3. Invariant Trace Fields.

§3.1. Trace Fields for Kleinian Groups of Finite Covolume.

<u>Recall</u> $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm I\}$

$\Rightarrow$ Via the linear fractional action $z \longmapsto \frac{az+b}{cz+d}$, elements $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C})$ are biholomorphic maps of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

$\Rightarrow$ The action of each $\gamma \in PSL(2,\mathbb{C})$ on $\hat{\mathbb{C}}$ extends to an action on the upper half 3-space

$$\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid t > 0\}$$

via the Poincaré extension.

<u>Poincaré extension</u>

Let a be a unit vector in E^n and let t be a real number. Consider the <u>hyperplane</u> of E^n defined by

$$P(a,t) = \{x \in E^n \mid a \cdot x = t\}.$$

$\Rightarrow$ The <u>reflection</u> ρ of E^n in the plane $P(a,t)$ is defined by

$$\rho(x) = x + 2(t - a\cdot x)a.$$

Let $a \in E^n$ and r be a positive real number.

$\Rightarrow$ The sphere of E^n of radius r centered at a is the set

$$S(a,r) = \{x \in E^n \mid |x-a| = r\}.$$

$\Rightarrow$ The <u>reflection</u> (or <u>inversion</u>) σ of E^n in $S(a;r)$ is defined by $\sigma(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x-a)$.

We define $\hat{E}^n = E^n \cup \{\infty\}$ $\Rightarrow$ the one-point compactification.

& a <u>sphere</u> Σ of $\hat{E}^n$ is defined to be either $S(a,r)$ or an extended plane $\hat{P}(a,t) = P(a,t) \cup \{\infty\}$.

<u>Definition</u> A Möbius transformation of $\hat{E}^n$ is a finite composition of reflections of $\hat{E}^n$ in spheres.

(ρ for $P(a,t)$ is extended to $\hat{\rho}: \hat{E}^n \longrightarrow \hat{E}^n$ by setting $\rho(\infty)=\infty$
& σ for $S(a,t)$ 〃 $\hat{\sigma}: \hat{E}^n \longrightarrow \hat{E}^n$ by $\sigma(a)=\infty, \sigma(\infty)=a$)

<u>Theorem</u> If ϕ is a Möbius transformation of $\hat{E}^n$ that fixes each point of a sphere Σ of $\hat{E}^n$, then ϕ is either the identity map of $\hat{E}^n$ or the reflection in Σ. //

Under the identification of E^{n-1} with $E^{n-1} \times \{0\}$ in E^n, a point x of E^{n-1} corresponds to $\hat{x} = (x, 0)$ of E^n. Let ϕ be a Möbius transf. of $\hat{E}^{n-1}$. We shall extend ϕ to a Möbius transf. $\tilde{\phi}$ of $\hat{E}^n$ as follows:

ϕ : the reflection of $\hat{E}^{n-1}$ in $\hat{P}(a,t)$ $(\hat{S}(a,t))$ $\Rightarrow$ $\tilde{\phi}$: the reflection of $\hat{E}^n$ in $\hat{P}(\tilde{a},t)$ $(\hat{S}(\tilde{a},t))$.

$\Rightarrow$ In both cases, $\tilde{\phi}(x,0) = (\phi(x), 0)$ for all x in E^{n-1}.

$\Rightarrow$ $\tilde{\phi}$ extends ϕ & $\tilde{\phi}$ leaves $\hat{E}^{n-1}$ invariant.

It is also clear that $\tilde{\phi}$ leaves invariant upper half-space

$$U^n = \{(x_1, \cdots, x_n) \in E^n \mid x_n > 0\}.$$

Now assume that ϕ is an arbitrary Möbius transf. of $\hat{E}^{n-1}$. Then ϕ is the composition $\phi = \sigma_1 \cdots \sigma_m$ of reflections. Let $\tilde{\phi} = \tilde{\sigma}_1 \cdots \tilde{\sigma}_m$. Then $\tilde{\phi}$ extends ϕ and leaves U^n invariant. Supp. that $\tilde{\phi}_1, \tilde{\phi}_2$ are two such extensions of ϕ. Then $\tilde{\phi}_1 \tilde{\phi}_2^{-1}$ fixes each point of $\hat{E}^{n-1}$ and leaves U^n invariant.

$\Rightarrow$ By the above thm, we have that $\tilde{\phi}_1 \tilde{\phi}_2^{-1}$ is the identity and so $\tilde{\phi}_1 = \tilde{\phi}_2$. $\Rightarrow$ $\tilde{\phi}$ depends only on ϕ and not on the decomposition $\phi = \sigma_1 \cdots \sigma_m$. $\Rightarrow$ $\tilde{\phi}$ is called the Poincaré extension of ϕ.

Theorem A Möbius transf. ϕ of $\hat{E}^n$ leaves upper half-space U^n invariant iff ϕ is the Poincaré extension of a Möbius transf. of $\hat{E}^{n-1}$.

Prop. Every linear fractional transf. of $\hat{\mathbb{C}}$ is an orientation preserving Möbius transf. of $\hat{\mathbb{C}}$. //

Recall Subgroups of $PSL(2,\mathbb{C})$. $\gamma \neq Id \in PSL(2,\mathbb{C})$.

- γ is elliptic if $tr\,\gamma \in \mathbb{R}$ and $|tr\,\gamma| < 2$.
- γ is parabolic if $tr\,\gamma = \pm 2$.
- γ is loxodromic otherwise.

Def. 1.2.1 Let Γ be a subgroup of $PSL(2,\mathbb{C})$.

- The group Γ is reducible if all elements have a common fixed point in their action on $\hat{\mathbb{C}}$. Otherwise, Γ is irreducible.
- The group Γ is elementary if it has a finite orbit in its action on $\mathbb{H}^3 \cup \hat{\mathbb{C}}$. Otherwise, Γ is non-elementary.

Def 1.2.5 A Kleinian group Γ is a discrete subgroup of $PSL(2,\mathbb{C})$.

Def. 3.1.1 Let Γ be a non-elementary subgp of $PSL(2,\mathbb{C})$. Let $\hat{\Gamma} = P^{-1}(\Gamma)$, where $P: SL(2,\mathbb{C}) \longrightarrow PSL(2,\mathbb{C})$. Then the trace field of Γ, denoted $\mathbb{Q}(tr\,\Gamma)$, is the field :

$$\mathbb{Q}(tr\,\hat{\gamma} : \hat{\gamma} \in \hat{\Gamma}).$$

Note that for any $\gamma \in PSL(2,\mathbb{C})$, the traces of any lifts to $SL(2,\mathbb{C})$ will only differ by $\pm$.

$\Rightarrow$ We simply write $\mathbb{Q}(tr\,\Gamma) = \mathbb{Q}(\pm tr\,\gamma \mid \gamma \in \Gamma)$. Of course, $\mathbb{Q}(tr\,\Gamma)$ is a conjugacy invariant. //

Thm 3.1.2 Let Γ be a Kleinian group of finite covolume. Then the field $\mathbb{Q}(\mathrm{tr}\,\Gamma)$ is a finite extension of $\mathbb{Q}$. //

Lemma 3.1.3 If $X \in SL(2,\mathbb{C})$, then $X^n = p_n(\mathrm{tr}X)X - q_n(\mathrm{tr}X)I$, where p_n and q_n are monic integral polynomials of degree $n-1$ and $n-2$, respectively.

