# 1 Introduction

### About this lecture

- $PSL(2, C)$ and hyperbolic 3-spaces.
- Subgroups of $PSL(2, C)$
- Hyperbolic manifolds and orbifolds
- Examples
- 3-manifold topology and Dehn surgery
- Rigidity
- Volumes and ideal tetrahedra
- Part 1: 1.1-1.4 Kleinian group theory
- Part 2: 1.5-1.7 Topology

### Some helpful references

- Ratcliffe, Foundations of hyperbolic manifolds, Springer (elementary)
- K. Matsuzaki, M. Taniguchi, Hyperbolic manifolds and Kleinian groups, Oxford (complete but technical)
- A. Marden, The geometry of finitely generated Kleinian groups, Ann of Math, 99 (1974) 299-323. (nice but more advanced)
- K. Ohshika, Discrete groups, AMS
- A. Adem, j. Leida, ... Orbifolds and stringly topology, Cambridge.
- W. Thurston, Three-dimensional geometry and topology I, Princeton University Press.
- W. Thurston, Lecture notes, (This is hard to read and incomplete)

### Some helpful references

- `http://www.math.uiuc.edu/~nmd/computop/index.html` These include many computational tools for finding hyperbolic manifolds. (SnapPy, originally Snappea by J. Weeks)
- `http://www.geom.uiuc.edu/~crobles/hyperbolic/` Interactive Javalets for experiments.
- `http://www.geometrygames.org/SnapPea/`

- `http://www.ms.unimelb.edu.au/~snap/orb.html` Snap, Orb (exact alg. computations, computations for orbifolds)
- `http://www.neverendingbooks.org/index.php/the-dedekind-tessellation.html` Modular groups

# 2 General introduction

**The field of geometry and topology: geometric structures**

- Basically, we try to understand the relationship between manifolds (orbifolds, varieties, ...) with discrete subgroup of Lie groups acting on homogeneous (or nice) spaces.
- Algebraic representations are often possible (Geometrization)
- Often such representations might be unique (rigidity, Margulis, Mostow) (Arithematicity places an important role here.)
- If not, we have moduli spaces. (Teichmuller spaces)
- We obtain invariants in this way (volume, eta invariants, numerical invariants,....)
- Properties of groups can be studied using topological and geometric methods (group decompositions and Gromov hyperbolicity)
- Thus, there are some correspondences between topology and algebra here.

**Manifolds**

- Manifolds: Hausdorff, covered by countable euclidean open balls. ($2, 3$-dim only)
- Main objectives is to make sense of their variety.
- Examples:
  - knot complements. The variety of these are suprisingly many. (still cannot classify)
  - Surfaces: classified by orientation, genus, and number of holes ( homology theory is needed)
  - 3-manifolds: Geomerization now makes the field into something of "algebraic problems".

**Geometrization of Manifolds**

- If $M$ is an orientable surface, then $M$ can be written $\mathbb{H}^2/\Gamma$, $E^2/\Gamma$, or $S^2/\Gamma$ by the uniformization theorem. This is not unique. So we need Teichmuller spaces.
- If $M$ is an orientable compact 3-manifold, then $M$ can be canonically decomposed by spheres, disks into irreducible 3-manifolds.
- Irreducible 3-manifolds decomposes along tori into open or closed submanifolds admitting one of eight geomeric structures $S^3/\Gamma$, $E^3/\Gamma$, $\mathbb{H}^3/\Gamma$, $Nil/\Gamma$, $Sol/\Gamma$, $\widetilde{SL}(2,\mathbb{R})/\Gamma$, $\mathbb{H}^2 \times \mathbb{R}/\Gamma$, and $\mathbf{S}^2 \times \mathbb{R}/\Gamma$.
- The hyperbolic pieces are most varied.

