4 Examples

We will illustrate the results and methods of the preceding chapter by calculating the invariant trace fields and quaternion algebras of some familiar examples.

4.1 Bianchi Groups

$O_d :=$ the ring of integers in $\mathbb{Q}(\sqrt{-d})$, where d is a positive square-free integer.

↳ a lattice in $\mathbb{C}$ with $\mathbb{Z}$-basis $\{1, \sqrt{-d}\}$ when $d \equiv 1, 2 \pmod 4$ and $\{1, \frac{1+\sqrt{-d}}{2}\}$ when $d \equiv 3 \pmod 4$.

$\Gamma_d :=$ The Bianchi group $PSL(2, O_d)$

↳ (arithmetic) Kleinian groups of finite covolume.

For every $\alpha \in O_d$, $g := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $h := \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in \Gamma_d$

$g^2 h^2 = \begin{pmatrix} 1+4\alpha & 2 \\ 2\alpha & 1 \end{pmatrix} \in \Gamma_d^{(2)} \Rightarrow tr(g^2 h^2) = 2 + 4\alpha \in k\Gamma_d$

$\Rightarrow \alpha \in k\Gamma_d \Rightarrow k\Gamma_d = \mathbb{Q}(\sqrt{-d})$.

Since Γ_d contains parabolic elements,

by Thm 3.3.8 $A\Gamma_d = M_2(\mathbb{Q}(\sqrt{-d}))$.

4.2 Knot and Link Complements

For knot and link complements in general, the invariant trace field coincides with the trace field.

As Thm 4.2.1 shows, this holds in a more general class of manifolds.

Thm 4.2.1 Let $M = \mathbb{H}^3/\Gamma$ be a manifold.
If $\mathrm{Coker}(H_1(\partial \overline{M}; \mathbb{Z}) \to H_1(\overline{M}; \mathbb{Z}))$ is finite of odd order, then $k\Gamma = \mathbb{Q}(\mathrm{tr}\,\Gamma)$.

Pf. $P :=$ the subgroup of Γ generated by parabolic elements

$\Rightarrow$ Γ/P is isomorphic to $\mathrm{Coker}(H_1(\partial \overline{M}; \mathbb{Z}) \to H_1(\overline{M}; \mathbb{Z}))$.

Now $\Gamma/\Gamma^{(2)}P$ has exponent 2.

$\Rightarrow$ The condition of Thm is equivalent to: $\Gamma = \Gamma^{(2)}P$.

Choose a finite set of parabolic elements $p_1, p_2, \dots, p_n$ such that these generate Γ modulo $\Gamma^{(2)}$

$\Rightarrow$ $\Gamma = \{ p_1^{\varepsilon_1} p_2^{\varepsilon_2} \cdots p_n^{\varepsilon_n} \Gamma^{(2)} \mid \varepsilon_i \in \{0,1\} \}$.

Now for a parabolic element p, $p^2 - 2p + I = 0$

so that $\operatorname{tr}(p_1^{\varepsilon_1}\cdots p_n^{\varepsilon_n}) = \frac{1}{2^n}\operatorname{tr}((p_1^2+I)^{\varepsilon_1}\cdots(p_n^2+I)^{\varepsilon_n}) \in k\Gamma\setminus\{0\}$.

From (3.14), $\operatorname{tr}(t^2\gamma) = \operatorname{tr}(t)\operatorname{tr}(t\gamma) - \operatorname{tr}(\gamma)$.

If $\gamma \in \Gamma^{(2)}$ then $\operatorname{tr}(t)\operatorname{tr}(t\gamma) \in k\Gamma$, so

if $\operatorname{tr}(t) \in k\Gamma\setminus\{0\}$ then $\operatorname{tr}(t\gamma) \in k\Gamma$.

$\Rightarrow \mathbb{Q}(\operatorname{tr}\Gamma) \subset k\Gamma$. $\Rightarrow \mathbb{Q}(\operatorname{tr}\Gamma) = k\Gamma$ ▨

Cor 4.2.2 If $M = \mathbb{H}^3/\Gamma$ is the complement of a link in a $\mathbb{Z}/2$-homology sphere,

then $k\Gamma = \mathbb{Q}(\operatorname{tr}\Gamma)$ and $A\Gamma = M_2(\mathbb{Q}(\operatorname{tr}\Gamma))$.

Pf. The first part follows from Thm 4.2.1 and the second from the fact that M is non-compact. ▨

Thm 4.2.3 A non-cocompact finite volume Kleinian group Γ has a faithful discrete representation in $PSL(2, \mathbb{Q}(\operatorname{tr}\Gamma))$.

Pf. Choose a lift of a cusp of Γ to be at ∞ and normalise so that the parabolic element $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$.
With further normalisation, let $h \in \Gamma$ be such that $h(0) = 0$.
$\Rightarrow$ Γ also contains an element of the form $h = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$.
$\Rightarrow$ $z \in \mathbb{Q}(\operatorname{tr}(\Gamma))$

Since $\operatorname{tr}[g,h] = 2 + z^2$, $\langle g, h \rangle$ is irreducible.
By Cor 3.2.3, $A_0(\Gamma) = \mathbb{Q}(\operatorname{tr}(\Gamma))[I, g, h, gh]$
$\subset M_2(\mathbb{Q}(\operatorname{tr}(\Gamma)))$ ▨

4.3 Hyperbolic Fibre Bundles

If Γ is the covering group of a finite-volume hyperbolic 3-orbifold which fibres over the circle with fibre a 2-orbifold of negative Euler characteristic, then $\exists$ a short exact sequence $1 \longrightarrow F \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow 1$ where F is isomorphic to the fundamental group of the 2-orbifold and F is geometrically infinite.

A conjecture of Thurston is that all finite-volume hyperbolic manifolds are finitely covered by a hyperbolic surface bundle.

<u>Note</u>: The invariant trace field and quaternion algebra are commensurability invariants.
The important feature is that these invariants are determined by the fibre.

Thm 4.3.1 If Δ is a finitely generated non-elementary normal subgroup of the finitely generated Kleinian group Γ, then $k\Delta = k\Gamma$ and $A\Delta = A\Gamma$.

Pf. $k\Delta = \mathbb{Q}(\mathrm{tr}\,\Delta^{(2)}) \subset \mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)}) = k\Gamma$.
Choose a pair of elements in $\Delta^{(2)}$ generating an irreducible subgroup $\Rightarrow$ $A\Gamma = A\Delta . k\Gamma$ by Cor 3.2.3.
(As Thm 3.3.4) By conjugation, each $\gamma \in \Gamma$ induces

an automorphism of $A\Delta$ ($\because \Delta^{(2)}$ is normal in Γ).
which is necessarily inner, by the Skolem Noether Thm.

$\Rightarrow \exists\ \delta \in A\Delta^*$ such that $\delta^{-1}\gamma$ commutes with all the elements of $A\Delta$.

$\Rightarrow \delta^{-1}\gamma = aI$ for some $a \in \mathbb{C}$.

$\Rightarrow a^2 = \det(\delta^{-1}\gamma) = \det(\delta)^{-1}$

Now $\det(\delta)\, I = \mathrm{tr}(\delta)\,\delta - \delta^2 \in A\Delta$

$\Rightarrow a^2 \in k\Delta \quad \Rightarrow \gamma^2 = a^2\delta^2 \in A\Delta$

$\therefore \Gamma^{(2)} \subset A\Delta \Rightarrow k\Gamma = k\Delta \Rightarrow A\Gamma = A\Delta.$ $\boxtimes$

Cor 4.3.2 If Γ is the covering group of a hyperbolic fibre bundle as above, then $kF = k\Gamma$ and $AF = A\Gamma$.

