1 Introduction

About this lecture:

- 3-manifold topology and Dehn surgery
  - 2-and 3-manifold topology
  - Understanding hyperbolic 3-manifolds
  - hyperbolic Dehn surgery
  - Snappea
- The details can be really huge.... But feel free to ask...
- Rigidity
- Volumes and ideal tetrahedra (The last two to be done the next time)

Some helpful references

- Ratcliffe, Foundations of hyperbolic manifolds, Springer (elementary)
- W. Thurston, Lecture notes, (This is hard to read and incomplete) `http://www.msri.org/communications/books/gt3m`
- Morgan, Bass, The Smith Conjecture, Academic Press
- J. Weeks, A computation of hyperbolic structures in knot theory, in Handbook of knot theory
- W. Neumann, Zagier, Volumes of hyperbolic three-manifolds, Topology Volume 24, Issue 3, 1985, Pages 307-332
- M. Kapovich, Hyperbolic manifolds and discrete groups. (English summary) Progress in Mathematics, 183. Birkh?user Boston, Inc., Boston, MA, 2001.
- J.P. Otal, The hyperbolization theorem for fibered 3-manifolds. Translated from the 1996 French original by Leslie D. Kay. SMF/AMS Texts and Monographs, 7.

Some helpful references

- Cooper, Hodgson, Kerckhoff, Three-dimensional orbifolds and cone-manifolds. (English summary) With a postface by Sadayoshi Kojima. MSJ Memoirs, 5. (Geometrization of 3-orbifolds)
- Boileau, Michel, Maillot, Sylvain,; Porti, Joan(E-BARA) Three-dimensional orbifolds and their geometric structures. (English, French summary) Panoramas et Synth?ses [Panoramas and Syntheses], 15. SMF. (Geometrization of 3-orbifolds)

- Geometric structures on 2-orbifolds, a book. Preliminary version downloadable at `mathsci.kaist.ac.kr/~schoi/research.html` (mostly on 2-orbifold and geometry)

Some computer programs

- `http://www.math.uiuc.edu/~nmd/computop/index.html` These include many computational tools for finding hyperbolic manifolds. Windows XP, mac, linux, unix(SnapPy, originally Snappea by J. Weeks)
- `http://www.geometrygames.org/SnapPea/`
- `http://www.ms.unimelb.edu.au/~snap/orb.html` Snap, Orb (exact alg. computations, computations for orbifolds)
- `http://www.math.sci.osaka-u.ac.jp/~wada/OPTi/index.html` M. Wada (drawing isometric spheres for figure eight knot complements)

2 2- and 3-manifold topology

2-manifolds

- Orientability of objects are assumed (to avoid ....)
- Subject: A compact 2-manifold with possibly nonempty boundary.
- The fundamental group is finitely presented.
- Connected sum: Given one or two 2-manifolds, remove a pair of the interiors of closed disks in it and glue the resulting circle boundary components by a homeomorphism.
- Then $M = M_1 \# M_2$ and is unique up to diffeomorphism regardless of the choices involved.
- Conversely, given an imbeded circle (not bounding a 3-ball) in a 3-manifold M, $M = M_1 \# M_2$.
- In general $M = M_1 \# \ldots \# M_n$ and each M_i is $\mathbf{S}^1 \times \mathbf{S}^1$.

2-manifolds and geometric structures

- $g = 0$ case $\mathbf{S}^2$ has an elliptic structure.
- $g = 1$. Torus: Euclidean structure.
- $g > 1$. Take a regular $4g$-cone in $\mathbb{H}^2$. Use Poincare polyhedron theorem.

- The deformation space of hyperbolic (complex) structures on S is said to be the Teichmuller space

$$\{\mu|\mu \text{ is a hyperbolic structure }\}/\text{isotopies}$$

- We know that T^g is $6g-6$-dimensional space with a complex structure.
- $M_g = Mod(S)$ acts on T^g, and T^g/M_g is the Riemann moduli space.