Proof) The result follows from repeated use of

$$X^2 = (\mathrm{tr}X)X - I,$$

from which we see that $p_n(x) = x p_{n-1}(x) - q_{n-1}(x)$ and $q_n(x) = p_{n-1}(x)$. ▨

Corollary 3.1.4 $\mathrm{tr}(X^n)$ is a monic integral polynomial of degree n in $\mathrm{tr}(X)$.

Lemma 3.1.5 V : an algebraic variety defined over an alg. number field k. & has dimension 0.

$\Rightarrow$ V is a single point and its coordinates are algebraic numbers.

Proof) (Recall) By a complex alg. set we mean a subset S of $\mathbb{C}^n$ which is the vanishing set of a system S_p of poly's in $\mathbb{C}[X_1, \cdots, X_n]$. By Hilbert's basis thm, the ideal gen. by S_p, $I(S_p)$, is fin. gen. & $I(S_p) \subset k[X_1, \cdots, X_n] \Rightarrow S$ is defined $/k$. // $I(S_p)$

When S is irr., S is a variety V & $I(V)$ is a prime ideal.

$\Rightarrow \mathbb{C}[X]/I(V) = \mathbb{C}[V]$ is an int. domain & its field of quotients is the function field $\mathbb{C}(V)$.

Def 1.6.1 V : an alg. variety.

$\Rightarrow$ The dim. of V := the transcendental deg. of $\mathbb{C}(V)/\mathbb{C}$ //

In this case, $\dim V = 0 \Rightarrow \mathbb{C}(V) = \mathbb{C}$ since $\mathbb{C}$ is alg. closed.

Hence, $\mathbb{C}[V] = \mathbb{C}$. Let $\mathbf{x} = (x_1, \cdots, x_n) \in V$. The max. ideal defined by $m_{\mathbf{x}} = \{ f \in \mathbb{C}[V] \mid f(\mathbf{x}) = 0 \}$ must be trivial ideal $\{0\}$. Now the ftn f_i, obtained from the polynomial $X_i - x_i$, lies in $m_{\mathbf{x}}$.

$$\left(\begin{array}{ccc} \mathbb{C}[X_1, \cdots, X_n] & \longmapsto & \mathbb{C}[V] = \mathbb{C}[X_1, \cdots, X_n]/I(V) \\ X_i - x_i & \longmapsto & f_i \end{array} \right)$$

$\Rightarrow X_i - x_i \in I(V)$.

$\Rightarrow$ Thus $I(V)$ contains $X_1 - x_1, \cdots, X_n - x_n$ and so its vanishing set is a single point.

& For the alg. closed fld $\bar{k} = \bar{\mathbb{Q}}$, $\bar{k}(V) = \bar{k}$.

$\Rightarrow$ as above, $X_i - x_i \in \bar{k}[X]$ lies in $I(V)$. $\Rightarrow x_i \in \bar{k}$. ▨

Cor. 3.1.6 (of Thm 3.1.2) Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold which has finite volume. Then $\mathbb{Q}(\mathrm{tr}\,\Gamma)$ is a topological invariant of M. //

§3.2. Quaternion Algebras for Subgroups of $SL(2,\mathbb{C})$.

Γ : a non-elementary subgp of $SL(2,\mathbb{C})$.

Let $A_0\Gamma = \{ \sum a_i\gamma_i \mid a_i \in \mathbb{Q}(\mathrm{tr}\Gamma),\ \gamma_i \in \Gamma \}$, where only fin. many of the a_i are non-zero.

Theorem 3.2.1 $A_0\Gamma$ is a quaternion algebra over $\mathbb{Q}(\mathrm{tr}\Gamma)$.

Proof) We need to show that $A_0\Gamma$ is 4-dim, central simple over $\mathbb{Q}(\mathrm{tr}\Gamma)$.

(Recall) Thm 1.2.2 Every non-elementary subgp of $PSL(2,\mathbb{C})$ contains infinitely many loxodromic elements, no two of which have a common fixed point.

(Recall) Lemma 1.2.4 Let $x, y \in PSL(2,\mathbb{C})$. The group $\langle x, y\rangle$ is irreducible iff the vectors I, X, Y and XY in $M_2(\mathbb{C})$ are lin. independent. //

Since Γ is non-elementary, it contains a pair of loxodromic elts, say g and h, s.t. $\langle g, h\rangle$ is irreducible (by Thm 1.2.2), and so the vectors I, g, h and gh in $M_2(\mathbb{C})$ are lin. indep. by Lem 1.2.4.

$\Rightarrow A_0\Gamma\mathbb{C}$ is a ring of dim ≥ 4 over $\mathbb{C}$. $\Rightarrow A_0\Gamma\mathbb{C} = M_2(\mathbb{C})$.

Note also that $A_0\Gamma$ is central for if a lies in the center of $A_0\Gamma$, then it lies in the center of $M_2(\mathbb{C})$. ($\because$ a commutes with I, g, h, gh & $\{I, g, h, gh\}$ is a basis for $M_2(\mathbb{C})$.)

$\Rightarrow a$ is a multiple of the identity.

It will now be shown that $A_0\Gamma$ is 4-dim. over $\mathbb{Q}(\mathrm{tr}\Gamma)$.

Let T denote the trace form on $M_2(\mathbb{C})$ so that

$$T(a,b) = \mathrm{tr}(ab).$$

$\Rightarrow$ A dual basis of $M_2(\mathbb{C})$, $\{I^*, g^*, h^*, (gh)^*\}$, is therefore well-defined. (w.r.t. T)

$\Rightarrow$ If $\gamma \in \Gamma$, then $\gamma = x_0 I^* + x_1 g^* + x_2 h^* + x_3 (gh)^*$, $x_i \in \mathbb{C}$.

$\Rightarrow$ If $\gamma_i \in \{I, g, h, gh\}$, then

$T(\gamma, \gamma_i) = \operatorname{tr}(\gamma\gamma_i) = x_j$ for some $j \in \{0,1,2,3\}$.

Hence as $\gamma\gamma_i \in \Gamma$, $\operatorname{tr} \gamma\gamma_i \in \mathbb{Q}(\operatorname{tr}\Gamma)$, and so we deduce that $x_0, \cdots, x_3 \in \mathbb{Q}(\operatorname{tr}\Gamma)$. Thus

$$\mathbb{Q}(\operatorname{tr}\Gamma)[I, g, h, gh] \subset A_0\Gamma \subset \mathbb{Q}(\operatorname{tr}\Gamma)[I^*, g^*, h^*, (gh)^*].$$

Thus $A_0\Gamma$ is 4-dim. over $\mathbb{Q}(\operatorname{tr}\Gamma)$.