**Orbifolds**

- Orbifolds: Hausdorff, covered by countable quotients of open balls by finite linear group actions that remember the action and the open balls. (Manifolds "are" orbifolds)
- Orbifolds are of form $M/\Gamma$ where $M$ is a universal covering orbifold and $\Gamma$ is a properly discontinous action (not free).
- Given an orbifold $O$, we can always find $M$ and $\Gamma$ and the orbifold structure is equivalent to the pair $(M, \Gamma)$.
- If $M$ is a manifold, then $M/\Gamma$ is a *good orbifold*.
- If $M$ is a compact 2-dim orbifold, then $M$ is classified by orbifold Euler characteristic.
- If $M$ is a compact 3-dim orbifod, then $M$ satisfies the geometrization.

**Orbifolds**

- Sometimes $O = N/\Gamma$ for a manifold $N$ and $\Gamma$ finite. The $O$ is *very good*.
- Selberg's Lemma: If $\Gamma$ is a finitely generated subgroup of $GL(N, \mathbb{C})$, then it has a torsion-free finite index subgroup.
- Most orbifolds here are very good.

# 3 $PSL(2, \mathbb{C})$ and hyperbolic 3-space

- `http://www.geom.uiuc.edu/docs/forum/hype/model.html`
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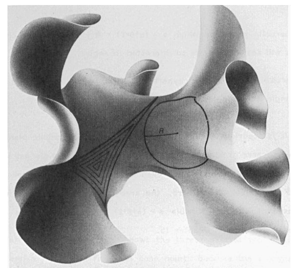
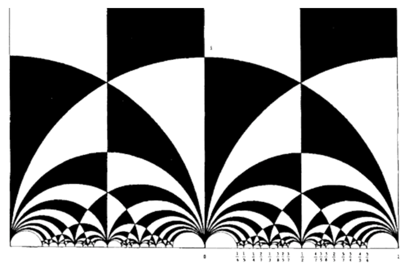




$PSL(2, \mathbb{C})$ **and hyperbolic** $3$**-space**

- $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$.
- $\mathrm{PSL}(2, \mathbb{C})$ acts on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by $z \mapsto \frac{az+b}{cz+d}$.
- $\mathbb{H}^3$ is defined as $\{(x, y, t) \in \mathbb{R}^3 | t > 0\}$.
- $t = 0$ plane is identified with $\mathbb{C}$.
- $\mathbb{H}^3$ compactifies to a closed ball with boundary $\hat{\mathbb{C}}$ in the compactification of $\mathbb{R}^3$ as $\hat{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}$.
- The boundary set is the sphere of infinity $\mathbf{S}^{2,\infty} := \hat{\mathbb{C}}$ with a complex structure.
- Each Mobius transformation on $\mathbf{S}^{2,\infty}$ extends to an action in $\mathbb{R}^3$ (Poincare extension) This is obtained by inversions in spheres perpendicular to $t = 0$ or the the planes perpendicular to $t = 0$.

- The Mobius transformations form the isometry group of the Riemannian metric given by $\delta_{ij}/t^2$.
- The angles are same as the euclidean angles.
- $\mathrm{PSL}(2, \mathbb{C})$ is isomorphic to $\mathrm{Isom}^+(\mathbb{H}^3)$. (Lie group)
- Geodesics are half circles perpendicular to $\mathbf{S}^{2,\infty}$ or a straight line parallel to the $t$-axis.

- Totally geodesic subspaces are either hemispheres or half-spaces parallel to $t$-axis.
- Horospheres are given by $t = const$ or its images under isometries. The images are spheres tangent to $t = 0$ or planes.
- In fact all isometries are generated by reflections. (Mobius type inversion actually)
- Volume form $dx \wedge dy \wedge dt$.
- Models `http://www.geom.uiuc.edu/~crobles/hyperbolic/`

$\mathbb{H}^2$

- Consider setting $y = 0$. Then we obtain $\mathbb{H}^2$ with metric $\delta_{ij}/t^2$.
- This is a totally geodesic subspace. In fact, any other $2D$-totally geodesic subspace is isometric to it.
- $\mathrm{PSL}(2, \mathbb{R})$ isomorphic to $\mathrm{Isom}^+(\mathbb{H}^2)$.
- the angles, geodesics, subspaces.
- Isometries are generated by reflections.
- Volume form $dx \wedge dt$.
- The boundary is a circle $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ in $\hat{\mathbb{R}}^3$.
- Geodesics: `http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/uhp/uhpjava.html`
- Distances: `http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/uhp/eq.html`

