1 Introduction

About this lecture

- $PSL(2, C)$ and hyperbolic 3-spaces.
- Subgroups of $PSL(2, C)$
- Hyperbolic manifolds and orbifolds
- Examples
- 3-manifold topology and Dehn surgery
- Rigidity
- Volumes and ideal tetrahedra
- Part 1: 1.1-1.4 Kleinian group theory
- Part 2: 1.5-1.7 Topology

Some helpful references

- Ratcliffe, Foundations of hyperbolic manifolds, Springer (elementary)
- K. Matsuzaki, M. Taniguchi, Hyperbolic manifolds and Kleinian groups, Oxford (complete but technical)
- A. Marden, The geometry of finitely generated Kleinian groups, Ann of Math, 99 (1974) 299-323. (nice but more advanced)
- K. Ohshika, Discrete groups, AMS
- A. Adem, j. Leida, ... Orbifolds and stringly topology, Cambridge.
- W. Thurston, Three-dimensional geometry and topology I, Princeton University Press.
- W. Thurston, Lecture notes, (This is hard to read and incomplete) `http://www.msri.org/communications/books/gt3m`
- B. Fine, Algebraic theory of the Bianchi groups, Marcel Dekker 1989
- C. Series, A crash course on Kleinian groups `http://www.dmi.units.it/~rimut/volumi/37/series.ps`

Some computer programs

- `http://www.math.uiuc.edu/~nmd/computop/index.html` These include many computational tools for finding hyperbolic manifolds. (SnapPy, originally Snappea by J. Weeks)
- `http://www.geom.uiuc.edu/~crobles/hyperbolic/` Interactive Javalets for experiments.
- `http://www.geometrygames.org/SnapPea/`
- `http://www.ms.unimelb.edu.au/~snap/orb.html` Snap, Orb (exact alg. computations, computations for orbifolds)
- `http://www.math.sci.osaka-u.ac.jp/~wada/OPTi/index.html` M. Wada (drawing isometric spheres for figure eight knot complements)

2 Poincare theorem

2.0.1 Convex polyhedrons

Convex subsets

Convex subsets

A *convex subset* of H^3 is a subset such that for any pair of points, there is a unique geodesic segment between them and it is in the subset. For example, a pair of antipodal point in $\mathbf{S}^n$ is convex.

Convex subsets

Let us state some facts about convex sets:

- The dimension of a convex set is the least integer m such that C is contained in a unique m-plane $\hat{C}$ in H^3.
- The interior C^o, the boundary ∂C are defined in $\hat{C}$.
- The closure of C is in $\hat{C}$. The interior and closures are convex. They are homeomorphic to an open ball and a contractible domain of dimension equal to that of $\hat{C}$ respectively.
- A side C is a nonempty maximal convex subset of ∂C.
- A convex polyhedron is a nonempty closed convex subset such that the set of sides is locally finite in H^3.

2.0.2 Convex polytopes

Convex polytopes

Convex hulls

Using the Beltrami-Klein model, the open unit ball B, i.e., the hyperbolic space, is a subset of an affine patch $\mathbb{R}^n$. In $\mathbb{R}^n$, one can talk about convex hulls.

- A *convex polytope* in $B = H^n$ is a convex polyhedron with finitely many vertices and is the convex hull of its vertices in $B = H^n$.
- A polyhedron P in $B = H^n$ is a *generalized convex polytope* if its closure is a polytope in the affine patch. A generalized polytope may have ideal vertices.

Convex polytopes

- A compact simplex which convex hull of $n + 1$ points in $B = H^n$ is an example of a convex polytope.
  - Take an origin in B, and its tangent space $T_O B$.
  - Start from the origin O in $T_O B$ expand the infinitesimal euclidean polytope from an interior point radially. That is a map sending $x \to sx$ for $s > 0$ and x is the coordinate vector of an affine patch using in fact any vector coordinates. Now map the vertices of the convex polytope by an exponential map to B.
  - The convex hull of the vertices is a convex polytope.
  - Thus for any Euclidean polytope, we obtain a one parameter family of hyperbolic polytopes.

Convex polytopes

2.0.3 Side pairings and Poincare fundamental polyhedron theorem

Side pairings and Poincare fundamental polyhedron theorem

- A *tessellation* of $\mathbb{H}^3$ is a locally-finite collection of polyhedra covering $\mathbb{H}^3$ with mutually disjoint interiors.
- A convex fundamental polyhedron with some conditions provides examples of exact tessellations.
- For such a convex fundamental polyhedron P, $\mathbb{H}^3$ is a union $\bigcup_{g\in\Gamma} g(P)$.