Finally, we show that $A_0\Gamma$ is simple. For if J is a nonzero two-sided ideal, then $J\mathbb{C}$ is a nonzero two-sided ideal in $M_2(\mathbb{C})$. $\Rightarrow J\mathbb{C} = M_2(\mathbb{C})$ & J has dim. 4 over $\mathbb{C}$.

$\Rightarrow$ It must have dim. ≥ 4 over $\mathbb{Q}(\operatorname{tr}\Gamma)$ so that $J = A_0\Gamma$ ▨

Note The multiplication in $A_0(\Gamma)$ is just the restriction of matrix multi. in $M_2(\mathbb{C})$.

Cor. 3.2.2 | Γ : a non-elementary subgp of $SL(2,\mathbb{C})$.

$g, h \in \Gamma$: a pair of loxodromic elts s.t. $\langle g, h\rangle$ is irr.

$\Rightarrow A_0\Gamma = \mathbb{Q}(\operatorname{tr}\Gamma)[I, g, h, gh]$.

Cor 3.2.3 | Γ : a subgp of $SL(2,\mathbb{C})$ containing two elements g, h s.t. $\langle g, h\rangle$ is irr.

$\Rightarrow A_0\Gamma$ is a quaternion alg. over $\mathbb{Q}(\operatorname{tr}\Gamma)$

& $A_0\Gamma = \mathbb{Q}(\operatorname{tr}\Gamma)[I, g, h, gh]$. //

<u>Note</u> We normalize g, h so that

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad h = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}, \quad c \neq 0.$$

If $k = \mathbb{Q}(\mathrm{tr}\,\Gamma)$, then the eigenvalue λ satisfies a quad. over k and so $K = k(\lambda)$ is an extension of degree 1 or 2 over k. Since $a+d$ and $\lambda a + \lambda^{-1} d$ $(= \mathrm{tr}(gh)) \in k$, it follows that a, d and $c = ad - 1$ all lie in $k(\lambda)$.

$\Rightarrow A_0\Gamma \subset M_2(k(\lambda))$ after conjugation.

<u>Cor 3.2.4</u> With Γ, g, h and λ as described above, Γ is conjugate to a subgp of $SL(2, k(\lambda))$.

<u>Cor 3.2.5</u> If Γ is a non-elementary subgp of $SL(2, \mathbb{C})$ s.t. $\mathbb{Q}(\mathrm{tr}\,\Gamma)$ is a subset of $\mathbb{R}$, then Γ is conjugate to a subgp of $SL(2, \mathbb{R})$.

Proof) g is loxodromic & $\mathrm{tr}\,g \in \mathbb{R}$.

$\Rightarrow \lambda + \lambda^{-1} \in \mathbb{R}$ means that $\lambda \in \mathbb{R}$. (simple calculation.) ∎

3.3. Invariant Trace Fields and Quaternion Algebras

Example 3.3.1 Γ : the subgroup of $PSL(2,\mathbb{C})$ gen. by the images of A and B, where $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $B=\begin{pmatrix}1&0\\-\omega&1\end{pmatrix}$.

Here $\omega=(-1+\sqrt{-3})/2$ so that the ring of integers O_3 in the field $\mathbb{Q}(\sqrt{-3})$ is $\mathbb{Z}[\omega]$. Clearly all the entries of the matrices in Γ lie in O_3. Thus Γ is discrete and $\mathbb{Q}(\mathrm{tr}\,\Gamma)=\mathbb{Q}(\sqrt{-3})$. If $X=\begin{pmatrix}i&0\\0&-i\end{pmatrix}$, then one easily sees that the image of X normalizes Γ and its square is the identity.

$\Rightarrow \Gamma_1=\langle\Gamma, PX\rangle$ contains Γ as a subgp of index 2.

(PX : the image of X of the map $P: SL(2,\mathbb{C})\longrightarrow PSL(2,\mathbb{C})$

& $(PX)\Gamma(PX)^{-1}=\Gamma$ & $(PX)^2=I$.)

Now Γ_1 contains the image of $XBA=\begin{pmatrix}i&i\\i\omega&-i+i\omega\end{pmatrix}$ so that i lies in the trace fld of Γ_1.

($\mathrm{tr}\,XBA=i\omega\in\mathbb{Q}(\mathrm{tr}\,\Gamma_1)$ & $\mathbb{Q}(\mathrm{tr}\,\Gamma)=\mathbb{Q}(\omega)\subseteq\mathbb{Q}(\mathrm{tr}\,\Gamma_1)$

$\Rightarrow (i\omega)\cdot(\omega^{-1})=i\in\mathbb{Q}(\mathrm{tr}\,\Gamma_1)$. //)

Let Γ be a fin. gen. non-elementary subgroup of $SL(2,\mathbb{C})$.

Def. 3.3.2 Let $\Gamma^{(2)}=\langle\gamma^2\mid\gamma\in\Gamma\rangle$.

Lemma 3.3.3 $\Gamma^{(2)}$ is a finite index normal subgroup of Γ whose quotient is an elementary abelian 2-group.

Proof) $\Gamma^{(2)}$ is obviously normal in Γ and such that all elts in the quotient $\Gamma/\Gamma^{(2)}$ have order 2. Since Γ is fin. generated, it follows that $\Gamma/\Gamma^{(2)}$ is a fin. elementary abelian 2-group. //

Thm 3.3.4 Let Γ be a fin. generated non-elementary subgroup of $SL(2,\mathbb{C})$. The field $\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)})$ is an invariant of the commensurability class of Γ.

Proof) Claim: If Γ_1 has finite index in Γ, then $\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)}) \subset \mathbb{Q}(\mathrm{tr}\,\Gamma_1)$.

With this claim, the theorem will follows. To see this, suppose Δ is commensurable with Γ.

$\rightarrow$ By Lemma 3.3.3, $\Gamma^{(2)}$ and $\Delta^{(2)}$ are commensurable.

$(\because (\Delta \cap \Gamma)^{(2)}$ has finite index in $\Delta \cap \Gamma$ & $(\Delta \cap \Gamma)^{(2)} \subseteq \Delta^{(2)} \cap \Gamma^{(2)}.)$

$\Rightarrow \Gamma^{(2)} \cap \Delta^{(2)}$ has finite index in both Γ and Δ.

$\Rightarrow$ Assuming the above claim, we have the following inclusions:

- $\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)}) \subset \mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)} \cap \Delta^{(2)})$
- $\mathbb{Q}(\mathrm{tr}\,\Delta^{(2)}) \subset \mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)} \cap \Delta^{(2)})$

By definition, $\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)} \cap \Delta^{(2)}) \subset \mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)})$.

$\Rightarrow$ the above inclusions are all equalities. In particular,

$\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)}) = \mathbb{Q}(\mathrm{tr}\,\Delta^{(2)})$, as required.

Proof of the claim

First note that we can assume, in addition, that Γ_1 is a normal subgp of finite index in Γ. :

If we set $C = \bigcap_{g \in \Gamma} g\Gamma_1 g^{-1}$ (the normal core of Γ_1 in Γ),

then C is a normal subgp of fin. index in Γ.