**Alternative view as a hyperboloid in the Lorentzian $4$-space**

- Let $V$ be a four dimensional space with a quadratic form $q(\vec{x}) = x_1^2 + x_2^2 + x_3^2 - x_4^2$.
- $V$ is decomposed into three parts $q > 0$, the positive open cone $C^+$ with $q < 0, x_4 > 0$, the negative open cone $q < 0, x_4 < 0$, and the null cone $q = 0$.
- The vectors are called spacelike, positive timelike, negative timelike, or null.
- A hyperboloid is given by $q = -1$.
- We take the upper part $\Lambda$. Then the $q$ restricts to Riemannian metric. Then $\Lambda$ is isometric with $\mathbb{H}^3$.
- Define $O^+(V, q) = O^+(1, 3)$ be the orthogonal map preserving $C^+$.

- This group is generated by the Lorenztian reflection through time-like hyperplanes.
- $\mathrm{Isom}\Lambda = PO^+(V,q)$ and $\mathrm{Isom}^+\Lambda = PSO^+(V,q)$. These are isomorphic to the previous groups.

**Poincare model**

- Consider the unit ball $B^3$ in $\mathbb{R}^3$. There is a inversion sending $\mathbb{H}^3$ onto $B^3$.
- The metric is given by $4\delta_{ij}/(1-|r|^2)^2$.
- Again the isometry group is generated by reflections in spheres orthogonal to $\partial B^3$.
- The unit disk $B^2$ is identified with the hyperbolic plane.

**The hyperbolic trigonometry**

- hyperbolic law of sines:
$$\sin A/\sinh a = \sin B/\sinh b = \sin C/\sinh c$$
- hyperbolic law of cosines:
$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$$
$$\cosh c = (\cosh A \cosh B + \cos C)/\sinh A \sinh B$$
- The triangles behave in a funny way... `http://www.math.ksu.edu/~bennett/gc/tri.html`

# 4 Subgroups of $\mathrm{PSL}(2,\mathbb{C})$

**The classifications of elements**

- Assume $\gamma \neq \mathrm{I}$.
- $\gamma$ is elliptic if $|tr\gamma| < 2$.
- $\gamma$ is parabolic of $tr\gamma = \pm 2$.
- $\gamma$ is loxodromic otherwise.
- $\gamma$ is elliptic if and only if it fixes a unique geodesic and if and only if it is conjugate to $z \mapsto e^{i\theta}z$ for $\theta \neq 0$.
- $\gamma$ is loxodromic if and only if it acts on a unique geodesic and if and only it is conjugate to $z \mapsto vz$ where $v$ is a complex number whose length is not $1$. (hyperbolic if $v$ is a positive real number)
- $\gamma$ is parabolic if and only if it acts on horospheres and if and only if it is conjugate to $(x,y,t) \to (x+a, y+b, t)$ for some real numbers $a, b$ not both zero. (This fixes a unique point of the tangency)

**Some more general theory in terms of symmetric space theory**

- $\mathbb{H}^3$ is a symmetric space of the Lie group $\mathrm{PSL}(2, \mathbb{C})$ with maximal compact group $PSU(2, \mathbb{C})$ isomorphic to $\mathrm{SO}(3, \mathbb{R})$.
- A parabolic subgroup is a subgroup fixing an infinity and acts on leaves of foliation given by a disjoint collection of horospheres. This is conjugate to a group of upper triangular matrices.
- $\mathbb{H}^3$ can be compactified by adding one point for each parabolic subgroup.
- A geodesic ends at a point of infinite and Busemann function gives us a parameter of horospheres. `http://eom.springer.de/b/b120550.htm`
- This description agrees with the above.
- Reference: Eberlein, Spaces of nonpositive curvature, Chicago