Cor 4.3.3 If Γ is the covering group of a hyperbolic fibre bundle as above and F_1 is a subgroup of finite index in F, which lies in $\Gamma^{(2)}$, then

$$kF = \mathbb{Q}(\mathrm{tr}\, F_1) = k\Gamma \quad \text{and} \quad AF = A_0F_1 = A\Gamma.$$

Pf. $F_1 \subset \Gamma^{(2)} \Rightarrow \mathbb{Q}(\mathrm{tr}\, F_1) \subset k\Gamma$

$\Rightarrow A\Gamma = A_0F_1 . k\Gamma$ as in the Pf. of Thm 4.3.1

Moreover, $kF = \mathbb{Q}(\mathrm{tr}\, F^{(2)}) \subset \mathbb{Q}(\mathrm{tr}\, F_1) \subset k\Gamma = kF$

(as Thm 3.3.4)

4.4 Figure 8 Knot Complement

The image of the knot group has index 12 in the Bianchi group $PSL(2, O_3)$.

$\Rightarrow k\Gamma = \mathbb{Q}(\sqrt{-3})$ and $A\Gamma = M_2(\mathbb{Q}(\sqrt{-3}))$.

It is instructive to consider how to calculate these invariants directly from the various ways of constructing this well-studied manifold.

4.4.1 Group Presentation

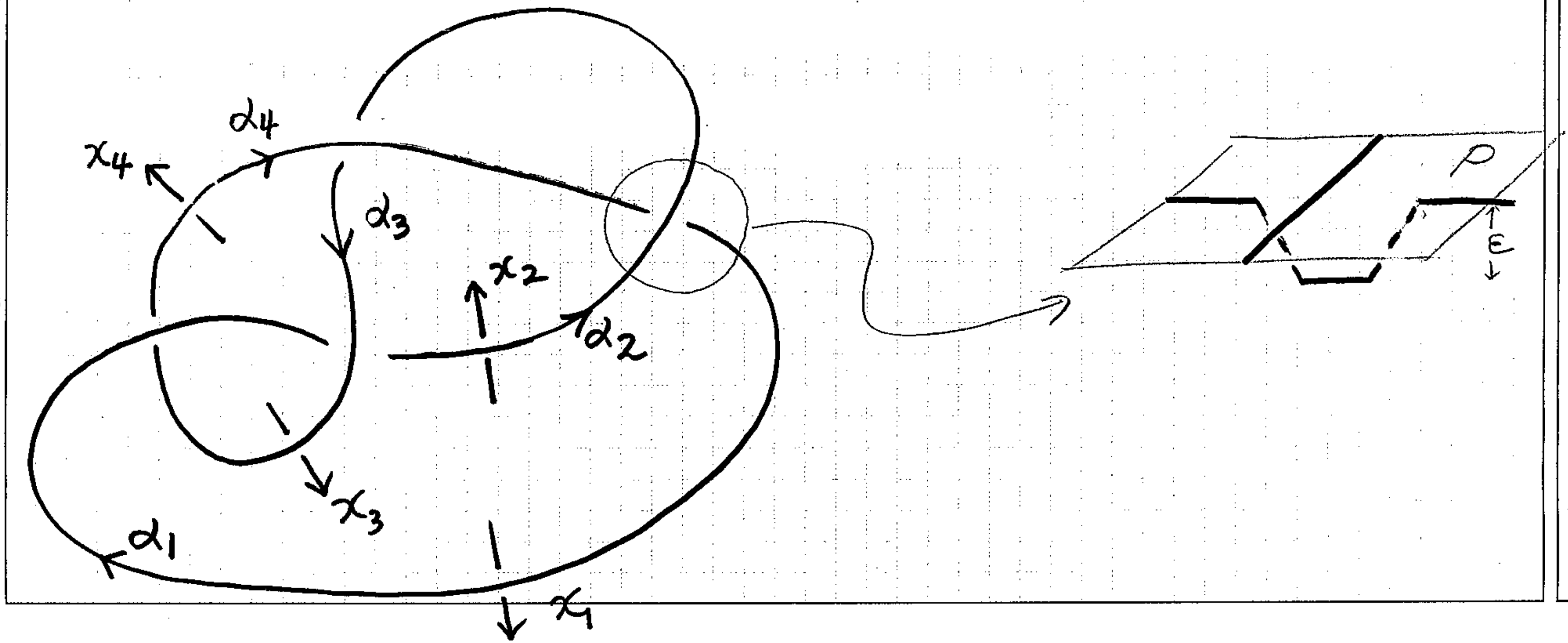
From the Wirtinger presentation,

$$\pi_1(S^3 \setminus K) = \langle x, y \mid xy^{-1}x^{-1}yx = yxy^{-1}x^{-1}y \rangle.$$

[D. Rolfsen, Knots and Links] p. 56 - p. 59

The Wirtinger Presentation.

This describes a procedure for writing down a presentation of the group of a knot K in $\mathbb{R}^3$, given a 'picture' of the knot.



The Algorithm.

We assume for convenience that the α_i are oriented (assigned a direction) compatibly with the order of their subscripts.

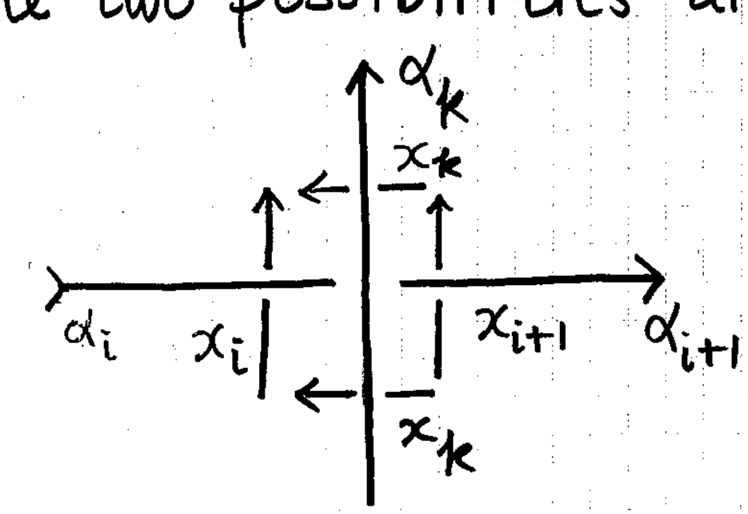
Draw a short arrow labelled x_i passing under each α_i in a right-left direction.

This is supposed to represent a loop in $\mathbb{R}^3 - K$ as follows.

