1 Introduction

About this lecture:

- Rigidity
- Volumes of ideal tetrahedra
- Dehn surgery and volumes
- The volume spectrum of hyperbolic 3-manifolds

Some helpful references

- Ratcliffe, Foundations of hyperbolic manifolds, Springer (elementary)
- W. Thurston, Lecture notes, (This is hard to read and incomplete) `http://www.msri.org/communications/books/gt3m`
- J. Weeks, A computation of hyperbolic structures in knot theory, in Handbook of knot theory
- W. Neumann, Zagier, Volumes of hyperbolic three-manifolds, Topology Volume 24, Issue 3, 1985, Pages 307-332
- M. Kapovich, Hyperbolic manifolds and discrete groups. (English summary) Progress in Mathematics, 183. Birkh?user Boston, Inc., Boston, MA, 2001.

Some helpful references

- Milnor, John, Hyperbolic geometry: the first 150 years. Bull. Amer. Math. Soc. (N.S.) 6 (1982), no. 1, 9–24.
- Benedetti, Riccardo; Petronio, Carlo, Lectures on hyperbolic geometry. Universitext. Springer-Verlag, Berlin, 1992. (This contains a lot on Dehn surgery of hyperbolic 3-manifolds)
- Gromov, M, Hyperbolic manifolds (according to Thurston and Jorgensen). Bourbaki Seminar, Vol. 1979/80, pp. 40–53,

Some computer programs

- `http://www.math.uiuc.edu/~nmd/computop/index.html` These include many computational tools for finding hyperbolic manifolds. Windows XP, mac, linux, unix(SnapPy, originally Snappea by J. Weeks)
- `http://www.geometrygames.org/SnapPea/`
- `http://www.ms.unimelb.edu.au/~snap/orb.html` Snap, Orb (exact alg. computations, computations for orbifolds)
- `http://www.math.sci.osaka-u.ac.jp/~wada/OPTi/index.html` M. Wada (drawing isometric spheres for figure eight knot complements)

2 Rigidity (Mostow-Prasad)

2-manifolds

- A *complex algebraic set*, we means a subset S in $\mathbb{C}^n$ that is a vanishing set of a sysetem of n-variable complex polynomials in $\mathbb{C}[X_1, ..., X_n]$.
- $I(S)$ the ideal generated by the polynomials in $\mathbb{C}[X_1, .., X_n]$.
- If $I(S) \subset k[X_1, .., X_n]$ for a subfield k, then we say that S is defined over k.
- S is *irreducible* if S is not a union of two nontrivial algebraic subsets.
- If S is irreducible, then S is a variety V and $I(V)$ is a prime ideal.
- $\mathbb{C}[X]/I(V) = \mathbb{C}[V]$ is an integral domain and the fields of quotients $\mathbb{C}(V)$ is the *function field*.
- The dimension of V is the transcendence degree of $\mathbb{C}(V)$.

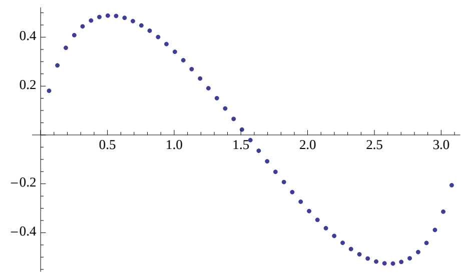
Representation space

- Γ a group with generators $\gamma_1, .., \gamma_n$ and relations $R_1(\gamma_1, ..., \gamma_n) = \cdots = R_m(\gamma_1, .., \gamma_n) = I$, words...
- $Hom(\Gamma, SL(2, \mathbb{C})) = \{\rho : \rho : \Gamma \to SL(2, \mathbb{C})\}$.
- $A_j = \rho(\gamma_j)$ is a 2×2-matrix

  $$\begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix}.$$

  $R_k(A_1, .., A_n)$ corresponds to $4n$-variable polynomial with integer coefficients.
- $R_1(A_1, .., A_n) = R_2(A_1, .., A_n) = \cdots = R_m(A_1, ..., A_n) = I$ is a defines a subset in $SL(2, \mathbb{C})^n \subset \mathbb{C}^{4n}$ defined by $4m$-equations and $x_i w_i - y_i z_i = 1$ for $i = 1, ..., n$.