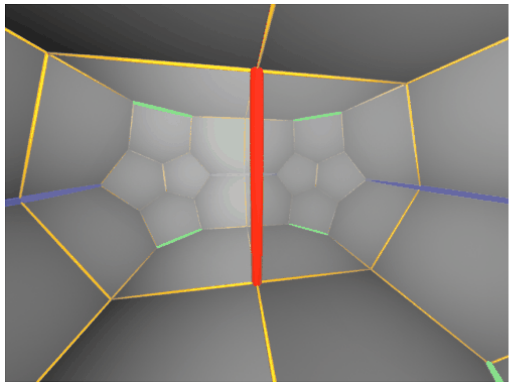

Figure 1: Regular dodecahedron with all edge angles $\pi/2$ as seen from inside (Geometry center).

Side pairings and Poincare fundamental polyhedron theorem

- Given a side S of an exact convex fundamental domain P, there is a unique element g_S such that $S = P \cap g_S(P)$. And $S' = g_S^{-1}(S)$ is also a side of P.
- $g_{S'} = g_S^{-1}$ since $S' = P \cap g_S^{-1}(P)$.
- Γ-side-pairing is the set of g_S for sides S of P.
- The equivalence class at P is generated by $x \cong x'$ if there is a side-pairing sending x to x' for $x, x' \in P$.
- $[x]$ is finite and $[x] = P \cap \Gamma$.

Side pairings and Poincare fundamental polyhedron theorem

- Cycle relations:
  - Let $S_1 = S$ for a given side S. Choose the side R of S_1. Obtain S_1'. Let S_2 be the side adjacent to S_1' so that $g_{S_1}(S_1' \cap S_2) = R$.
  - Let S_{i+1} be the side of P adjacent to S_i' such that $g_{S_i}(S_i' \cap S_{i+1}) = S_{i-1}' \cap S_i$.
- Then we obtain
  - There is an integer l such that $S_{i+l} = S_i$ for each i.
  - $\sum_{i=1}^{l} \theta(S_i', S_{i+1}) = 2\pi/k$.
  - $g_{S_1} g_{S_2} \ldots g_{S_l}$ has order k.
- The period l is the number of sides of codimension one coming into a given side R of codimension two in X/Γ.

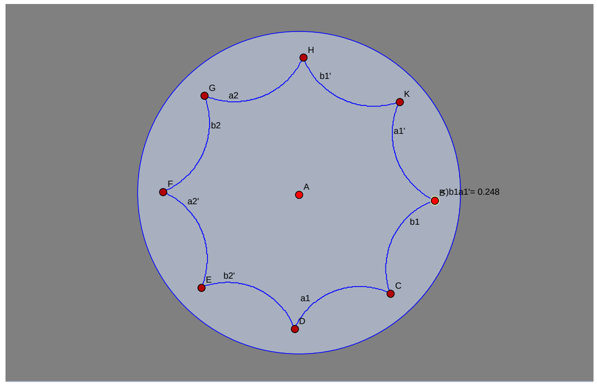

Figure 2: Example: the octahedron in the hyperbolic plane giving genus 2-surface. There are the cycle $(a1, D), (a1', K), (b1', K), (b1, B), (a1', B), (a1, C), (b1, C)$, the cycle $(b1', H), (a2, H), (a2', E), (b2', E), (b2, F), (a2', F), (a2, G)$, and the cycle $(b2, G), (b2', D), (a1, D), (a1', K), ...$

Theorem 1. *If P is an exact convex fundamental polyhedron of a discrete group Γ of isometries acting on $\mathbb{H}^3$, then Γ is generated by $\Phi = \{g_S \in \Gamma | P \cap g_S(P)$ is a side S of $P\}$ and is finitely presented by cyclic relations $(g_{S_1} g_{S_2} .... g_{S_l})^k$*

- To see this, let g be an element of Γ, and let us choose a frame at a point of P and consider its image in $g(P)$.
- Then we choose a path of frames from the intial from to the terminal frame.
- We perturb the path so that it meets only the interiors of the sides of the tessellating polyhedrons.
- Each time the path crosses a side S, we take the side-pairing g_S obtained as below.
- Then multiplying all such side-pairings in the reverse order to what occured, we obtain an element $g' \in \Gamma$ so that $g'(P) = g(P)$ as $h g_S h^{-1}$ moves $h(P)$ to the image of P adjacent in the side $h(S)$ for every $h \in \Gamma$.
- Since P is a fundamental domain, $g^{-1}g'$ is the identity element of Γ.