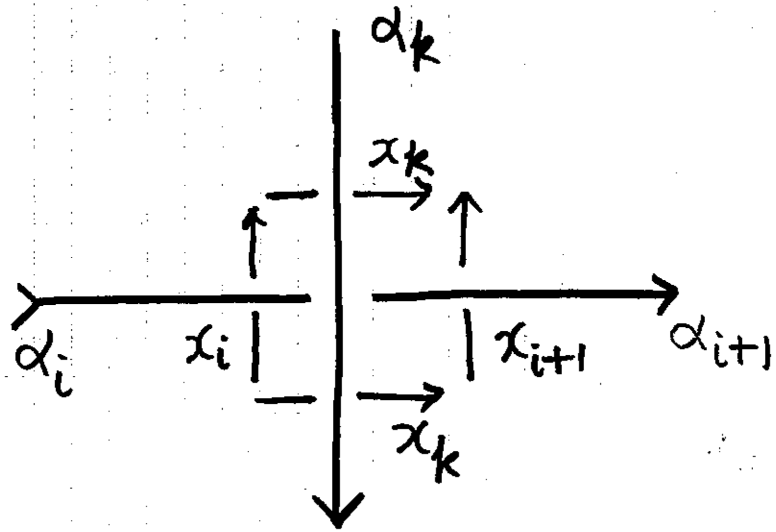
The point $(0,0,1) = *$ is taken as basepoint (best imagined as the eye of the viewer), and the loop consists of the oriented triangle from $*$ to the tail of x_i, along x_i to the head, thence back to $*$.

Now at each crossing, there is a certain relation among the x_i's which obviously must hold.

The two possibilities are:



$r_i : x_k x_i = x_{i+1} x_k$



$x_i x_k = x_k x_{i+1}$

Here α_k is the arc passing over the gap from α_i to α_{i+1} ($k=i$ or $i+1$ is possible).

Let r_i denote whichever of the two equations holds.

In all, there are exactly n relations $r_1, \ldots, r_n$ which may be read off this way.

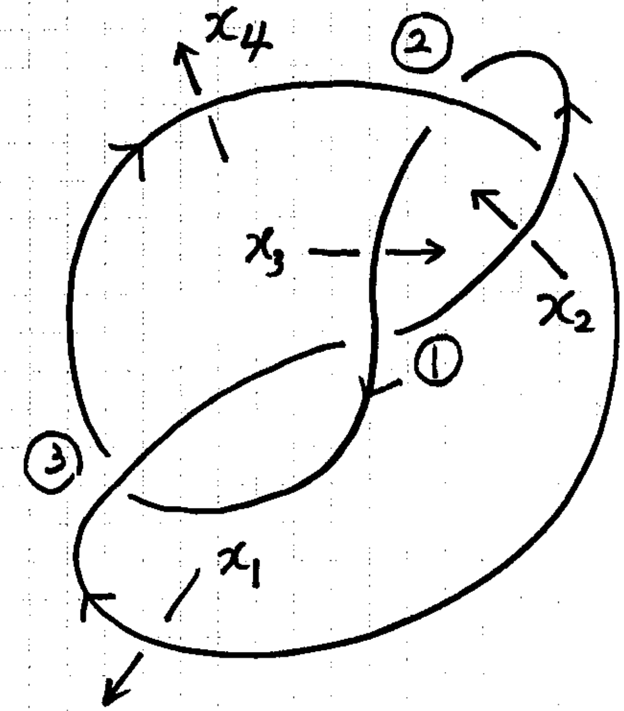
We will see that these comprises a complete set of relations.

Theorem:

The group $\pi_1(\mathbb{R}^3 - K)$ is generated by the x_i and has presentation $\pi_1(\mathbb{R}^3 - K) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$.

Moreover, any one of the r_i may be omitted and the above remains true.

For the figure-eight knot, we have a presentation with generators x_1, x_2, x_3, x_4 and relations



(1) $x_1 x_3 = x_3 x_2$

(2) $x_4 x_2 = x_3 x_4$

(3) $x_3 x_1 = x_1 x_4$

We may simplify, using (1) and (3) to eliminate $x_2 = x_3^{-1} x_1 x_3$ and $x_4 = x_1^{-1} x_3 x_1$ and substitute into (2) to obtain the equivalent presentation

$$\pi_1(\mathbb{R}^3 - \text{figure-eight}) = \langle x_1, x_3 \mid x_1^{-1} x_3 x_1 x_3^{-1} x_1 x_3 = x_3 x_1^{-1} x_3 x_1 \rangle.$$

The images of x and y are parabolic elements and by conjugation can be taken to be $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. Substituting in the defining relation for the group gives

$$\begin{pmatrix} 1-z+z^2 & 1+z^2 \\ z^2 & z^2+z+1 \end{pmatrix} = \begin{pmatrix} 1-z+z^2 & z \\ z+z^3 & z^2+z+1 \end{pmatrix}.$$

$\Rightarrow$ $z^2-z+1=0$ $\Rightarrow$ $z = e^{\pm\pi i/3}$

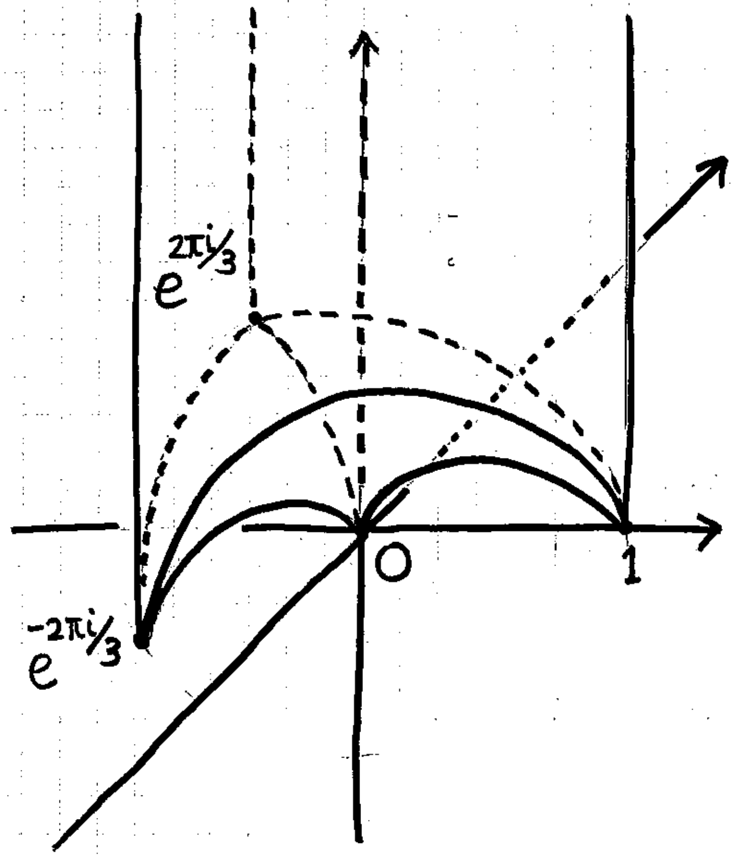
Modulo complex conjugation, we have a unique such representation with image Γ necessarily a finite-covolume group such that $\mathbb{H}^3/\Gamma$ is isometric to the figure 8 knot complement by Mostow Rigidity.

$\Rightarrow$ $k\Gamma = \mathbb{Q}(\sqrt{-3})$ and $A\Gamma = M_2(\mathbb{Q}(\sqrt{-3}))$.

4.4.2 Ideal Tetrahedra

The figure 8 knot complement can be seen to be a finite-volume hyperbolic manifold by suitably gluing together two regular ideal hyperbolic tetrahedra with dihedral angles $\pi/3$.