- Γ finite covolume, torsion free Kleinian group
- $< A_1, A_2 >$ irreducible. Conjugate uniquely so that A_1 fix $0, \infty$ and A_2 fix 1. Equivalently $y_1 = z_1 = 0$ and $x_2 + y_2 = z_2 + w_2$. (To fix conjugations)
- If Γ contains a parabolic element, then we add $tr^2 U_i = 4$ and $U_i V_i = V_i U_i$ for U_i, V_i parabolics in Γ.
- Use these equations and the above to define $V(\Gamma) \subset \mathbb{C}^{4n}$.
- $V(\Gamma)$ an irreducible component of the algebraic set, and it classifies representations $\Gamma \to SL(2, \mathbb{C})$ up to conjugacies.



Weil-Garland, Mostow-Prasad rigidity

Theorem 1. *(Weil-Garland) (infinitesimal rigidity) An inclusion* $\iota : \Gamma \to SL(2, \mathbb{C})$ *for Kleinian group* Γ*. Then for* $\rho \in V(\Gamma)$ *sufficiently close to* ι*,* $\rho(\Gamma)$ *is finite covolume Kleinian group. Or more precisely,* $H^1(\Gamma, sl(2, \mathbb{C})) = 0$*.*

Theorem 2. *(Mostow-Prasad) (Global uniqueness) if* Γ_1, Γ_2 *two finite covolume Kleinian groups and they are isomorphic by* $\phi : \Gamma_1 \to \Gamma_2$*. Then there exists* $g \in SL(2, \mathbb{C})$ *so that* $\phi(\gamma_1) = g\gamma_1 g^{-1}$*.*

Thus, if $V(\Gamma)$ contains Γ, then $V(\Gamma)$ is an isolated point. Since conjugation was fixed by the choices above of A_1, A_2. (See Weil-Garland rigidity again)

3 Volumes of ideal tetrahedra

Lobachevski function

- See Section 11.4 Ratcliffe for details. (These are from Chapter 7 of Thurston ("a good reading"). He also presents the volume of $H^3/PSL(2, O_d)$ for square free ds at the end.)
- For $\theta \neq n\pi$, define $\mathcal{L}(\theta) = -\int_0^\theta ln|2sinu|du$ and $\mathcal{L}(n\pi) = 0$. Continuous at $n\pi$.
- $\mathcal{L}$ is periodic of period π and is odd.
- Complex dilogarithm function $\phi(z) = \sum_{n=1}^\infty z^n/n^2$ for $|z| \leq 1$.
- For $|z| < 1$, $z\phi'(z) = -ln(1-z)$. Thus, $\phi(z) = -\int_0^z ln(1-w)/wdw$. Extend to $|z| = 1$ and $\phi(e^{2i\theta}) - \phi(1)$.
- uniformly convergent Fourier expansion: $\mathcal{L}(\theta) = 1/2 \sum_{n=1}^\infty \sin(2n\theta)/n^2$.

Lobachevski function

Volumes of ideal tetrahedra

- $T_{\alpha,\beta,\gamma}$ an ideal tetrahedron with dihedral angles α, β, γ. The opposite edges have the same angles here. We have $\alpha + \beta + \gamma = \pi$ always.
- $Vol(T_{\alpha,\beta,\gamma}) = \mathcal{L}(\alpha) + \mathcal{L}(\beta) + \mathcal{L}(\gamma)$.
- Using a complex parameterization of ideal tetrahedra by z for $Imz > 0$, we obtain $\mathcal{L}(arg(z)) + \mathcal{L}(arg((z-1)/z)) + \mathcal{L}(arg(1/(1-z))$.
- The maximal volume ideal tetrahedron occurs for dihedral angles all $\pi/3$; To maximize V subject to $\alpha + \beta + \gamma = \pi$, we must have $\mathcal{L}'(\alpha) = \mathcal{L}'(\beta) = \mathcal{L}'(\gamma)$. This implies $\alpha = \beta = \gamma = \pi/3$ since $\mathcal{L}'(x) = -ln|2sinx|$.
- Thus, if one knows the ideal triangulations, then we can compute the volumes of any hyperbolic 3-manifolds.
- The figure 8 knot complement is composed of two regular ideal tetrahedra. The volume is $6\mathcal{L}(\pi/3) \approx 2.029...$.