Poincare fundamental polyhedron theorem
The Poincare fundamental polyhedron theorem is the converse. We claim that the theorem holds for geometries (X, G) with notions of m-planes. (See Kapovich P. 80–84):

Theorem 2. *Given a convex polyhedron P in $\mathbb{H}^3$ with side-pairing isometries satisfying the above relations, then P is the fundamental domain for the discrete subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *generated by the side-pairing isometries.*

Manifold case

If every k equals 1, then the result of the face identification is a manifold. Otherwise, we obtain orbifolds. The results are always complete. (See Jeff Weeks `http://www.geometrygames.org/CurvedSpaces/index.html` for an examples of hyperbolic or spherical manifold as seen from "inside".)

Reflection groups

- We will be particularly interested in reflection groups.
- Suppose that X has notions of angles between m-planes.
- A *discrete reflection group* is a discrete subgroup in G generated by reflections in X about sides of a convex polyhedron. Then all the dihedral angles are submultiples of π.
- The side pairing is such that each face is glued to itself by a reflection satisfies the Poincare fundamental theorem.
- The reflection group has presentation $\{S_i : (S_iS_j)^{k_{ij}}\}$ where $k_{ii} = 1$ and $k_{ij} = k_{ji}$. which are examples of Coxeter groups.

Reflection groups

- Andreev gave a combinatorial condition for the existence of acute-angled ($> 0, \leq \pi/2$) convex polytope in $\mathbb{H}^3$. Such polytope is unique upto isometry. Conditions are long: Basically, the sum of angles around a vertex is less $> \pi$. Prismatic circuits...
- When angles of form π/n, then we obtain a reflection group based on the sides of P.
- By the Poincare theorem, the group is a Coxeter group generated by reflections r_i and $(r_ir_j)^{e_{ij}} = I$.
- In many cases, these are classified. For example P is a tetrahedron.
- Among these, there are only finitely many maximal arithmetic ones. (Agol)

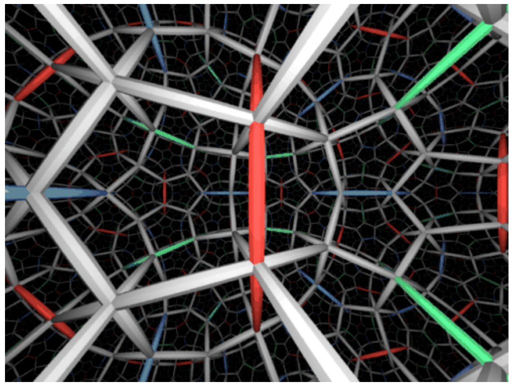

Figure 3: The dodecahedral reflection group as seen by an insider: One has a regular dodecahedron with all edge angles $\pi/2$ and hence it is a fundamental domain of a hyperbolic reflection group. From Geometry center

3 1.4. Examples

Bianchi groups

- A discrete subring R of $\mathbb{C}$: $\mathrm{PSL}(2, R)$ is a discrete subgroup.
- Let $R = O_d$ the ring of integers in $\mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}$. Let $\Gamma = \mathrm{PSL}(2, O_d)$.
- $1, \omega$ a basis of O_d. $\omega = \sqrt{-d}$ if $d \neq 1 mod 4$ and $\omega = (1 + \sqrt{-d})/2$ for $d = 1 mod 4$.
- Then translation by 1 and ω fixes ∞ and form $\mathbb{Z} + \mathbb{Z}$ abelian group. They correspond to a cusp point ∞. They form the cusp group Γ_∞.
- Then we define a "Ford domain" exterior to all "isometric spheres" for $\gamma \in \Gamma$ and intersect it with the fundamental domain for Γ_∞. (See Fig. 1.1)
- The polytope gives us the fundamental domain and Poincare side pairing transformations are as follows
- $$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W = \begin{pmatrix} i & -1 \\ 0 & -i \end{pmatrix}$$

Isometric spheres

General Bianchi groups

General Bianchi groups

Also see the paper of Hatcher `http://www.math.cornell.edu/~hatcher/bianchi.html` for examples of Bianchi groups. See also B. Fine.

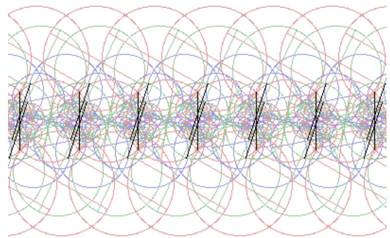

Figure 4: drawn by Opti.

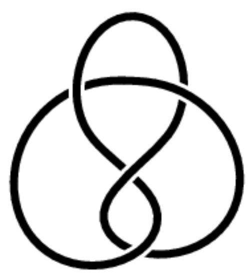

Figure eight knot complement

- A knot complement is a compact manifold with a boundary homeomorphic to a torus.
- $\pi_1(S^3 - N_\epsilon(K)) =< x_1, x_1 | w x_1 w^{-1} = x_2, w = x_1^{-1} x_2 x_1 x_2^{-1} >$
- $\tilde{w} := x_1 x_2^{-1} x_1^{-1} x_2$.

