1 Introduction

About this lecture

- $PSL(2, C)$ and hyperbolic 3-spaces.
- Subgroups of $PSL(2, C)$
- Hyperbolic manifolds and orbifolds
- Examples
- 3-manifold topology and Dehn surgery
- Rigidity
- Volumes and ideal tetrahedra
- Part 1: 1.1-1.4 Kleinian group theory
- Part 2: 1.5-1.7 Topology

Some helpful references

- Ratcliffe, Foundations of hyperbolic manifolds, Springer (elementary)
- K. Matsuzaki, M. Taniguchi, Hyperbolic manifolds and Kleinian groups, Oxford (complete but technical)
- K. Ohshika, Discrete groups, AMS
- A. Adem, j. Leida, ... Orbifolds and stringly topology, Cambridge.
- W. Thurston, Three-dimensional geometry and topology I, Princeton University Press.
- W. Thurston, Lecture notes, (This is hard to read and incomplete)

Some helpful references

- http://www.math.uiuc.edu/~nmd/computop/index.html These include many computational tools for finding hyperbolic manifolds. (SnapPy, originally Snappea by J. Weeks)
- http://www.geom.uiuc.edu/~crobes/hyperbolic/ Interactive Javalets for experiments.
- http://www.geometrygames.org/SnapPea/
The field of geometry and topology: geometric structures

- Basically, we try to understand the relationship between manifolds (orbifolds, varieties, ...) with discrete subgroup of Lie groups acting on homogeneous (or nice) spaces.
- Algebraic representations are often possible (Geometrization)
- Often such representations might be unique (rigidity, Margulis, Mostow) (Arithmeticity places an important role here.)
- If not, we have moduli spaces. (Teichmuller spaces)
- We obtain invariants in this way (volume, eta invariants, numerical invariants,....)
- Properties of groups can be studied using topological and geometric methods (group decompositions and Gromov hyperbolicity)
- Thus, there are some correspondences between topology and algebra here.

Manifolds

- Manifolds: Hausdorff, covered by countable euclidean open balls. (2,3-dim only)
- Main objectives is to make sense of their variety.
- Examples:
  - knot complements. The variety of these are suprisingly many. (still cannot classify)
  - Surfaces: classified by orientation, genus, and number of holes ( homology theory is needed)
  - 3-manifolds: Geomerization now makes the field into something of “algebraic problems”.
Geometrization of Manifolds

- If $M$ is an orientable surface, then $M$ can be written $\mathbb{H}^2/\Gamma, E^2/\Gamma$, or $S^2/\Gamma$ by the uniformization theorem. This is not unique. So we need Teichmüller spaces.

- If $M$ is an orientable compact 3-manifold, then $M$ can be canonically decomposed by spheres, disks into irreducible 3-manifolds.

- Irreducible 3-manifolds decomposes along tori into open or closed submanifolds admitting one of eight geometric structures $S^3/\Gamma, E^3/\Gamma, \mathbb{H}^3/\Gamma, Nil/\Gamma, Sol/\Gamma, \tilde{SL}(2, \mathbb{R})/\Gamma, \mathbb{H}^2 \times \mathbb{R}/\Gamma$, and $S^2 \times \mathbb{R}/\Gamma$.

- The hyperbolic pieces are most varied.

Orbifolds

- Orbifolds: Hausdorff, covered by countable quotients of open balls by finite linear group actions that remember the action and the open balls. (Manifolds “are” orbifolds)

- Orbifolds are of form $M/\Gamma$ where $M$ is a universal covering orbifold and $\Gamma$ is a properly discontinuous action (not free).

- Given an orbifold $O$, we can always find $M$ and $\Gamma$ and the orbifold structure is equivalent to the pair $(M, \Gamma)$.

- If $M$ is a manifold, then $M/\Gamma$ is a good orbifold.

- If $M$ is a compact 2-dim orbifold, then $M$ is classified by orbifold Euler characteristic.

- If $M$ is a compact 3-dim orbifold, then $M$ satisfies the geometrization.

Orbifolds

- Sometimes $O = N/\Gamma$ for a manifold $N$ and $\Gamma$ finite. The $O$ is very good.

- Selberg’s Lemma: If $\Gamma$ is a finitely generated subgroup of $GL(N, \mathbb{C})$, then it has a torsion-free finite index subgroup.