3-manifolds

- Subject: A compact 3-manifold with possibly nonempty boundary.
- The fundamental group is finitely presented.
- Connected sum: Given one or two 3-manifolds, remove a pair of the interiors of closed balls in it and glue the resulting sphere boundary components by a homeomorphism.
- Then $M = M_1 \# M_2$ and is unique up to diffeomorphism regardless of the choices involved.
- Conversely, given an imbeded 2-sphere (not bounding a 3-ball) in a 3-manifold M, $M = M_1 \# M_2$.
- In general $M = M_1 \# \ldots \# M_n$ and M_i are either irreducible or is a sphere bundle.
- M is *irreducible* if all spheres bound 3-balls.
- Such a decomposition is uniquely determined up to diffeomorphisms.

Haken manifolds

- An imbedded surface $f : S \to M^3$ is *incompressible* if $\pi_1(S) \to \pi_1(M^3)$ is injective or S is a sphere and not bound a three-ball.
- A *Haken* manifold is a 3-manifold containing an incompressible surface. They are unique up to homotopies. (some sense canonical by normal surface theory)
- 3-manifolds with nonempty boundary are Haken.
- Cutting along an incompressible surface, we obtain a simpler manifold. The fundamental group is an amalgamation or HNN-extensions.
- M is *atoroidal* if any incompressible $f : T^2 \to M$ is homotopic to a map into the boundary.

Haken manifolds

- Mapping torus case: M diffeomorphic to $S \times I/\sim$ where $(x, 0) \sim (\phi(x), 1)$ for $\phi : S \to S$. M is then Haken.
- If ϕ is of infinite order and do not preserve any collection of disjoint circles, then ϕ is "pseudo-Anosov". In this case M is atoroidal.
- Haken manifolds are understood by algorithms. A sequence of cutting along incompressible surfaces yields cells and are reversible (Haken)
- There is a method of classifying Haken manifolds including knot complements. Too slow...

Dehn surgery

- M a compact 3-manifold with a boundary component T^2. We distinguish meridian m and longitude l.
- $\mathbf{S}^1 \times D^2$ has also torus boundary.
- We identify $\partial(\mathbf{S}^1 \times D^2)$ with T^2 in M.
- $o \times \partial D^2$ maps to $m^p l^q$ for relatively prime integers p, q.
- (p, q) classify the resulting manifolds up to diffeomorphism. (Often these are nonHaken)
- This is called (p, q)-Dehn surgery.

Dehn surgery

- In fact, knot or link complements in $\mathbf{S}^3$ are compact 3-manifolds with torus boundary.
- Dehn surgery creates new manifolds. This is the so-called Dehn surgery on the link.
- In fact all 3-manifolds can be obtained by a Dehn surgery on a link in $\mathbf{S}^3$.
- There many be more than one way to create the same manifold.
- Kirby moves identifies all such equivalent Dehn surgeries.

3 Understanding hyperbolic 3-manifolds

Hyperbolic 3-manifolds

- M compact Haken atoroidal. $\pi_1(M)$ contains no abelian subgroup of finite index. Then M is hyperbolizable. (M^o admits a complete hyperbolic structure.)
- These include the pseudo-Anosov bundles over circles.
- K a nontrivial prime knot and not a satellite knot and not a torus knot. Then $S^3 - K$ has a complete hyperbolic structure of finite volume.
- The unigueness by Mostow-Prasad rigidity.
- This is the so-called Monster theorem. The proofs are scattered. See Kapovich and Otal. For 3-orbifolds, see Cooper, Hodgson, Kerckhoff or Boileau, Maillot, Porti.
- Proof involves complex analysis, skinning maps, compactness, and so on...
- This settled the Smith conjecture with help of many other theories...

Hyperbolic 3-manifolds

- By Perelmann's proof of Geometrization conjecture, we know that evenwhen M is not Haken and if M is not Seifert or contains an essential torus, then M is hyperbolic.
- Two classes of hyperbolic manifolds: either Haken or is arithmetic (H. Bass) See the book "the Smith conjecture" Academic Press.
- Neumann,; Reid, Alan W, Arithmetic of hyperbolic manifolds. Topology '90 (Columbus, OH, 1990), 273–310, This shows that arithmetic hyperbolic manifolds has interesting fundamental group related to a prime.