If we locate the tetrahedra with their vertices at $1, e^{2\pi i/3}, e^{-2\pi i/3}, \infty$ and $1, e^{2\pi i/3}, e^{-2\pi i/3}, 0$, then the face pairing transformations from the first tetrahedron to the second, carry, respectively,



$$1,\ e^{2\pi i/3},\ \infty \quad \text{to} \quad 0,\ e^{-2\pi i/3},\ 1$$
$$e^{2\pi i/3},\ e^{-2\pi i/3},\ \infty \quad \text{to} \quad e^{2\pi i/3},\ 0,\ 1$$
$$1,\ e^{-2\pi i/3},\ \infty \quad \text{to} \quad 0,\ e^{-2\pi i/3},\ e^{2\pi i/3}$$

These identifications are carried out by the matrices

$$\tau\begin{pmatrix} 1 & -1 \\ 1 & e^{-2\pi i/3}-2e^{2\pi i/3} \end{pmatrix},\quad \tau\begin{pmatrix} 1 & -e^{-2\pi i/3} \\ 1 & 1-2e^{2\pi i/3} \end{pmatrix},\quad \tau\begin{pmatrix} 1 & -1 \\ e^{-2\pi i/3} & 1-2e^{2\pi i/3} \end{pmatrix}$$

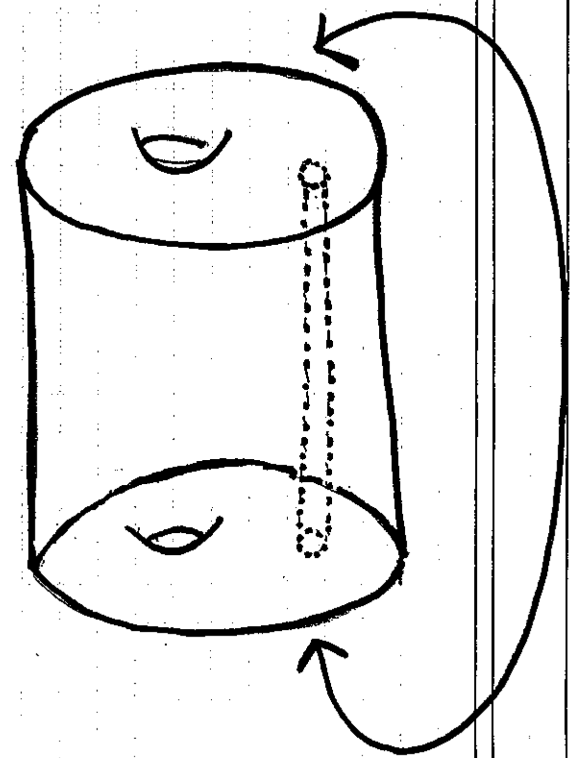
where $\tau = (e^{2\pi i/3}-1)^{-1}$.

Since the group is generated by these matrices we see that the group lies in $SL(2, \mathbb{Q}(\sqrt{-3}))$ and, again, the result follows.

4.4.3 Once-Punctured Torus Bundle

We now give a third approach in which the invariant trace field is determined without first obtaining a matrix representation. Let $M = \mathbb{H}^3/\Gamma$ denote the hyperbolic manifold of finite volume which is the figure 8 knot complement.

Now M can be described as a fibre bundle over the circle with fibre a once-punctured torus T_0.



There is thus an exact sequence

$$1 \longrightarrow \pi_1(T_0) \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow 1$$

and the monodromy of the bundle is given by the element RL in the mapping class group of T_0.

[J. Hempel, 3-Manifolds] p.162 and p.121.

The Mapping Class Group

$MC(T)$:= the mapping class group of a closed orientable surface T

:= the group of homeomorphisms of T modulo those which are isotopic to the identity.

Every automorphism of $\pi_1(T)$ is induced by a homeomorphism of T.

Two homeomorphisms of T whose induced automorphism of $\pi_1(T)$ differ by an inner automorphism are freely homotopic, hence, isotopic.

Thus $MC(T)$ is isomorphic to the group of outer automorphism of $\pi_1(T)$.

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Surface bundles over S^1

Let F be a compact surface and $\phi : F \to F$ a homeomorphism. Then the space M_ϕ obtained from $F \times I$ by identifying $(x, 0) \in F \times 0$ with $(\phi(x), 1) \in F \times 1$ is a fiber bundle over S^1 with fiber F.

Every such bundle is so obtained.

If $\phi_1, \phi_2 : F \to F$ are isotopic, then an isotopic between 1 and $\phi_2^{-1}\phi_1$ induces a map $F \times I \to F \times I$ which in turn induces a fiber preserving homeomorphism $M_{\phi_1} \longrightarrow M_{\phi_2}$.

It is known that homotopic homeomorphisms of F are isotopic.

Two homeomorphisms of F which induce the same automorphism of $\pi_1(F)$, modulo an inner automorphism, are homotopic.

Thus M_ϕ is determined by the outer automorphism of $\pi_1(F)$ induced by ϕ.

We can present $\pi_1(M_\phi) = \pi_1(F) * \langle t \mid - \rangle / N$ where N is the smallest normal subgroup containing

$$\{ tat^{-1}\phi_*(a^{-1}) : a \in \pi_1(F) \}$$

where $\phi_* : \pi_1(F) \to \pi_1(F)$ is induced by ϕ and t is represented by a loop meeting each fiber transversely in a single point.

p162-p169.

[W. Magnus, A. Karrass, and D. Solitar, Combinatorial Group Theory]

The Automorphism Groups Φ_n of Free Groups

$F_n :=$ the free group of rank n.

$\Lambda_n :=$ the group of n-dimensional lattice transformation

↳ can be described as the multiplicative group of n-by-n matrices with determinant ± 1 and integers as entries.

It is obvious that there exists a homomorphic mapping of Φ_n into Λ_n.

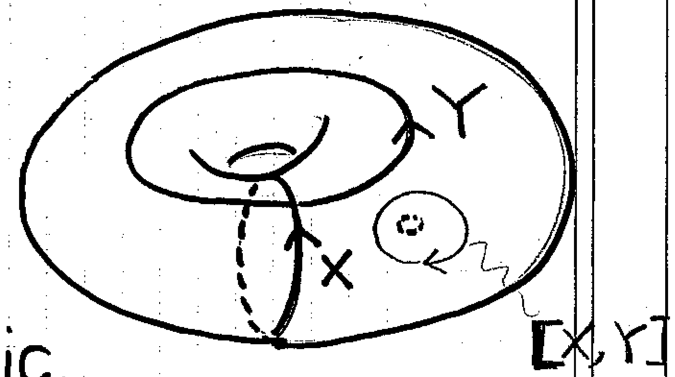
$$\begin{tikzcd} F_n \arrow[r] \arrow[d] & F_n \arrow[d] & \in \Phi_n \\ \mathbb{Z}^n \arrow[r] & \mathbb{Z}^n & \in \Lambda_n \end{tikzcd}$$

The kernel of the homomorphic mapping $\Phi_2 \twoheadrightarrow \Lambda_2$ consists of the inner automorphisms of F_2.

This group is isomorphic to the orientation-preserving subgroup of the outer automorphism group of $\pi_1(T_0) = F = \langle X, Y \rangle$, the free group on two generators, and so is isomorphic to $SL(2, \mathbb{Z})$.

Then $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is induced by the automorphism ρ where $\rho(X) = X$, $\rho(Y) = YX$ and $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ by λ, where $\lambda(X) = XY$, $\lambda(Y) = Y$.

The commutator $[X, Y]$ is represented by a simple closed loop round the puncture of T_0 so that $[X, Y]$ is parabolic.