$(\because hCh^{-1} = \bigcap_{g\in\Gamma} hg\Gamma_1 g^{-1}h^{-1} = \bigcap_{g\in\Gamma} (hg)\Gamma_1(hg)^{-1} = C$. for all $h \in \Gamma$.

& $\boxed{\text{FACT}}$ $[\Gamma : C] \leq [\Gamma : \Gamma_1]! < \infty$.)

$\Rightarrow$ Since $\mathbb{Q}(\mathrm{tr}\,C) \subseteq \mathbb{Q}(\mathrm{tr}\,\Gamma_1)$, it suffices to show that

$\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)}) \subset \mathbb{Q}(\mathrm{tr}\,C)$.

Let $A_0\Gamma_1 = \{ \sum a_i \gamma_i \mid a_i \in \mathbb{Q}(\mathrm{tr}\,\Gamma_1),\ \gamma_i \in \Gamma_1 \}$.

$\Rightarrow$ By Thm 3.2.1, $A_0\Gamma_1$ is a quaternion algebra over $\mathbb{Q}(\mathrm{tr}\,\Gamma_1)$.

We next claim that given any $g \in \Gamma$, $g^2 \in A_0\Gamma_1$. Notice that since Γ_1 is normal, any $g \in \Gamma$ induces by conjugation an automorphism of Γ_1 and hence an autom. ϕ_g of $A_0\Gamma_1$.

$\Rightarrow$ By the Skolem Noether thm (Cor 2.9.9), ϕ_g is an inner autom.

$\Rightarrow \exists\ a \in (A_0\Gamma_1)^*$ s.t. $\phi_g(x) = axa^{-1}$ for all $x \in A_0\Gamma_1$.

$\Rightarrow$ If we consider ϕ_g in $A_0\Gamma\mathbb{C} = M_2(\mathbb{C})$, then $\phi_g = gxg^{-1}$ for $\forall x \in A_0\Gamma\mathbb{C}$. $\Rightarrow gxg^{-1} = axa^{-1}$ for $\forall x \in A_0\Gamma\mathbb{C}$

$\Rightarrow g^{-1}ax = x \cdot g^{-1}a$ for $\forall x \in A_0\Gamma\mathbb{C} = M_2(\mathbb{C})$.

$\Rightarrow g^{-1}a$ is in the center of $M_2(\mathbb{C}) \Rightarrow g^{-1}a = yI$ for some $y \in \mathbb{C}$.

$\Rightarrow y^2 = \det(g^{-1}a) = \det(g^{-1})\cdot\det(a) = \det(a)$ since $g \in \Gamma < SL(2,\mathbb{C})$.

Now $(\det a)I = a^2 - \mathrm{tr}(a)\cdot a \in A_0(\Gamma_1)$ so that $y^2 \in \mathbb{Q}(\mathrm{tr}\,\Gamma_1)$.

Hence, $g^2 = y^{-2}a^2 \in A_0\Gamma_1$, as claimed. Since g was chosen arbitrarily from Γ, $\Gamma^{(2)} \subset A_0\Gamma_1$ and hence $\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)}) \subset \mathbb{Q}(\mathrm{tr}\,\Gamma_1)$ ▨

Cor 3.3.5 If Γ is a fin. gen. non-elementary subgp. of $SL(2,\mathbb{C})$, then the quat. alg. $A_0\Gamma^{(2)}$ is an invariant of the commensurability class of Γ.

Proof) If Γ and Δ are commensurable, then $\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)}) = \mathbb{Q}(\mathrm{tr}\,\Delta^{(2)})$. Now choose an irr. pair of loxodromic elements in $\Gamma^{(2)} \cap \Delta^{(2)}$. Then by Cor. 3.2.2, $A_0\Gamma^{(2)} = A_0\Delta^{(2)}$. ▨

Remark "Wide commensurability": Γ and Δ are in the same wide commensurability class if $\exists\ t \in SL(2,\mathbb{C})$ s.t. $t\Gamma t^{-1}$ and Δ are commensurable. $\Rightarrow \mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)})$ is invariant under the wide comm'ble class.

Def 3.3.6 $\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)})$ is denoted by $k\Gamma$ and referred to as the invariant field of Γ. & the quat. alg. $A_0\Gamma^{(2)}$ over $\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)})$ will be denoted by $A\Gamma$ and referred to as the invariant quaternion algebra of Γ.

Thm 3.3.7 If Γ is a Kleinian gp of finite covolume, then its invariant trace field is a finite non-real extension of $\mathbb{Q}$.

Proof) $k\Gamma$ is a fin. ext. of $\mathbb{Q}$ by Thm 3.1.2. Supp. that $k\Gamma$ is real

$\Rightarrow$ By Cor. 3.2.5, $\Gamma^{(2)}$ is conjugate to a subgp of $SL(2,\mathbb{R})$. Then, $\Gamma^{(2)}$ cannot have finite covolume. ▨

Thm 3.3.8 Γ : a non-elementary gp which contains parabolic elements. Then $A_0\Gamma = M_2(\mathbb{Q}(\mathrm{tr}\,\Gamma))$. In particular, if Γ is a Kleinian gp s.t. $\mathbb{H}^3/\Gamma$ has finite volume but is non-compact, then $A\Gamma = M_2(k\Gamma)$.

Proof) $\gamma \in \Gamma$: a parabolic elt. $\Rightarrow \gamma$ has a fixed points.

$\Rightarrow \gamma - I$ is noninvertible in $A_0\Gamma$

$\Rightarrow A_0\Gamma$ cannot be a division algebra over $\mathbb{Q}(\mathrm{tr}\,\Gamma)$.

$\Rightarrow A_0\Gamma = M_2(\mathbb{Q}(\mathrm{tr}\,\Gamma))$ by Thm 2.1.7.

$\mathbb{H}^3/\Gamma$ has finite volume but is non-compact.

$\Rightarrow \Gamma$ is of finite covolume but is not cocompact.

$\Rightarrow$ By Thm 1.2.12, Γ has a parabolic element. ▨

3.4. Trace Relations.

(3.10) $\operatorname{tr} XY = \operatorname{tr} ZXYZ^{-1}$ for $X, Y \in M_2(\mathbb{C})$, $Z \in GL(2,\mathbb{C})$.

(3.11) $\operatorname{tr} XY = \operatorname{tr} YX$, $\operatorname{tr} X_1 X_2 \cdots X_n = \operatorname{tr} X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(n)}$

for any cyclic permutation σ of $1, 2, \cdots, n$.

Recall that for $X \in SL(2,\mathbb{C})$, $X^2 = (\operatorname{tr} X)X - I$, (3.12)

from which we deduce $\operatorname{tr} X^2 = \operatorname{tr}^2 X - 2$ (3.13).

