathbb{R}$ and $0 < ab < 4$

and $(ab)^2 - 4(ab) < 0$ $\therefore$ $a \notin \mathbb{R}$.

Thus if $k_n = \mathbb{Q}(\cos\frac{\pi}{n}, ab, a)$ then $b, c \in k_n$ and so $\mathbb{Q}(\mathrm{tr}\,F_n) = k_n$

Since $F_n \subset \Gamma_n^{(2)}$, $k\Gamma_n = k_n$ and $A\Gamma_n = A_0 F_n$

Using (3.37), $A\Gamma_n \cong \left(\frac{\mathrm{tr}^2 X - 4,\ \mathrm{tr}[X,Y] - 2}{k_n}\right) = \left(\frac{a^2 - 4,\ -2 - 2\cos\frac{\pi}{n}}{k_n}\right)$

Note that as $n \to \infty$, $[k_n : \mathbb{Q}] \to \infty$.

Theorem 4.8.1 There exist infinitely many commensurability classes of compact hyperbolic 3-manifolds.