A presentation of Γ is obtained as

$$\Gamma = \langle X, Y, T \mid TXT^{-1} = XYX,\ TYT^{-1} = YX \rangle.$$

$(\because\ \rho(\lambda(X)) = \rho(XY) = XYX$ and $\rho(\lambda(Y)) = \rho(Y) = YX\ )$.

Now $\Gamma^{(2)} = \langle X, Y, T^2 \rangle$. Let $a = \mathrm{tr}\, X$, $b = \mathrm{tr}\, Y$, $c = \mathrm{tr}\, XY$.

From the defining relations for Γ,

$b = \mathrm{tr}\, Y = \mathrm{tr}\, TYT^{-1} = \mathrm{tr}\, YX = \mathrm{tr}\, XY = c$ and

$a = \mathrm{tr}\, X = \mathrm{tr}\, TXT^{-1} = \mathrm{tr}\, XYX \underset{(3.14)}{=} (\mathrm{tr}\, XY)(\mathrm{tr}\, X) - \mathrm{tr}\, XYX^{-1} = ac - b$.

Since $[X, Y]$ is parabolic,

$$-2 = \mathrm{tr}[X, Y] \underset{(3.15)}{=} \mathrm{tr}^2 X + \mathrm{tr}^2 Y + \mathrm{tr}^2 XY - \mathrm{tr}\, X\, \mathrm{tr}\, Y\, \mathrm{tr}\, XY - 2$$
$$= a^2 + b^2 + c^2 - abc - 2$$

From these three equations, $a = (3 \pm \sqrt{-3})/2$ and $b = (3 \mp \sqrt{-3})/2$.

From (3.25), $\mathbb{Q}(\mathrm{tr}\, F) = \mathbb{Q}(\mathrm{tr}\, X, \mathrm{tr}\, Y, \mathrm{tr}\, XY) = \mathbb{Q}(\sqrt{-3})$.

Since F has parabolic elements, $A_0 F \cong M_2(\mathbb{Q}(\sqrt{-3}))$ by Thm 3.3.8.

By Cor 4.3.3, $k\Gamma = \mathbb{Q}(\mathrm{tr}\, F) = \mathbb{Q}(\sqrt{-3})$ and

$$A\Gamma = A_0 F = M_2(\mathbb{Q}(\sqrt{-3})).$$

4.6 Once-Punctured Torus Bundles

If $M = \mathbb{H}^3/\Gamma$ is a once-punctured torus bundle, then the fibre group $F = \langle X, Y \rangle$ is a free group.

The monodromy of the bundle, as an element of the mapping class group $SL(2, \mathbb{Z})$, is a hyperbolic element can be taken to have the form $(-1)^{\varepsilon} R^{n_1} L^{n_2} R^{n_3} \cdots L^{n_{2k}}$, where $n_i \geq 1$ and $\varepsilon \in \{0, 1\}$.

This is induced by the automorphism $\theta = i^{\varepsilon} \rho^{n_1} \lambda^{n_2} \cdots \lambda^{n_{2k}}$ where ρ and λ are as defined in §4.4.3 and

$$i(X) = X^{-1}, \quad i(Y) = Y^{-1}.$$

The group Γ has presentation

$$\langle X, Y, T \mid TXT^{-1} = \theta(X),\ TYT^{-1} = \theta(Y) \rangle.$$

If $a = \mathrm{tr}\, X$, $b = \mathrm{tr}\, Y$ and $c = \mathrm{tr}\, XY$, then since $[X, Y]$ is parabolic, $a^2 + b^2 + c^2 = abc$.

A. Monodromy $-RL$

$\Gamma = \langle X, Y, T \mid TXT^{-1} = X^{-1}Y^{-1}X^{-1},\ TYT^{-1} = Y^{-1}X^{-1} \rangle$

$\Rightarrow$ a, b and c satisfy exactly the same equations as in the case of monodromy RL.

$\Rightarrow$ $k\Gamma = \mathbb{Q}(\sqrt{-3})$ and $A\Gamma = M_2(\mathbb{Q}(\sqrt{-3}))$.

The manifold that arises is the "sister" of the figure 8 knot complement. and is commensurable with the figure 8 knot complement as these two complements can be shown to have a common double cover.

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Thus the above deductions are immediate from the commensurability invariance.

For the same reasons, the bundles with monodromies of the form $(RL)^m$ have the same invariant trace field and quaternion algebra.

B. Monodromy R^2L

$$\Gamma = \langle X, Y, T \mid TXT^{-1} = XYX^2,\ \ TYT^{-1} = YX^2 \rangle$$

$$= \langle X, Y, T \mid T^{-1}XT = XY^{-1},\ \ TYT^{-1} = YX^2 \rangle$$

$$(\because\ TXT^{-1} = XYX^2 = XTYT^{-1} \Rightarrow TX = XTY\ )$$

Moreover, $F_1 = \langle X^2, Y, XYX^{-1} \rangle$ is a subgroup of index 2 in F.

$(\because\ F_1$ is a normal subgroup of F and $F/F_1 = \langle X, Y \mid X^2, Y, XYX^{-1} \rangle = \langle X \mid X^2 \rangle \cong \mathbb{Z}_2)$

and $F_1 \subset \Gamma^{(2)}$.

By Cor 4.3.3, $k\Gamma = \mathbb{Q}(\operatorname{tr} F_1)$ and $A\Gamma = M_2(\mathbb{Q}(\operatorname{tr} F_1))$.

By (3.26), $\mathbb{Q}(\operatorname{tr} F_1) = \mathbb{Q}(a^2, b, ac, c^2)$.

$$
\begin{aligned}
(\because\ & \operatorname{tr} X^2 = \operatorname{tr}^2 X - 2 = a^2 - 2, \quad \operatorname{tr} Y = b, \quad \operatorname{tr} XYX^{-1} = b \\
& \operatorname{tr} X^2 Y = \operatorname{tr} XYX = \operatorname{tr} XY \operatorname{tr} X - \operatorname{tr} XYX^{-1} = ac - b \\
& \operatorname{tr} X^3 Y X^{-1} = \operatorname{tr} X^2 Y = ac - b \\
& \operatorname{tr} YXYX^{-1} = \operatorname{tr} XYX^{-1} Y = \operatorname{tr} XYX^{-1} \operatorname{tr} Y - \operatorname{tr} [X, Y] = b^2 + 2 \\
& \operatorname{tr} X^2 YXYX^{-1} = \operatorname{tr} XYXY = \operatorname{tr}^2 XY - 2 = c^2 - 2\ )
\end{aligned}
$$

By (4.4) and (4.5), $b = ac - b$, $a = ab - c$, $a^2 + b^2 + c^2 = abc$.

$\Rightarrow b = \dfrac{a^2}{a^2 - 2}$, $c = \dfrac{2a}{a^2 - 2}$ $\Rightarrow$ $\mathbb{Q}(\operatorname{tr} F_1) = \mathbb{Q}(a^2)$.

$\Rightarrow a^4 - 5a^2 + 8 = 0 \Rightarrow a^2 = \dfrac{5 \pm \sqrt{-7}}{2}$

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$\Rightarrow k\Gamma = \mathbb{Q}(\operatorname{tr} F_1) = \mathbb{Q}(\sqrt{-7})$.

For future reference, we note that $a^2, b, ac \in O_7$.

This furnishes an example of a non-compact manifold where the invariant trace field is not the trace field.