The other basic identities for elements $X, Y \in SL(2,\mathbb{C})$ are

$$\operatorname{tr} XY = (\operatorname{tr} X)(\operatorname{tr} Y) - \operatorname{tr} XY^{-1}, \quad \operatorname{tr} X = \operatorname{tr} X^{-1}. \quad (3.14).$$

By repeated application of these relations, we can obtain the followings.

$$\operatorname{tr}[X,Y] = \operatorname{tr}(XYX^{-1}Y^{-1})$$
$$= \operatorname{tr}^2 X + \operatorname{tr}^2 Y + \operatorname{tr}^2 XY - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} XY - 2 \quad (3.15)$$

$$\Big(\operatorname{tr}(XYX^{-1}Y^{-1}) = \operatorname{tr}(XY)\operatorname{tr}(X^{-1}Y^{-1}) - \operatorname{tr}(XY \cdot YX) \text{ by } (3.14)$$
$$= \operatorname{tr}^2(XY) - \operatorname{tr}(X^2Y \cdot Y)$$
$$= \operatorname{tr}^2(XY) - \left(\operatorname{tr}(X^2Y) \cdot \operatorname{tr} Y - \operatorname{tr} X^2\right)$$
$$= \operatorname{tr}^2(XY) - \left(\operatorname{tr}(XYX) \cdot \operatorname{tr} Y - \operatorname{tr} X^2\right)$$
$$= \operatorname{tr}^2(XY) - \left(\left(\operatorname{tr}(XY)\operatorname{tr}(X) - \operatorname{tr} Y\right)\operatorname{tr} Y - \operatorname{tr} X^2\right)$$
$$= \operatorname{tr}^2(XY) - \left(\operatorname{tr} X \operatorname{tr} Y \operatorname{tr} XY - \operatorname{tr}^2 Y - (\operatorname{tr}^2 X - 2)\right)$$
$$= \operatorname{tr}^2 X + \operatorname{tr}^2 Y + \operatorname{tr}^2 XY - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} XY - 2 \;/\!/ \Big)$$

$$\operatorname{tr} XYXZ = \operatorname{tr} XY \operatorname{tr} XZ - \operatorname{tr} YZ^{-1} \quad (3.16)$$

$$\operatorname{tr} XYX^{-1}Z = \operatorname{tr} XY \operatorname{tr} X^{-1}Z - \operatorname{tr} X^2YZ^{-1} \quad (3.17)$$

$$\operatorname{tr} X^2YZ = \operatorname{tr} X \operatorname{tr} XYZ - \operatorname{tr} YZ. \quad (3.18).$$

$$\operatorname{tr} XYZ + \operatorname{tr} YXZ + \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} Z = \operatorname{tr} X \operatorname{tr} YZ + \operatorname{tr} Y \operatorname{tr} XZ + \operatorname{tr} Z \operatorname{tr} XY \quad (3.19).$$

$$2\operatorname{tr} XYZW = \operatorname{tr} X \operatorname{tr} YZW + \operatorname{tr} Y \operatorname{tr} ZWX + \operatorname{tr} Z \operatorname{tr} WXY + \operatorname{tr} W \operatorname{tr} XYZ$$
$$+ \operatorname{tr} XY \operatorname{tr} ZW - \operatorname{tr} XZ \operatorname{tr} YW + \operatorname{tr} XW \operatorname{tr} YZ - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} ZW$$

In the quaternion algebra $A = M_2(\mathbb{C})$, the pure quaternions A_0 are the matrices of trace 0 and the norm form induces a bilinear from B on A_0 given by

$$B(X,Y) = \frac{-1}{2}(XY+YX) = -\frac{1}{2}\operatorname{tr}(XY) \qquad (3.21)$$

$(XY+YX)$ ↓ elements in the quat. alg. $\operatorname{tr}(XY)$ ↓ elements in $M_2(\mathbb{C})$.

⇒ For $X, Y, Z \in A_0$, $\operatorname{tr} XYZ = \operatorname{tr}\left([(\operatorname{tr}XY)I - YX]Z\right)$

$$= \operatorname{tr}(XY)\cdot\operatorname{tr}(Z) - \operatorname{tr}(YXZ)$$
$$= \operatorname{tr}(XY)\cdot 0 - \operatorname{tr}(YXZ)$$
$$= -\operatorname{tr}(YXZ).$$

((3.21) means that $XY+YX = (\operatorname{tr}XY)I$ since the constant field $\mathbb{C}$ in $A \hookrightarrow \mathbb{C}\cdot I$ in $M_2(\mathbb{C})$.

⇒ $XY = (\operatorname{tr}XY)I - YX$.)

⇒ If we define F on A_0^3 by $F(X,Y,Z) = \operatorname{tr}XYZ$, then F is an alternating trilinear form.

⇒ If X', Y' and Z' also lie in A_0, then

$$\operatorname{tr}XYZ\,\operatorname{tr}X'Y'Z' = c\cdot\det\begin{pmatrix} B(X,X') & B(X,Y') & B(X,Z') \\ B(Y,X') & B(Y,Y') & B(Y,Z') \\ B(Z,X') & B(Z,Y') & B(Z,Z') \end{pmatrix}$$

for some const. c.

⇒ Using (3.21), and choosing suitable matrices, [e.g., $X = X' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = Y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = Z' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$], we obtain $c = 4$ and

$$\operatorname{tr}XYZ\cdot\operatorname{tr}X'Y'Z' = -\frac{1}{2}\det\begin{pmatrix} \operatorname{tr}XX' & \operatorname{tr}XY' & \operatorname{tr}XZ' \\ \operatorname{tr}YX' & \operatorname{tr}YY' & \operatorname{tr}YZ' \\ \operatorname{tr}ZX' & \operatorname{tr}ZY' & \operatorname{tr}ZZ' \end{pmatrix}. \qquad (3.22)$$

Now if we take X, Y, Z, X', Y' and Z' in $M_2(\mathbb{C})$, then their projections in A_0 are of the form $X_1 = X - 1/2\,(tr(X))\,I$ & satisfy (3.22). (I is orthogonal to $A_0 = \text{span}\left\{\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$ w.r.t. B & the projection of X to the hyperplane A_0 orthogonal to I w.r.t. B is,

$$\text{proj}_I(X) = X_1 = X - \frac{B(X,I)}{B(I,I)} I$$
$$= X - \frac{-\frac{1}{2} tr X}{-\frac{1}{2}\cdot 2} \cdot I$$
$$= X - \frac{1}{2} tr(X)\cdot I \quad)$$

$\Rightarrow$ For any X, Y, Z, X', Y', and Z' in $M_2(\mathbb{C})$,

$$tr XYZ\, tr X'Y'Z' + P'\, tr XYZ + P\, tr X'Y'Z' + Q = 0 \qquad (3.23)$$

where P', P and Q are rational poly's in the traces of these six matrices and their products taken in pairs.

$\Rightarrow$ Now choose $X = X'$, $Y = Y'$, $Z = Z'$, where $X, Y, Z \in SL(2, \mathbb{C})$.