4 Dehn surgery and volumes of hyperbolic 3-manifolds

Dehn surgery and volumes

- When we are doing Dehn sugery on complete hyperbolic manifolds, mostly the geometry in the cusp regions are changing.
- Concentrate on one-cusped case.
- In the Dehn sugery space $Z \times Z$, the volume is a decreasing function away from (∞, ∞).
- In fact the volume can be defined in $\mathbb{R}^2$ containing $Z \times Z$.
- In face the volume is a real part of a complex analytic function on $\mathbb{C} = \mathbb{R}^2$.

Dehn surgery and volumes

- There is a lower-bound to the volume of hyperbolic 3-orbifolds (manifolds). (closed or not)
- Proof: The bounds on volumes and curvatures and injectivity radii and give a compactness. If there are no lower bound on injectivity radius, then there is a large thin part...
- There are only finitely many hyperbolic 3-orbifolds (manifolds) of same volume.
- Given an infinite sequence of bounded volume hyperbolic manifolds M_j, then there exists a finitely many hyperbolic 3-manifolds $X_1, ..., X_k$ so that M_js are results of Dehn surgery on one of the boundary modified X_i.
- The proof of the above two: the geometric convergence theory of Gromov.

5 The volume spectrum of hyperbolic 3-manifolds

The volume spectrum of hyperbolic 3-manifolds

- See S. Finch "Volumes of hyperbolic 3-manifolds".
- The volume is a topological invariant by Mostow-Prasad rigidity.
- Let us collect all volumes of hyperbolic n-manifolds $spc(n) \subset \mathbb{R}^+$.
- $spc(2) = \{2\pi k | k \geq 1\}, spc(4) = \{4\pi^2/3k | k \geq 1\}$. discrete for $n \geq 4$.
- $spc(3)$ is discrete well ordered and has ordinal type (countable) ω^ω (Jorgensen-Thurston):

$$v_1 < v_2 < ... < v_\omega < v_{\omega+1} < .... < v_{2\omega} < v_{2\omega+1} < ...$$

$$... < v_{3\omega} < v_{3\omega+1} < ... < v_{\omega^2} < v_{\omega^2+1} < ... < v_{\omega^3} < v_{\omega^3+1} < ...$$

- hyperbolic 3-orbifolds.. same type...

Explanations

- The map from the set of hyperbolic 3-manifolds to $spc(3)$ is finite-to-one (no global bound).
- v_1 is the least volume of closed hyperbolic 3-manifolds. $v_1 = Im[Li_2(z_0) + ln(|z_0|)ln(1 - z_0)] = 0.9427073...$, Li_2 dilogarithm. z_0 a root of $z^3 - z^2 + 1$ with $Imz_0 > 0$.
- v_2 is the next smallest volume of closed hyperbolic 3-manifolds
- v_ω the first limit point. v_ω is the smallest volume of one-cusped hyperbolic manifolds.
- $v_{2\omega}$ is the next smallest volume of one-cusped hyperbolic manifolds.
- v_{ω^2} is the first limit point of limit points and is the smallest volume of two-cusped hyperbolic manifolds.
- The ordinal type ω^ω.

Recent progress

- Gabai, Meyerhoff, Milley (GMM): Mom-n structures, they showed some finite class of hyperbolic manifolds with cusp generates every one-cusped hyperbolic manifolds of volume < 2.848
- Mom-n-structure is a handle decomposition with complexity conditions, derived from Matveev-Fomenko's work.

- Matveev, Fomenko, Weeks manifold has the minimum volume v_1. $(2,1)$-surgery on $m003$.
- classification of 1-cusped hyperbolic manifolds of volume < 2.848
- See GMM "Minimum volume hyperbolic 3-manifolds".
- GMM "Mom technology and hyperbolic 3-manifolds"
- GMM "Mom technology and volumes of hyperbolic 3-manifolds"