($\because$ $a = \operatorname{tr} X \in \mathbb{Q}(\operatorname{tr}\Gamma)$, but $a \notin k\Gamma = \mathbb{Q}(\sqrt{-7})$).

With reference to Thm 4.2.1,

$$\operatorname{Coker}(\underset{\mathbb{Z}\oplus\mathbb{Z}}{\underset{\cong}{H_1(\partial\overline{M};\mathbb{Z})}} \longrightarrow \underset{\mathbb{Z}\oplus\mathbb{Z}_2}{\underset{\cong}{H_1(\overline{M};\mathbb{Z})}}) \cong \mathbb{Z}_2$$

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4.7 Polyhedral Groups

Many examples of hyperbolic 3-manifolds and orbifolds are constructed using a fundamental domain in $\mathbb{H}^3$.

Combinatorial and geometric conditions provided by <u>Andreev</u> allow one to construct polyhedra in $\mathbb{H}^3$. $\longrightarrow$

If the polyhedron satisfies Poincaré's requirements with respect to face pairing transformations, then these transformations generate a discrete group whose fundamental domain is the polyhedron.

<u>Coxeter groups</u> which are generated by reflections in the faces of suitable polyhedra are special cases of this.

[E.M. Andreev, On convex polytopes in Lobachevskii spaces and On convex polytopes of finite volume in Lobachevskii space]

$C :=$ an abstract 3-dimensional polyhedron

$C^* :=$ its dual.

A simple closed curve γ is a k-circuit if it consists of k edges of C^*.

A circuit γ is prismatic if all of the endpoints of the edges of C which γ meets are different.

Suppose that C is not a tetrahedron and non-obtuse angles $\theta_{ij} \in (0, \frac{\pi}{2}]$ are given corresponding to each edges $F_{ij} = F_i \cap F_j$ of C, where F_i are the faces of C.

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Then the following conditions (A1)-(A4) are necessary and sufficient for the existence of a compact 3-dimensional hyperbolic polyhedron P which realizes C with dihedral angle θ_{ij} at each edge F_{ij}.

(A1) If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of C then
$$\theta_{ij} + \theta_{jk} + \theta_{ki} > \pi.$$

(A2) If F_i, F_j, F_k form a prismatic 3-circuit, then
$$\theta_{ij} + \theta_{jk} + \theta_{ki} < \pi.$$

(A3) If F_i, F_j, F_k, F_l form a prismatic 4-circuit, then
$$\theta_{ij} + \theta_{jk} + \theta_{kl} + \theta_{li} < 2\pi.$$

(A4) If C is a triangular prism with triangular faces F_1 and F_2, then

$$\theta_{13} + \theta_{14} + \theta_{15} + \theta_{23} + \theta_{24} + \theta_{25} < 3\pi.$$

Furthermore, this polyhedron is unique up to hyperbolic isometries.

The following conditions $(\widetilde{A1})$ – $(\widetilde{A6})$ are necessary and sufficient for the existence of a 3-dimensional hyperbolic polyhedron P of finite volume which realizes C with dihedral angle $\theta_{ij} \in (0, \frac{\pi}{2}]$ at each edge F_{ij}.

$(\widetilde{A1})$ If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of C, then

$$\theta_{ij} + \theta_{jk} + \theta_{ki} \geq \pi.$$

$(\widetilde{A2})$ is the same as $(A2)$

$(\widetilde{A3})$ is the same as $(A3)$

$(\widetilde{A4})$ is same as $(A4)$

$(\widetilde{A5})$ If $F_{ijkl} = F_i \cap F_j \cap F_k \cap F_l$ is a vertex of C, then

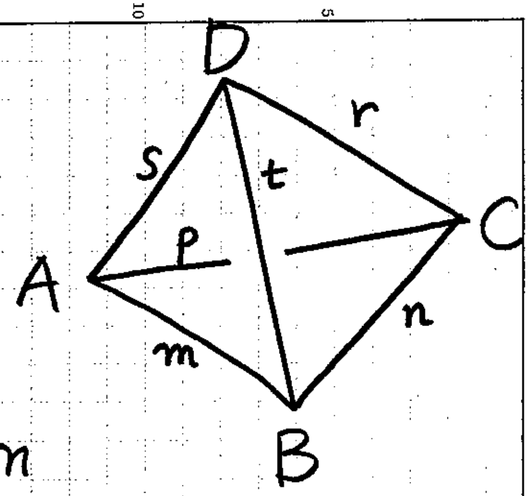
$$\theta_{ij} + \theta_{jk} + \theta_{kl} + \theta_{li} = 2\pi.$$

$(\widetilde{A6})$ If F_i, F_j, F_k are faces with F_i and F_j adjacent, F_j and F_k adjacent, and F_i and F_k are not adjacent but meet in a vertex not in F_j, then

$$\theta_{ij} + \theta_{jk} < \pi.$$

Note that if the vertices of C are all trivalent, then conditions $(\widetilde{A5})$ and $(\widetilde{A6})$ are not needed.

The index 2 subgroups consisting of orientation-preserving isometries in the groups generated by reflections in the faces of these polyhedra referred to as polyhedral groups.
Examples of particular interest arise when the polyhedron is a tetrahedron.
There are 9 compact hyperbolic tetrahedra whose dihedral angles are submultiples of π and there are a further 23 with at least 1 ideal vertex (i.e., vertex on the sphere at ∞), which have finite volume.
We represent these tetraheda schematically.



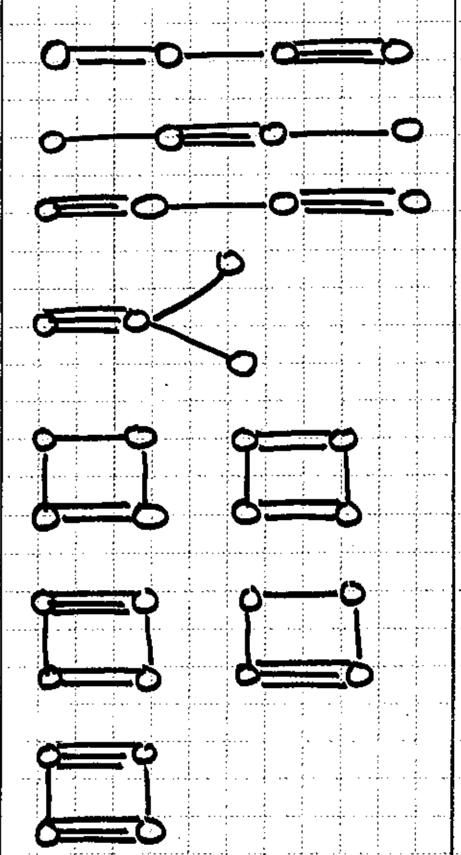
The edge labelling (e.g., p) indicates the dihedral angle (e.g., π/p) along that edge.