$\Rightarrow$ A tedious calculation shows that $tr XYZ$ satisfies the following quad. polynomial :

$$\begin{aligned} x^2 &- (tr XY\, tr Z + tr YZ\, tr X + tr ZX\, tr Y - tr X\, tr Y\, tr Z)\, x \\ &+ tr XY\, tr YZ\, tr ZX + (tr^2 XY + tr^2 YZ + tr^2 ZX) \\ &- (tr X\, tr Y\, tr XY + tr Y\, tr Z\, tr YZ + tr Z\, tr X\, tr ZX) \\ &+ (tr^2 X + tr^2 Y + tr^2 Z) - 4 \end{aligned} \qquad (3.24)$$

Note the coeffs are integral poly. in the traces of the three matrices involved and their products taken in pairs. //

3.5. Generators for Trace Fields.

Let Γ be generated by $\gamma_1, \cdots, \gamma_n$. Let P denote the collection

$$\{ \gamma_{j_1} \cdots \gamma_{j_t} \mid t \geq 1 \text{ and all } j_i \text{ are distinct} \}$$

Let Q denote the collection

$$\{ \gamma_{i_1} \cdots \gamma_{i_r} \mid r \geq 1 \text{ and } 1 \leq i_1 < \cdots < i_r \leq n \}.$$

Let R denote the collection

$$\{ \gamma_i, \gamma_{j_1}\gamma_{j_2}, \gamma_{k_1}\gamma_{k_2}\gamma_{k_3} \mid 1 \leq i \leq n, 1 \leq j_1 < j_2 \leq n, 1 \leq k_1 < k_2 < k_3 \leq n \}.$$

We show successively that $\mathbb{Q}(\mathrm{tr}\,\Gamma)$ is gen. over $\mathbb{Q}$ by the traces of the elements in P, then in Q, and finally in R.

For each $\gamma \in \Gamma$, define the length of γ [w.r.t. $\gamma_1, \cdots, \gamma_n$]

$$\ell(\gamma) := \min \left\{ \sum_{i=1}^{s} |\alpha_i| \;\middle|\; \gamma = \gamma_{k_1}^{\alpha_1} \cdots \gamma_{k_s}^{\alpha_s} \right\}$$

where the min. is taken over all repn's of γ in terms of $\gamma_1, \cdots, \gamma_n$.

<u>Lemma 3.5.1</u> Let $\gamma \in \Gamma$. Then $\mathrm{tr}\,\gamma$ is an integral poly in $\{ \mathrm{tr}\,\delta \mid \delta \in P \}$

Proof) Use the induction on the length of γ. From (3.13), (3.14), the result is clearly true if $\ell(\gamma) = 1$ or 2. So supp. $\ell(\gamma) \geq 3$ and the result holds for all elts of length $< \ell(\gamma)$.

If $\gamma \notin P$, then either $k_i = k_j$ for $i \neq j$ or some $\alpha_i \neq 1$.

If $k_i = k_j$, then γ, after conjugation (does not change the trace), has the form $XYXZ$ or $XYX^{-1}Z$.

$\Rightarrow$ the result follows from (3.16) ~ (3.18).

In the same way, if some $|\alpha_i| \geq 2$, then (3.18). & $\alpha_i = -1$

$\Rightarrow$ γ has the form $X\gamma_{k_i}^{-1}Y$, then

$$\mathrm{tr}\, X\gamma_{k_i}^{-1}Y = \mathrm{tr}\, YX\gamma_{k_i}^{-1} = \mathrm{tr}\, YX\, \mathrm{tr}\,\gamma_{k_i} - \mathrm{tr}\, YX\gamma_{k_i}.$$

$\Rightarrow$ By repeated application, the result follows. $\blacksquare$

Lemma 3.5.2 Let $\gamma \in \Gamma$. Then $tr\,\gamma$ is an int. poly. in $\{tr\,\delta \mid \delta \in Q\}$.

Proof) For each permutation τ of S_n, define

$$\tau^*(Q) = \{\gamma_{\tau(i_1)} \cdots \gamma_{\tau(i_r)} \mid 1 \le i_1 < \cdots < i_r \le n\}$$

$$\Rightarrow P = \bigcup_{\tau \in S_n} \tau^*(Q).$$

$\Rightarrow$ Each τ is a product of transpositions $(i\ i+1)$ and we define the length of τ to be the min. number of such transpositions required.

$\Rightarrow$ We need to show that if $\gamma \in \tau^*(Q)$, then $tr\,\gamma$ is an int. poly. in $\{tr\,\delta \mid \delta \in Q\}$. $\Rightarrow$ Induction on the length of τ.

If the length is 0, then the result is trivial.

$\Rightarrow$ let $\tau = \tau'\sigma$, $\sigma = (i, i+1)$ for some i.

$\Rightarrow$ the length of τ' < the length of τ.

$\Rightarrow$ Using (3.19) & (3.13), we obtain the result. ▨

Note From Lemma 3.5.2, we can see :

$\Gamma = \langle g, h \rangle \Rightarrow \mathbb{Q}(tr\,\Gamma) = \mathbb{Q}(tr\,g, tr\,h, tr\,gh)$

$\Gamma = \langle f, g, h \rangle \Rightarrow \mathbb{Q}(tr\,\Gamma) = \mathbb{Q}(tr\,f, tr\,g, tr\,h, tr\,fg, tr\,fh, tr\,gh, tr\,fgh)$

Lemma 3.5.3 $\gamma \in \Gamma$. $\Rightarrow tr\,\gamma$ is a rational poly in $\{tr\,\delta \mid \delta \in R\}$.

proof) Lemm 3.5.2 & (3.20). //

Def. 3.5.4 Γ : a non-elementary subgp of $SL(2, \mathbb{C})$, with generators $\gamma_1, \cdots, \gamma_n$. Define Γ^{SQ}, w.r.t. $\gamma_1, \cdots, \gamma_n$ by

$$\Gamma^{SQ} = \langle \gamma_1^2, \gamma_2^2, \cdots, \gamma_n^2 \rangle.$$

Lemma 3.5.5 With Γ as above and $tr\,\gamma_i \ne 0$ for $i = 1, 2, \cdots, n$, then

$k\Gamma = \mathbb{Q}(tr\,\Gamma^{SQ})$

Proof) Clearly $\Gamma^{SQ} \subset \Gamma^{(2)}$ so that $\mathbb{Q}(\mathrm{tr}\,\Gamma^{SQ}) \subset k\Gamma$.

$\Rightarrow$ Now from (3.12): $X^2 = (\mathrm{tr}X)X - I$, if $\mathrm{tr}\,\gamma \neq 0$, then $\gamma = (\mathrm{tr}\,\gamma)^{-1}(\gamma^2 + I)$ in $M_2(\mathbb{C})$.

$\Rightarrow$ Let $\gamma \in \Gamma^{(2)}$ so that $\gamma = \delta_1^2 \delta_2^2 \cdots \delta_r^2$ with $\delta_i \in \Gamma$.