Figure eight knot complement

- We show that $S^3 - K$ has a complete hyperbolic structure, i.e., $S^3 - K$ is diffeomorphic to $\mathbb{H}^3/\Gamma$ for the image $\pi \to \mathrm{PSL}(2, \mathbb{C})$.
- As a consequence of finding the hyperbolic structure:

$$\rho(x_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \rho(x_2) = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$$

- $\omega = (-1 + \sqrt{-3})/2$.
- Γ is an index 12 subgroup of $\mathrm{PSL}(2, O_3)$.

The figure 8 knot complement

- We glue two ideal tetrahedra in $\mathbb{H}^3$ as indicated in the picture of Ratcliffe.
- The dihedral angles are $\pi/3$. Six edges glue to one edge. There are two sets of these edges. Hence, they glue to a complete hyperbolic structure.
- We are interested in a complete or compact hyprbolic manifolds only.
- See Benedetti p. 218-219, Ratcliff p.444-447
- In general, many knot complements have a complete hyperbolic structures (non-satellite ones) as proved by Thurston.
- J. Weeks produced a numerical computer program Snappea: See "Computations of hyperbolic structures in knot theory" in Handbook of Geometric Topology.

Seifert Weber Dodecahedral space

- Take a regular dodecahedron with dihedral angle $2\pi/5$ obtained by "expansion".
- Glue opposite faces by $3/10$ turn. (misaligned by $1/10$ turn)
- See Weeks p.222-223.

4 3-manifolds and Dehn surgery

3-manifolds

- Orientability of objects are assumed.
- Subject: A compact 3-manifold with possibly nonempty boundary.
- The fundamental group is finitely presented.
- Connected sum: Given one or two 3-manifolds, remove a pair of the interiors of closed balls in it and glue the resulting sphere boundary components by a homeomorphism.
- Then $M = M_1 \# M_2$ and is unique up to diffeomorphism regardless of the choices involved.
- Conversely, given an imbeded 2-sphere (not bounding a 3-ball) in a 3-manifold M, $M = M_1 \# M_2$.
- In general $M = M_1 \# \ldots \# M_n$ and M_i are either irreducible or is a sphere bundle.
- M is *irreducible* if all spheres bound 3-balls.
- Such decomposition is uniquely determined up to diffeomorphisms.

Haken manifolds

- An imbedded surface $f : S \to M^3$ is *incompressible* if $\pi_1(S) \to \pi_1(M^3)$ is injective or S is a sphere and not bound a three-ball.
- A *Haken* manifold is a 3-manifold containing an incompressible surface.
- M is *atoroidal* if any incompressible $f : T^2 \to M$ is homotopic to a map into the boundary.
- Mapping torus case: M diffeomorphic to $S \times I/\sim$ where $(x, 0) \sim (\phi(x), 1)$ for $\phi : S \to S$. M is then Haken.
- If ϕ is of infinite order and do not preserve any collection of disjoint circles, then ϕ is "pseudo-Anosov". In this case M is atoroidal.

Hyperbolic 3-manifolds

- M compact Haken atoroidal. $\pi_1(M)$ contains no abelian subgroup of finite index. Then M is hyperbolizable. (M^o admits a complete hyperbolic structure.)
- These include the pseudo-Anosov bundles over circles.
- K a nontrivial prime knot and not a satellite knot and not a torus knot. Then $S^3 - K$ has a complete hyperbolic structure of finite volume.

Dehn surgery

- M a 3-manifold with a boundary component T^2. We distinguish meridian m and longitude l.
- $\mathbf{S}^1 \times D^2$ has also torus boundary.
- We identify $\partial(\mathbf{S}^1 \times D^2)$ with T^2 in M.
- $o \times \partial D^2$ maps to $m^p l^q$ for relatively prime integers p, q.
- (p, q) classify the resulting manifold up to diffeomorphism.
- This is called (p, q)-Dehn surgery

Dehn surgery and hyperbolic structure

Theorem 3. *M compact, orientable, incompressible torus boundary components $T_1, \cdots, T_n$. The interior of M admits a complete hyperbolic structure. Then except for only finitely many Dehn surgeries $((p_1, q_1), (p_2, q_2), ..., (p_n, q_n))$, the Dehn surgeries admit hyperbolic structures. (Good bounds like 12)*

Figure 8 knot complement
$S^3 - N_\epsilon(K)$ for a figure eight knot K. The only exceptions are $\{(1,0),(0,1),\pm(1,1),\pm(2,1),\pm(3,1),\pm(4,1)\}$. All other surgeries yield compact hyperbolic manifolds.

Snappea
J. Week's program does Dehn surgeries also and find many informations such as fundamental group presentations, volume, symmetry, and so on.