**Subgroups of** $\mathrm{PSL}(2, \mathbb{C})$

- A subgroup is *reducible* if it fixes a unique point in $\hat{\mathbb{C}}$.
- A subgroup is *elementary* if it has a finite orbit in its action on $\mathbb{H}^3 \cup \hat{\mathbb{C}}$. Otherwise it is *non-elementary*.
- Every non-elementary subgroup contains infinitely many loxodromic element, no two of which have a common fixed point.
- Let $x, y$ be elements of $\mathrm{PSL}(2, \mathbb{C})$. Then $< x, y >$ is reducible if and only if $tr[x, y] = 2$.
-

**Kleinian group**

- A *Kleinian* group is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.
- In this setting, the discreteness implies that $\Gamma$ acts properly discontinously (possibly with fixed points)
- Usually, we assume that it is non-elementary.
- $\mathbb{H}^3/\Gamma$ is a 3-dimensional orbifold (3-manifold if no torsion).
- We give two-dimensional examples. But they also act on $\mathbb{H}^3$ as a Kleinian group (called Fuchsian group).
- Fuchian group can be deformed to quasi-Fuchian groups.

**Triangle groups**

- Find a triangle in $\mathbb{H}^2$ with angles submultiples of $\pi$.
- We divide into three cases $\pi/a + \pi/b + \pi/c < 0$.
- In fact, given a surface (or 2-orbifold) $S$ with $\chi < 0$, we have $S = \mathbb{H}^2/\Gamma$ for a Fuchian group.
- Example: once-puctured torus group by Wada `http://vivaldi.ics.nara-wu.ac.jp/~wada/OPTi/index.html`
- $(2, 4, 8)$-triangle group

**The modular group $PSL(2, \mathbb{Z})$ action on $\mathbb{H}^2$.**

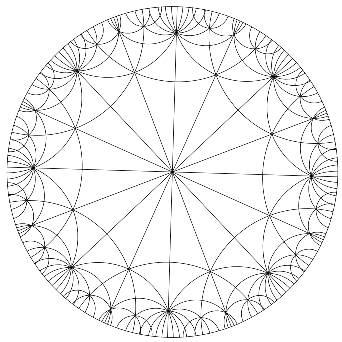
- Generated by $S : z \mapsto -1/z$, $T : z \mapsto z + 1$.
- $(2, 3, \infty)$-triangle group.

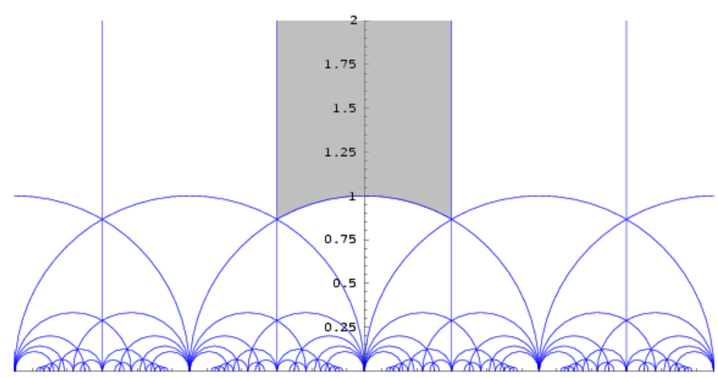
**Kleinian group**

- Let $\Gamma$ be a nonelementary Kleinian group.
- A stabilizer of a point of $\mathbb{H}^3$ is a finite subgroup.
- A stabilizer of a point of the sphere of infinity $\mathbf{S}^{2,\infty}$ can be conjugated to a subgroup $B$ with upper triangular matrices.
- $B$ can be of the following form:
  - Finite cyclic. (if it is finite)
  - A finite extension of an infinite cyclic group generated by a loxodromic or parabolic element. (if it contains a loxodromic)
  - A finite extension of $\mathbb{Z} \oplus \mathbb{Z}$ generated by two parabolic elements.
  - Essentially proved from 2-dim Bieberbach theorem.