The tetrahedral group then has presentation

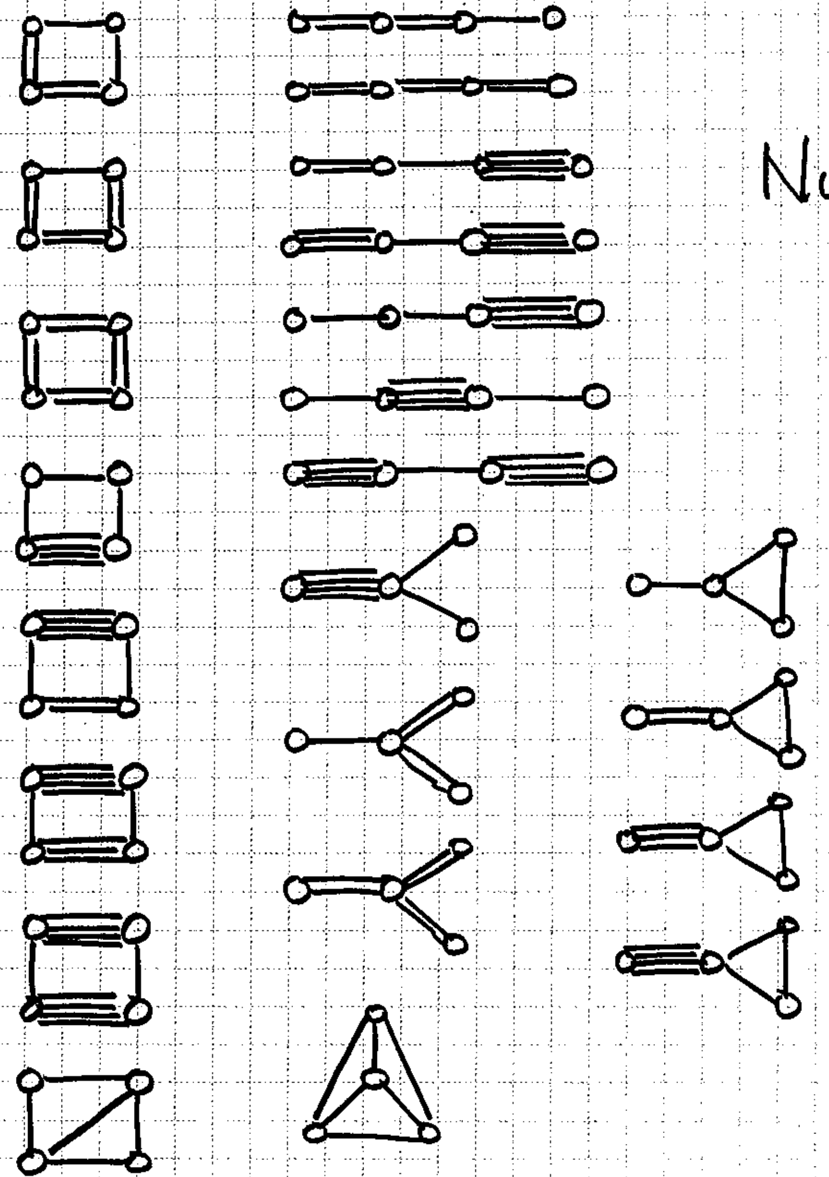
$$\langle x, y, z \mid x^m = y^n = z^p = (yz^{-1})^r = (zx^{-1})^s = (xy^{-1})^t = 1 \rangle.$$

These groups may also be described by the Coxeter symbol for the tetrahedron.

Compact



Non-compact



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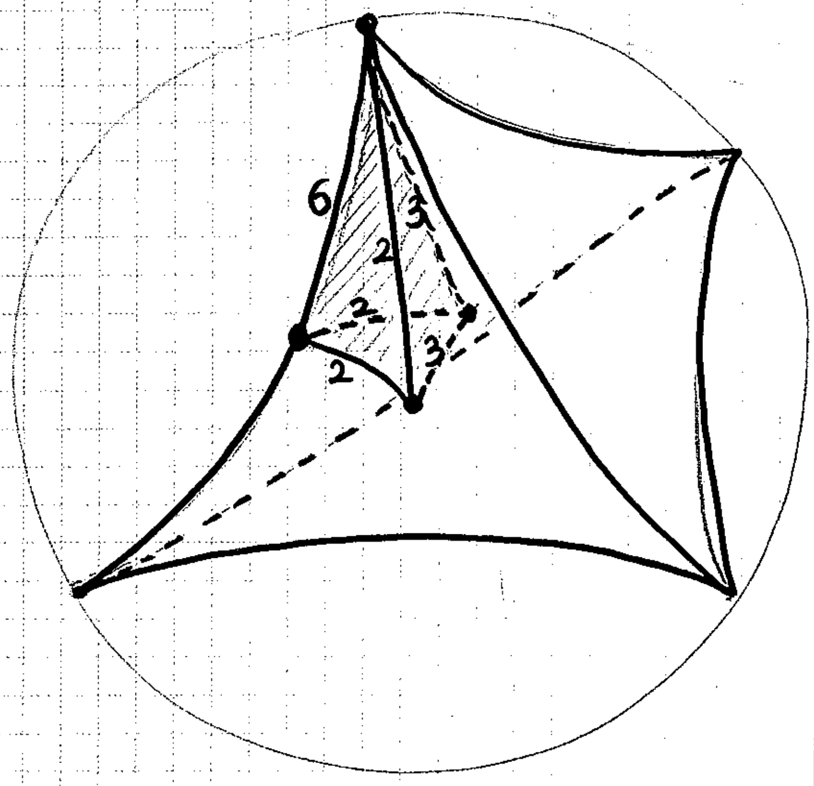
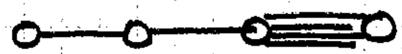
4.7.1 Non-compact Tetrahedra

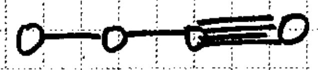
The figure 8 knot complement is described as the union of two regular ideal tetrahedra with dihedral angles $\pi/3$ by suitable face pairing.

These tetrahedra have all of their vertices on the sphere at ∞ and $\mathbb{H}^3$ admits a tesselation by such regular tetrahedra.





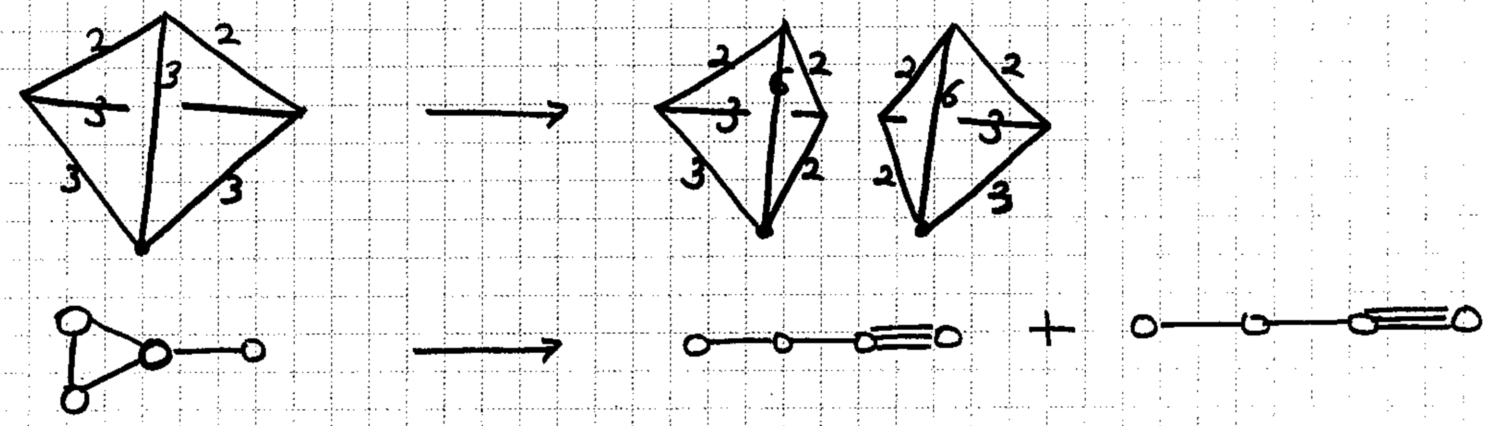
The full group of symmetries of this tesselation is the group generated by reflections in the faces of a tetrahedron which is a cell of the barycentric subdivision of the regular ideal tetrahedron.