- Most orbifolds here are very good.

3 \textbf{$PSL(2, \mathbb{C})$ and hyperbolic 3-space}

- \url{http://www.geom.uiuc.edu/docs/forum/hype/model.html}

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$PSL(2, \mathbb{C})$ and hyperbolic 3-space

- $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I\}$.
- $PSL(2, \mathbb{C})$ acts on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by $z \mapsto \frac{az+b}{cz+d}$.
- $\mathbb{H}^3$ is defined as $\{(x, y, t) \in \mathbb{R}^3 | t > 0\}$.
- $t = 0$ plane is identified with $\mathbb{C}$.
- $\mathbb{H}^3$ compactifies to a closed ball with boundary $\hat{\mathbb{C}}$ in the compactification of $\mathbb{R}^3$ as $\hat{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}$.
- The boundary set is the sphere of infinity $S^{2,\infty} := \hat{\mathbb{C}}$ with a complex structure.
- Each Mobius transformation on $S^{2,\infty}$ extends to an action in $\mathbb{R}^3$ (Poincare extension) This is obtained by inversions in spheres perpendicular to $t = 0$ or the the planes perpendicular to $t = 0$.

- The Mobius transformations form the isometry group of the Riemannian metric given by $\delta_{ij}/t^2$.
- The angles are same as the euclidean angles.
- $PSL(2, \mathbb{C})$ is isomorphic to $\text{Isom}^+(\mathbb{H}^3)$. (Lie group)
- Geodesics are half circles perpendicular to $S^{2,\infty}$ or a straight line parallel to the $t$-axis.
• Totally geodesic subspaces are either hemispheres or half-spaces parallel to $t$-axis.
• Horospheres are given by $t = const$ or its images under isometries. The images are spheres tangent to $t = 0$ or planes.
• In fact all isometries are generated by reflections. (Mobius type inversion actually)
• Volume form $dx \wedge dy \wedge dt$.
• Models http://www.geom.uiuc.edu/~crobles/hyperbolic/

$\mathbb{H}^2$
• Consider setting $y = 0$. Then we obtain $\mathbb{H}^2$ with metric $\delta_{ij}/t^2$.
• This is a totally geodesic subspace. In fact, any other 2D-totally geodesic subspace is isometric to it.
• $\text{PSL}(2, \mathbb{R})$ isomorphic to $\text{Isom}^+(\mathbb{H}^2)$.
• the angles, geodesics, subspaces.
• Isometries are generated by reflections.
• Volume form $dx \wedge dt$.
• The boundary is a circle $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ in $\mathbb{R}^3$.
• Geodesics: http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/uhp/uhpjava.html
• Distances: http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/uhp/eq.html

Alternative view as a hyperboloid in the Lorentzian $4$-space
• Let $V$ be a four dimensional space with a quadratic form $q(\vec{x}) = x_1^2 + x_2^2 + x_3^2 - x_4^2$.
• $V$ is decomposed into three parts $q > 0$, the positive open cone $C^+$ with $q < 0$, $x_4 > 0$, the negative open cone $q < 0$, $x_4 < 0$, and the null cone $q = 0$.
• The vectors are called spacelike, positive timelike, negative timelike, or null.
• A hyperboloid is given by $q = -1$.
• We take the upper part $\Lambda$. Then the $q$ restricts to Riemannian metric. Then $\Lambda$ is isometric with $\mathbb{H}^3$.
• Define $O^+(V, q) = O^+(1, 3)$ be the orthogonal map preserving $C^+$. 
• This group is generated by the Lorenztian reflection through time-like hyper-planes.

• \( \text{Isom}\Lambda = PO^+(V, q) \) and \( \text{Isom}^+\Lambda = PSO^+(V, q) \). These are isomorphic to the previous groups.

Poincare model

• Consider the unit ball \( B^3 \) in \( \mathbb{R}^3 \). There is an inversion sending \( \mathbb{H}^3 \) onto \( B^3 \).

• The metric is given by \( 4\delta_{ij}/(1 - |r|^2)^2 \).

• Again the isometry group is generated by reflections in spheres orthogonal to \( \partial B^3 \).

• The unit disk \( B^2 \) is identified with the hyperbolic plane.