Now $\delta_i = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{r_i}}$. Thus,

$$\delta_i = \prod_{j=1}^{r_i} (\mathrm{tr}\,\gamma_{i_j})^{-1} \prod_{j=1}^{r_i} (\gamma_{i_j}^2 + I)$$

$$\Rightarrow \delta_i^2 = \prod_{j=1}^{r_i} (\mathrm{tr}^2\gamma_{i_j})^{-1} \left(\prod_{j=1}^{r_i} (\gamma_{i_j}^2 + I) \right)^2$$

$$= \prod_{j=1}^{r_i} (\mathrm{tr}\,\gamma_{i_j}^2 + 2) \left(\prod_{j=1}^{r_i} (\gamma_{i_j}^2 + I) \right)^2$$

$\Rightarrow \mathrm{tr}\,\gamma_{i_j}^2 + 2 \in \mathbb{Q}(\mathrm{tr}\,\Gamma^{SQ})$ & $\gamma_{i_j}^2 + I \in A_0\Gamma^{SQ}$

$\Rightarrow {}^{\forall}\delta_i^2 \in A_0\Gamma^{SQ} \Rightarrow \gamma \in A_0\Gamma^{SQ} \Rightarrow \mathrm{tr}\,\gamma \in \mathbb{Q}(\mathrm{tr}\,\Gamma^{SQ})$ ▨

Lemma 3.5.6 Γ: a fin. gen. non-elementary subgp of $SL(2,\mathbb{C})$.

Let $k = \mathbb{Q}(\{\mathrm{tr}\,\gamma^2 \mid \gamma \in \Gamma\})$. Then $k = k\Gamma$.

Proof) Clearly, $\gamma^2 \in \Gamma^{(2)}$ for $\forall \gamma \in \Gamma$. $\Rightarrow k \subset k\Gamma$. Now choose a set of generators $\gamma_1, \cdots, \gamma_n$ of Γ s.t. $\mathrm{tr}\,\gamma_i \neq 0$, $\mathrm{tr}\,\gamma_i^2\gamma_j^2 \neq 0$ $\forall i,j$.

$\Rightarrow$ By Lemmas 3.5.3, 3.5.5, it suffices to show that

$\mathrm{tr}\,\gamma_i^2\gamma_j^2$, $\mathrm{tr}\,\gamma_i^2\gamma_j^2\gamma_k^2 \in k$ for $\forall i, j, k$.

$\Rightarrow$ We use the identity $\mathrm{tr}\,\gamma_i^2\gamma_j = \mathrm{tr}\,\gamma_i\,\mathrm{tr}\,\gamma_i\gamma_j - \mathrm{tr}\,\gamma_j$.

$\Rightarrow$ Squaring both sides:

$$(\mathrm{tr}\,\gamma_i^2\gamma_j)^2 = (\mathrm{tr}\,\gamma_i\,\mathrm{tr}\,\gamma_i\gamma_j - \mathrm{tr}\,\gamma_j)^2$$

$$= \mathrm{tr}^2\gamma_i\,\mathrm{tr}^2\gamma_i\gamma_j + \mathrm{tr}^2\gamma_j - 2\,\mathrm{tr}\,\gamma_i\,\mathrm{tr}\,\gamma_j\,\mathrm{tr}\,\gamma_i\gamma_j$$

$$\Rightarrow \mathrm{tr}\,\gamma_i\,\mathrm{tr}\,\gamma_j\,\mathrm{tr}\,\gamma_i\gamma_j = \frac{1}{2}\left(\mathrm{tr}^2\gamma_i\,\mathrm{tr}^2\gamma_i\gamma_j + \mathrm{tr}^2\gamma_j - \mathrm{tr}^2\gamma_i^2\gamma_j\right)$$

$\Rightarrow \operatorname{tr}\gamma_i \operatorname{tr}\gamma_j \operatorname{tr}\gamma_i\gamma_j = \operatorname{tr}\gamma_i\gamma_j \cdot \operatorname{tr}\gamma_j^{-1} \cdot \operatorname{tr}\gamma_i$

$= (\operatorname{tr}\gamma_i + \operatorname{tr}\gamma_i\gamma_j^2)\operatorname{tr}\gamma_i$ using (3.14)

$= \operatorname{tr}^2\gamma_i + \operatorname{tr}\gamma_j^2\gamma_i \cdot \operatorname{tr}\gamma_i$

$= \operatorname{tr}^2\gamma_i + \operatorname{tr}\gamma_j^2\gamma_i^2 + \operatorname{tr}\gamma_j^2$ //

$= \operatorname{tr}^2\gamma_i + \operatorname{tr}\gamma_j^2 + \operatorname{tr}\gamma_i^2\gamma_j^2 \in k$.

Since $\operatorname{tr}^2\gamma_i, \operatorname{tr}\gamma_j^2 \in k$, we get $\operatorname{tr}\gamma_i^2\gamma_j^2 \in k$.

& Using (3.14) again, $\operatorname{tr}\gamma_i^2\gamma_j^2\gamma_k^{-1} = \operatorname{tr}\gamma_i^2\gamma_j^2 \operatorname{tr}\gamma_k - \operatorname{tr}\gamma_i^2\gamma_j^2\gamma_k$.

$\Rightarrow$ Squaring both sides, then gives that $\operatorname{tr}\gamma_k \cdot \operatorname{tr}\gamma_i^2\gamma_j^2\gamma_k \in k$.

since $\operatorname{tr}\gamma_i^2\gamma_j^2 \neq 0$.

$\Rightarrow$ The result follows since $\operatorname{tr}\gamma_i^2\gamma_j^2\gamma_k^2 = \operatorname{tr}\gamma_k \operatorname{tr}\gamma_i^2\gamma_j^2\gamma_k - \operatorname{tr}\gamma_i^2\gamma_j^2$. ▨

Note Supp. $\Gamma = \langle g, h\rangle$ is a non-elementary gp. &

order of $g, h \neq 2$.

$\Rightarrow$ By Lemma 3.5.5 and (3.25), $k\Gamma = \mathbb{Q}(\operatorname{tr}g^2, \operatorname{tr}h^2, \operatorname{tr}g^2h^2)$.

Now $\operatorname{tr}g^2h^2 = \operatorname{tr}g \operatorname{tr}h \operatorname{tr}gh - \operatorname{tr}^2 g - \operatorname{tr}^2 h + 2$. by (3.13)

Lemma 3.5.7 Let $\Gamma = \langle g, h\rangle$, with $\operatorname{tr}g, \operatorname{tr}h \neq 0$, be a non-elementary subgroup of $SL(2,\mathbb{C})$. Then

$$k\Gamma = \mathbb{Q}(\operatorname{tr}g^2, \operatorname{tr}^2 h, \operatorname{tr}g \operatorname{tr}h \operatorname{tr}gh). //$$

Now supp. that $\operatorname{tr}h = 0$ so that h has order 2 (in $PSL(2,\mathbb{C})$).

$\Rightarrow \Gamma_1 = \langle g, hg^{-1}h^{-1}\rangle$ is a subgp of index 2 in Γ and so

$k\Gamma_1 = k\Gamma$ by Thm 3.3.4.