The face-pairing transformations which give rise to the figure 8 knot complement lie in the full group of symmetries of the tesselation, so that the tetrahedral group associated with  and the figure 8 knot group are commensurable.

$\Rightarrow$ This tetrahedral group's invariant trace field is $\mathbb{Q}(\sqrt{-3})$ and quaternion algebra is $M_2(\mathbb{Q}(\sqrt{-3}))$.

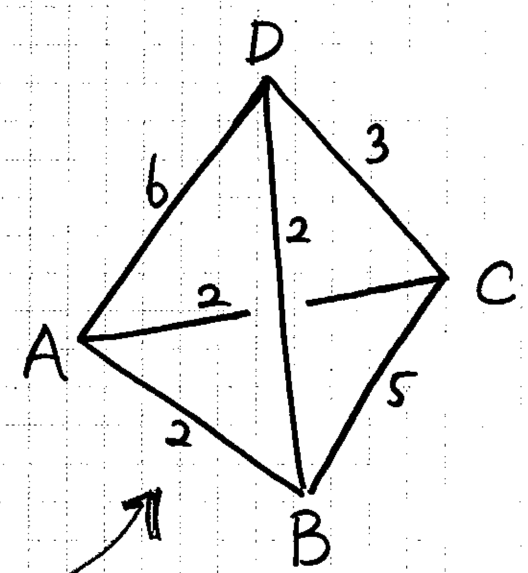
Several other tetrahedra with ideal vertices whose dihedral angles are submultiples of π can be obtained as unions of this tetrahedron, so that their associated tetrahedral groups have the same invariant trace field and quaternion algebra.

e.g.,

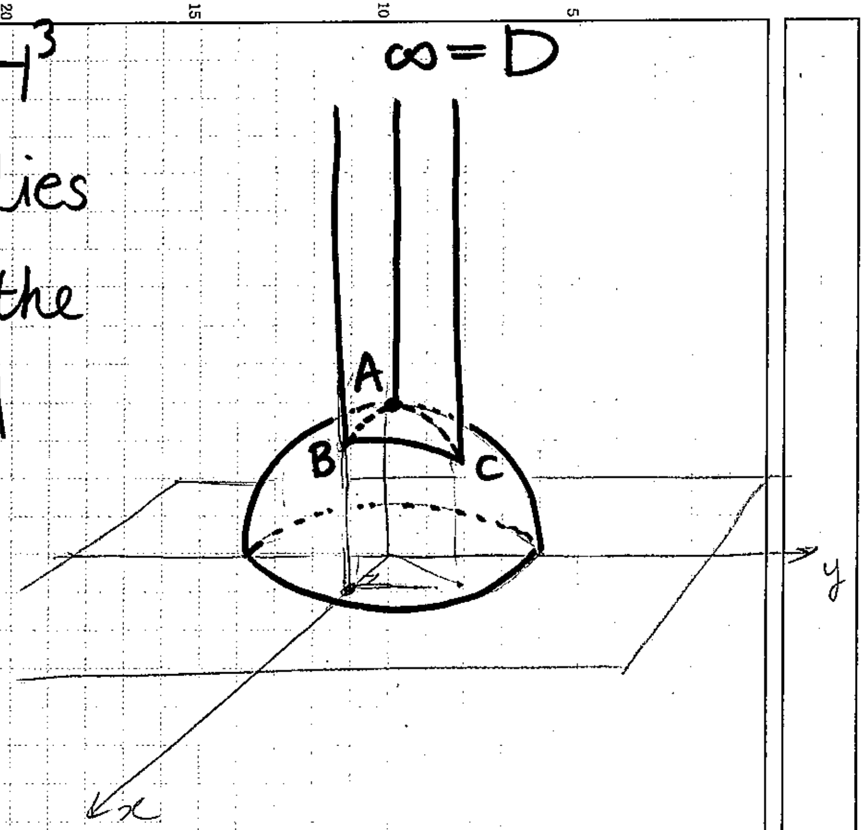


In an analogous way, $\mathbb{H}^3$ can be tesselated by regular ideal dodecahedra whose dihedral angles are $\pi/3$.

If we take the barycentric subdivision of one such regular ideal dodecahedra, we obtain the ideal tetrahedron



We can locate this tetrahedron in $\mathbb{H}^3$ such that D is at ∞ and ABC lies on the unit hemisphere centred at the origin with A the north pole and $B = (\cos \pi/5, 0, \sin \pi/5)$.



If we let x denote the rotation about AD, y the rotation about BD and z the rotation about AB, then

$$x = \begin{pmatrix} e^{\pi i/6} & 0 \\ 0 & e^{-\pi i/6} \end{pmatrix}, \quad y = \begin{pmatrix} 1 & \cos\frac{\pi}{5} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & -\cos\frac{\pi}{5} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} i & -2\cos\frac{\pi}{5} \\ 0 & -i \end{pmatrix}.$$

$$z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

To calculate the invariant trace field of Γ using Lemma 3.5.9,

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we change generators to $\gamma_1 = x$, $\gamma_2 = xy$ and $\gamma_3 = yz$, and obtain $k\Gamma = \mathbb{Q}(\sqrt{5}, \sqrt{-3})$ and $A\Gamma = M_2(k\Gamma)$.

($\because$ $k\Gamma$ is generated over $\mathbb{Q}$ by $\{ \mathrm{tr}^2\gamma_i,\ 1 \le i \le 3;$
$\mathrm{tr}\,\gamma_i\gamma_j\ \mathrm{tr}\,\gamma_i\ \mathrm{tr}\,\gamma_j,\ 1 \le i < j \le 3;\ \ \mathrm{tr}\,\gamma_1\gamma_2\gamma_3\ \mathrm{tr}\,\gamma_1\ \mathrm{tr}\,\gamma_2\ \mathrm{tr}\,\gamma_3 \}$, and

$\mathrm{tr}^2\gamma_1 = 3$, $\mathrm{tr}^2\gamma_2 = 1$, $\mathrm{tr}^3\gamma_3 = 4\cos^2(\frac{\pi}{5})$

$\mathrm{tr}\,\gamma_1\gamma_2\ \mathrm{tr}\,\gamma_1\ \mathrm{tr}\,\gamma_2 = 3$, $\mathrm{tr}\,\gamma_2\gamma_3\ \mathrm{tr}\,\gamma_2\ \mathrm{tr}\,\gamma_3 = 0$

$\mathrm{tr}\,\gamma_1\gamma_3\ \mathrm{tr}\,\gamma_1\ \mathrm{tr}\,\gamma_3 = 2\cos^2(\frac{\pi}{5})\,(3+\sqrt{-3})$.

$\mathrm{tr}\,\gamma_1\gamma_2\gamma_3\ \mathrm{tr}\,\gamma_1\ \mathrm{tr}\,\gamma_2\ \mathrm{tr}\,\gamma_3 = 0$).