The hyperbolic trigonometry

• hyperbolic law of sines:
  \[
  \frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}
  \]

• hyperbolic law of cosines:
  \[
  \cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C
  \]
  \[
  \cosh c = \left( \cosh A \cosh B + \cos C \right)/\sinh A \sinh B
  \]

• The triangles behave in a funny way... http://www.math.ksu.edu/~bennett/gc/tri.html

4 Subgroups of \( \text{PSL}(2, \mathbb{C}) \)

The classifications of elements

• Assume \( \gamma \neq I \).

• \( \gamma \) is elliptic if \( |tr\gamma| < 2 \).

• \( \gamma \) is parabolic of \( tr\gamma = \pm 2 \).

• \( \gamma \) is loxodromic otherwise.

• \( \gamma \) is elliptic if and only if it fixes a unique geodesic and if and only if it is conjugate to \( z \mapsto e^{i\theta}z \) for \( \theta \neq 0 \).

• \( \gamma \) is loxodromic if and only if it acts on a unique geodesic and if and only if it is conjugate to \( z \mapsto vz \) where \( v \) is a complex number whose length is not 1. (hyperbolic if \( v \) is a positive real number)

• \( \gamma \) is parabolic if and only if it acts on horospheres and if and only if it is conjugate to \( (x, y, t) \mapsto (x + a, y + b, t) \) for some real numbers \( a, b \) not both zero. (This fixes a unique point of the tangency)
Some more general theory in terms of symmetric space theory

- $\mathbb{H}^3$ is a symmetric space of the Lie group $\text{PSL}(2, \mathbb{C})$ with maximal compact group $\text{PSU}(2, \mathbb{C})$ isomorphic to $\text{SO}(3, \mathbb{R})$.
- A parabolic subgroup is a subgroup fixing an infinity and acts on leaves of foliation given by a disjoint collection of horospheres. This is conjugate to a group of upper triangular matrices.
- $\mathbb{H}^3$ can be compactified by adding one point for each parabolic subgroup.
- A geodesic ends at a point of infinite and Busemann function gives us a parameter of horospheres. [http://eom.springer.de/b/b120550.htm](http://eom.springer.de/b/b120550.htm)
- This description agrees with the above.
- Reference: Eberlein, Spaces of nonpositive curvature, Chicago

Subgroups of $\text{PSL}(2, \mathbb{C})$

- A subgroup is **reducible** if it fixes a unique point in $\hat{\mathbb{C}}$.
- A subgroup is **elementary** if it has a finite orbit in its action on $\mathbb{H}^3 \cup \hat{\mathbb{C}}$. Otherwise it is **non-elementary**.
- Every non-elementary subgroup contains infinitely many loxodromic elements, no two of which have a common fixed point.
- Let $x, y$ be elements of $\text{PSL}(2, \mathbb{C})$. Then $\langle x, y \rangle$ is reducible if and only if $\text{tr}[x, y] = 2$.

Kleinian group

- A **Kleinian** group is a discrete subgroup of $\text{PSL}(2, \mathbb{C})$.
- In this setting, the discreteness implies that $\Gamma$ acts properly discontinuously (possibly with fixed points)
- Usually, we assume that it is non-elementary.
- $\mathbb{H}^3/\Gamma$ is a 3-dimensional orbifold (3-manifold if no torsion).
- We give two-dimensional examples. But they also act on $\mathbb{H}^3$ as a Kleinian group (called Fuchsian group).
- Fuchian group can be deformed to quasi-Fuchian groups.
Triangle groups

- Find a triangle in $\mathbb{H}^2$ with angles submultiples of $\pi$.
- We divide into three cases $\pi/a + \pi/b + \pi/c < 0$.
- In fact, given a surface (or 2-orbifold) $S$ with $\chi < 0$, we have $S = \mathbb{H}^2/\Gamma$ for a Fuchian group.
- $(2, 4, 8)$-triangle group

The modular group $PSL(2, \mathbb{Z})$ action on $\mathbb{H}^2$.

- Generated by $S : z \mapsto -1/z, T : z \mapsto z + 1$.
- $(2, 3, \infty)$-triangle group.