**Cusp**

- A point $\zeta$ of $\mathbf{S}^{2,\infty}$ is a cusp point of $\Gamma$ if the stablizer consists of parabolic elements and the identity. (rank $= 2$)
- We will usually be working in finite volume case. So we do not need to know "limit set". the domain of discontinuity in $\mathbf{S}^{2,\infty}$.
- See ideal triangle examples.
- The ideal example `http://egl.math.umd.edu/software.html`

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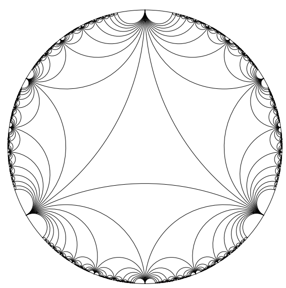

**Fundamental domain**

- A fundamental domain $F$ for a Kleinian group $\Gamma$ is a closed subset of $\mathbb{H}^3$ satisfying
  - $\bigcup_{\gamma\in\Gamma}\gamma F = \mathbb{H}^3$.
  - $F^o \cap \gamma F^o = \emptyset$ if $\gamma \neq \mathrm{I}$.
  - the boundary of $F$ has measure zero.
- $F$ is usually a polyhedron (compact or noncompact, finite or infinite sided)

**Dirichlet domains**

- A Kleinian group $\Gamma$, choose a point $p$ (not fixed)
- $D_p(\Gamma) := \{q \in \mathbb{H}^3 | d(q,p) \leq d(\gamma(q),p)$ for all $\gamma \in \Gamma\}$.
- This is a polyhedron with locally finite sides.
- Sides are bisectors of $p$ and $\gamma(p)$ for some $\gamma$.
- The triangles in the triangle reflection groups are Dirichlet domains.

**Geometrical finiteness**

- A Kleinian group is *geometrically finite* if it admits a finite sided Dirichlet domain for every (some) points. (Another way is the thick part of the convex hull is finite volume)
- $\Gamma$ is *cocompact* if $\mathbb{H}^3/\Gamma$ is compact. This is true if and only if all/some Dirichlet domain is compact. (See triangle groups)
- $\Gamma$ is of *finite covolume* if the volume of a Dirichlet domain is finite. (See ideal triangle groups)
- This is well-defined.
- If A Kleinian group $\Gamma$ is of finite covolume, then if $\Gamma$ is geometrically finite and hence finitely generated. (finitely presented) (Converse not true)
- This has a rather involved proof....

# 5 Hyperbolic manifolds and orbifolds

### Metric structures of hyperbolic manifolds in general

- Margulis constant $\epsilon > 0$.
- Given a hyperbolic manifold $M$, $M = \mathbb{H}^3/\Gamma$ for a torsion-free $\Gamma$, the set of thin parts is defined as $M_\epsilon = \{x \in \mathbb{H}^3/\Gamma | \exists x_1, x_2 \in x, d(x_1, x_2) \leq \epsilon\}$.
- The thick part is defined as $M_0 = \{x \in \mathbb{H}^3/\Gamma | \forall x_1, x_2 \in x, d(x_1, x_2) > \epsilon\}$.
- The thin part either an annulus, a Mobius band, a torus or a Klein bottle times interval (corresponds to cusps) or is a solid torus or solid Klein bottle.
- The thick part is a tame 3-manifold, i.e., either is compact or is the interior of a compact 3-manifold. (Hence, the topology is finite.)
- The orbifold versions are similar to this..(that is the thin parts are finite quotients of the above.)
- These work for general geometric manifolds...

### Commensurability

- $\Gamma_1, \Gamma_2$ subgroups of $\mathrm{PSL}(2, \mathbb{C})$. They are *directly commensurable* if $\Gamma_1 \cap \Gamma_2$ is of finite index in both $\Gamma_1$ and $\Gamma_2$.
- They are *commensurable* if $\Gamma_1$ and a conjugate of $\Gamma_2$ are directly commensurable.
- Two hyperbolic orbifolds $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$ are commensurable if their groups are so.
- Examples: Consider subgroups of a common Kleinian group.