Topology of hyperbolic 3-manifolds

- Thin parts: solid torus or a torus times $\mathbb{R}^+$ (called cusps).
- $\mathbb{H}^3$ covers M. Then the cusps correspond to horoballs given by images of $z > c$ for a const c.
- By taking a relatively small cusps, we can assume that the horoballs are disjoint.
- An abelian group of rank two is acting on each horoball.
- Taking the horoballs to maximal disjoint ones, we obtain the combinatorics of the Ford domain.
- In this way, we can build the canonical "triangulation" of the hyperbolic 3-manifold.

Snappea method of finding hyperbolic structures on knot or link complements

- On the practical side, we need to compute the hyperbolic structure... (See Weeks paper.)
- Start from a knot or link complement. General 3-manifolds later.
- This gives us a triangulation. (Fig. 8, 9) There is some simplification process here....
- Snappea choses possibly the simplest ones following its algorithm.

Snappea method of finding hyperbolic structures on knot or link complements

- Identify the tetrahedra with ideal tetrahedra. (This may not be the canonical one) (See J. Week's paper)
- The ideal tetrahedron is classified by a complex number z for $Imz \geq 0$. (Fig 12,13,14)
- The complex number gives us a transition map from one face to the next in the tetrahedron.
- We find a gluing equation around edges $z_1 z_2 .. z_n = 1$.
- The space of solutions is a complex algebraic set of complex dimension n for the number of link components n. (smooth near the complete cases Neumann-Zagier)
- The solutions with the positively oriented triangulations correspond to hyperbolic structures.

4 hyperbolic Dehn surgery

Cusps and complex parameters

- Each cusp C^i is $T^2 \times \mathbb{R}^+$ for $i = 1, ..., n$.
- The cusp is characterized by two matrices $h(\gamma_1^i)$, $h(\gamma_2^i)$ where two deck transformations γ_1^i and γ_2^i commute.
- Up to conjugations, two complex parameter is sufficient z_1^i, z_2^i. Thus, the total space is $\mathbb{C}^{2n}$.
- Then the space of solutions is a half-dimensional in $\mathbb{C}^{2n}$.
- The complete case correspond to $(1, 1, ..., 1)$. This is unique point and the space of solutions are smooth near it.

Cusps and complex parameters

- Snappea finds this point by Newton method.
- If no solution exists, then Snappea tries to find another triangulation or get broken down.
- If solution exists, Snappea proceeds to find canonical triangulation, Dirichlet domains, Cusp shapes, Ford domains, generators and relations and so on... (Sometimes, these steps can get stuck...)
- Everything is in Snappea source code and Week's papers...
- Hodgson and Weeks produced a census of hyperbolic 3-manifolds from smallest volume on... about 10,000. (accessible from Snappea)
- Knots can be classified also (upto 16 crossings. about 1,701,936 knots)

Dehn surgery and hyperbolic structure

Theorem 1. *M compact, orientable, incompressible torus boundary components $T_1, \cdots, T_n$. The interior of M admits a complete hyperbolic structure. Then except for only finitely many Dehn surgeries $((p_1, q_1), (p_2, q_2), ..., (p_n, q_n))$, the Dehn surgeries admit hyperbolic structures. (Good bounds like 12)*

Figure 8 knot complement
$S^3 - N_\epsilon(K)$ for a figure eight knot K. The only exceptions are $\{(1,0), (0,1), \pm(1,1), \pm(2,1), \pm(3,1), \pm(4,1)\}$. All other surgeries yield compact hyperbolic manifolds.

5 Snappea

Using Snappea

- J. Week's program does Dehn surgeries also and find many informations such as fundamental group presentations, volume, symmetry, horoballs, Ford domains, a few geodesics and so on.
- To install, go to SnapPy homepage. Download for WinXP or Mac,...
- Type Manifold? (or tab)...
- There is a census collected that can be used. Knots and links are listed.
- This also give us triangulations and canonical triangulation and the hyperbolicity equations.
- New knots and link can be input by a graphical method.
- Now I demonstrate....