Kleinian group

- Let $\Gamma$ be a nonelementary Kleinian group.
- A stabilizer of a point of $\mathbb{H}^3$ is a finite subgroup.
- A stabilizer of a point of the sphere of infinity $S^2, \infty$ can be conjugated to a subgroup $B$ with upper triangular matrices.
- $B$ can be of the following form:
  - Finite cyclic. (if it is finite)
  - A finite extension of an infinite cyclic group generated by a loxodromic or parabolic element. (if it contains a loxodromic)
  - A finite extension of $\mathbb{Z} \oplus \mathbb{Z}$ generated by two parabolic elements.
  - Essentially proved from 2-dim Bieberbach theorem.

Cusp

- A point $\zeta$ of $S^2, \infty$ is a cusp point of $\Gamma$ if the stabilizer consists of parabolic elements and the identity. (rank = 2)
- We will usually be working in finite volume case. So we do not need to know "limit set". the domain of discontinuity in $S^2, \infty$.
- See ideal triangle examples.
- The ideal example http://egl.math.umd.edu/software.html
Fundamental domain

- A fundamental domain $F$ for a Kleinian group $\Gamma$ is a closed subset of $\mathbb{H}^3$ satisfying
  - $\bigcup_{\gamma \in \Gamma} \gamma F = \mathbb{H}^3$.
  - $F^\circ \cap \gamma F^\circ = \emptyset$ if $\gamma \neq 1$.
  - the boundary of $F$ has measure zero.
- $F$ is usually a polyhedron (compact or noncompact, finite or infinite sided)

Dirichlet domains

- A Kleinian group $\Gamma$, choose a point $p$ (not fixed)
- $D_p(\Gamma) := \{ q \in \mathbb{H}^3 | d(q, p) \leq d(\gamma(q), p) \text{ for all } \gamma \in \Gamma \}$.
- This is a polyhedron with locally finite sides.
- Sides are bisectors of $p$ and $\gamma(p)$ for some $\gamma$.
- The triangles in the triangle reflection groups are Dirichlet domains.

Geometrical finiteness

- A Kleinian group $\Gamma$ is geometrically finite if it admits a finite sided Dirichlet domain for every (some) points. (Another way is the thick part of the convex hull is finite volume)
- $\Gamma$ is cocompact if $\mathbb{H}^3/\Gamma$ is compact. This is true if and only if all/some Dirichlet domain is compact. (See triangle groups)
- $\Gamma$ is of finite covolume if the volume of a Dirichlet domain is finite. (See ideal triangle groups)
- This is well-defined.
- If a Kleinian group $\Gamma$ is of finite covolume, then if $\Gamma$ is geometrically finite and hence finitely generated. (finitely presented) (Converse not true)
- This has a rather involved proof....
5 Hyperbolic manifolds and orbifolds

Metric structures of hyperbolic manifolds in general

- Margulis constant $\epsilon > 0$.
- Given a hyperbolic manifold $M$, $M = \mathbb{H}^3/\Gamma$ for a torsion-free $\Gamma$, the set of thin parts is defined as $M'_\epsilon = \{ x \in \mathbb{H}^3/\Gamma | \exists x_1, x_2 \in x, d(x_1, x_2) \leq \epsilon \}$.
- The thick part is defined as $M_0 = \{ x \in \mathbb{H}^3/\Gamma | \forall x_1, x_2 \in x, d(x_1, x_2) > \epsilon \}$.
- The thin part either an annulus, a Mobius band, a torus or a Klein bottle times interval (corresponds to cusps) or is a solid torus or solid Klein bottle.
- The thick part is a tame 3-manifold, i.e., either is compact or is the interior of a compact 3-manifold. (Hence, the topology is finite.)
- The orbifold versions are similar to this...(that is the thin parts are finite quotients of the above.)
- These work for general geometric manifolds...

Commensurability

- $\Gamma_1, \Gamma_2$ subgroups of $\text{PSL}(2, \mathbb{C})$. They are directly commensurable if $\Gamma_1 \cap \Gamma_2$ is of finite index in both $\Gamma_1$ and $\Gamma_2$.
- They are commensurable if $\Gamma_1$ and a conjugate of $\Gamma_2$ are directly commensurable.
- Two hyperbolic orbifolds $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$ are commensurable if their groups are so.
- Examples: Consider subgroups of a common Kleinian group.