$\Rightarrow$ From (3.15), we get the following:

<u>Lemma 3.5.8</u> $\Gamma = \langle g, h \rangle$ with $\mathrm{tr}\, h = 0$, be a non-elementary subgp of $SL(2,\mathbb{C})$. Then $k\Gamma = \mathbb{Q}(\mathrm{tr}^2 g, \mathrm{tr}[g,h])$.

<u>Note</u> The conjugacy class of an irr. Kleinian gp $\Gamma = \langle g, h \rangle$ is deter. by the three complex parameters

$$\beta(g) = \mathrm{tr}^2 g - 4,\ \beta(h) = \mathrm{tr}^2 h - 4,\ \gamma(g,h) = \mathrm{tr}[g,h] - 2.$$

$\Rightarrow$ In the case $\mathrm{tr}\, h = 0$, we have $k\Gamma = \mathbb{Q}(\gamma(g,h), \beta(g))$.

When $\mathrm{tr}\, g \neq 0$, $\mathrm{tr}\, h \neq 0$, $\mathrm{tr}g\,\mathrm{tr}h\,\mathrm{tr}gh$ satisfies the monic quad poly.

$$x^2 - (\beta(g)+4)(\beta(h)+4)x - (\beta(g)+4)(\beta(h)+4)(\gamma(g,h) - \beta(g) - \beta(h) - 4) = 0.$$

$\Rightarrow$ From Lem. 3.5.7, $[k\Gamma : \mathbb{Q}(\gamma(g,h), \beta(g), \beta(h))] \leq 2$.

<u>Lemma 3.5.9</u> $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ with $\mathrm{tr}\,\gamma_i \neq 0$ $\forall i$.

$\Rightarrow$ $k\Gamma$ is gen. over $\mathbb{Q}$ by $\{\mathrm{tr}^2\gamma_i, 1 \leq i \leq 3;\ \mathrm{tr}\gamma_i\gamma_j\,\mathrm{tr}\gamma_i\,\mathrm{tr}\gamma_j, 1 \leq i < j \leq 3;\ \mathrm{tr}\gamma_1\,\mathrm{tr}\gamma_2\,\mathrm{tr}\gamma_3\,\mathrm{tr}\gamma_1\gamma_2\gamma_3\}$. //

3.6. Generators for Invariant Quaternion Algebras.

Recall $\langle g, h\rangle$ is irr. subgp of $\Gamma^{(2)} \Rightarrow A\Gamma = k\Gamma[I, g, h, gh]$.

$\Rightarrow$ We require a standard basis of $A\Gamma$.

Now $A\Gamma\mathbb{C} = M_2(\mathbb{C})$. $\Rightarrow sl(2,\mathbb{C}) :=$ the pure quat. space of $A\Gamma\mathbb{C}$.

$\Rightarrow$ Let the assoc. symm. bilin. form be B so that for $C, D \in sl(2,\mathbb{C})$,

$$B(C,D) = -\frac{1}{2}\operatorname{tr}(CD).$$

$\Rightarrow C, D$ are orthogonal iff $CD = -DC$.

$\Rightarrow \{i, j, ij\}$ forms an ortho. basis $sl(2,\mathbb{C})$ w.r.t. B.

Thus given g, h as above, let $t_0 = \operatorname{tr} g$, $t_1 = \operatorname{tr} h$, $t_2 = \operatorname{tr} gh$.

$\Rightarrow$ Set $g' = g - (t_0/2)I$, $h' = h - (t_1/2)I$ ($\Rightarrow$ projections onto $sl(2,\mathbb{C})$)

so that $g', h' \in sl(2,\mathbb{C})$.

$\Rightarrow g'^2 = -n(g')$ & $B(g', \frac{t_0}{2}I) = n(g' + \frac{t_0}{2}I) - n(g') - n(\frac{t_0}{2}I)$

($\because g'$ is pure)

$$= n(g) - n(g') - \frac{t_0^2}{4}$$

$$= -\frac{1}{2}\operatorname{tr} \cdot g' \cdot \frac{t_0}{2}I = 0$$

$$(\because \operatorname{tr} g' = 0).$$

$$\therefore g'^2 = \frac{t_0^2}{4} - n(g) = \frac{t_0^2}{4} - 1 = \frac{t_0^2 - 4}{4}.$$

Similarly, $h'^2 = \dfrac{t_1^2 - 4}{4}$. Thus provided g and h are not parabolic,

$g'^2, h'^2 \in k\Gamma^*$. Assuming that g is not parabolic, set

$$h'' = h' - \frac{B(g', h')}{B(g', g')} g' \quad \text{so that } h'' \in sl(2,\mathbb{C}) \text{ and}$$

is ortho. to g'. Now, $h''^2 = -\dfrac{\operatorname{tr}[g,h] - 2}{t_0^2 - 4}$ (a routine calculation).

Note that since $\langle g, h\rangle$ is irr., g, h have no common fixed pt.

$\Rightarrow ghg^{-1}h^{-1}$ is not parabolic. $\Rightarrow \operatorname{tr}[g,h] - 2 \neq 0$.

$\Rightarrow$ With the Hilbert symbol, we have that

$$A\Gamma = \left(\frac{tr^2 g - 4, -(tr^2 g - 4)(tr[g,h] - 2)}{k\Gamma}\right)$$

$$= \left(\frac{tr^2 g - 4, tr[g,h] - 2}{k\Gamma}\right) \quad \S 2.1 \ \left(\frac{a,b}{F}\right) \cong \left(\frac{a,-ab}{F}\right)$$

<u>Thm 3.6.1</u> $g, h \in \Gamma^{(2)}$, non-elementary gp. & $\langle g,h \rangle$ is irreducible & g is not parabolic.

$$\Rightarrow A\Gamma = \left(\frac{tr^2 g - 4, tr[g,h] - 2}{k\Gamma}\right).$$

<u>Thm 3.6.2</u> g, h do not have order 2 in $PSL(2,\mathbb{C})$ & g is not parabolic

$$\Rightarrow A\Gamma = \left(\frac{tr^2 g(tr^2 g - 4), tr^2 g\, tr^2 h(tr[g,h] - 2)}{k\Gamma}\right).$$

<u>Cor 3.6.3</u> $g, h \in \Gamma$, $\langle g,h \rangle$ is a non-elementary subgp of Γ & $g \neq$ parabolic & $\neq$ order 2. & h has order 2.

$$\Rightarrow A\Gamma = \left(\frac{tr^2 g(tr^2 g - 4), (tr[g,h] - 2)(tr[g,h] - tr^2 g + 2)}{k\Gamma}\right)$$

<u>Note</u> If $\Gamma = \langle g,h \rangle$, then $k\Gamma = \mathbb{Q}(\gamma(g,h), \beta(g))$. Then,

<u>Cor 3.6.4</u> $\Gamma = \langle g,h \rangle$: a non-elementary, order of $h = 2$ & g is not parabolic.

$$\Rightarrow A\Gamma \cong \left(\frac{(\beta(g)+4)\beta(g), \gamma(g,h)(\gamma(g,h) - \beta(g))}{\mathbb{Q}(\beta(g), \gamma(g,h))}\right). \quad //$$