Equivariant cohomological and cohomological rigidity of toric hyperKahler manifolds

Shintarô Kuroki (KAIST)

The 8th KAIST Geometric Topology Fair (2010)

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1. Cohomological rigidity problems in Toric Topology

Cohomological Rigidity Problem

Are M and M' homeomorphic (or diffeomorephic) if $H^*(M) \simeq H^*(M')$?

In general, the answer is NO.
E.g., the Poincaré homology sphere and the standard sphere.

However, if we restrict the class of the manifolds, then the answer is sometimes affirmative.

Example (Hirzebruch surfaces)

$$S^3 \times_{S^1} (\mathbf{C}_k \oplus \underline{\mathbf{C}}) \hookleftarrow \mathbf{C} \oplus \underline{\mathbf{C}} \qquad \xrightarrow{projectify} \qquad S^3 \times_{S^1} \mathbb{P}(\mathbf{C}_k \oplus \underline{\mathbf{C}}) \hookleftarrow \mathbf{C}P^1$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathbf{C}P^1 \qquad\qquad\qquad\qquad\qquad\qquad \mathbf{C}P^1$$

Here, S^1 acts on $S^3 \subset \mathbf{C}^2$ naturally, on $\mathbf{C}_k$ by the k-times rotation $(k \in \mathbf{Z})$ and on $\underline{\mathbf{C}}$ trivially.

We call $H_k = S^3 \times_{S^1} \mathbb{P}(\mathbf{C}_k \oplus \underline{\mathbf{C}})$ the Hirzebruch surface.

Cohomological rigidity of Hirzebruch surfaces

Theorem 1 (Hirzebruch). *For all $k \in \mathbf{Z}$,*

$$H_k \cong H_{k+2}.$$

Moreover, we have

$$H^*(H_{2\ell}) \simeq \mathbf{Z}[\alpha, \beta]/\langle \alpha^2, \beta^2 \rangle$$

and

$$H^*(H_{2\ell+1}) \simeq \mathbf{Z}[\alpha, \beta]/\langle \alpha^2, \beta(\alpha + \beta) \rangle.$$

Hence, $H_{2\ell} \ncong H_{2\ell+1}$.

$$\Downarrow$$

Therefore, Hirzebruch surfaces satisfy cohomological rigidity.

What is the meaning of the integer k?

The Hirzebruch surface $H_k = S^3 \times_{S^1} \mathbb{P}(\mathbf{C}_k \oplus \underline{\mathbf{C}})$ has T^2-action by

$$(t_1, t_2) \cdot [(x, y),\ [z : w]] \longmapsto [(x, t_1 y),\ [z : t_2 w]],$$

where $(x, y) \in S^3 \subset \mathbf{C}^2$ and $[z : w] \in \mathbf{C}P^1$.

Theorem 2 ((essentially) Hirzebruch)**.**

$$k = k' \Longleftrightarrow (H_k, T^2) \cong (H_{k'}, T^2)$$

$$\Downarrow$$

Therefore, the integer k determines their equivariant types.

Invariant of Transformation Group Theory

Let M be a space with T-action, ET be the contractible space with free T-action and $BT = ET/T$. Then, T acts on $ET \times M$ freely. Take its quotient $ET \times_T M$ (the Borel construction).

Equivariant cohomology: $H_T^*(M) = H^*(ET \times_T M)$

Remark1: $H_T^*(M)$ is not only ring but also $H^*(BT)$-algebra by

$$\begin{array}{ccc} ET \times_T M & \hookleftarrow & M \\ \pi \downarrow & & \\ BT & & \end{array} \quad \overset{H^*}{\Longrightarrow} \quad \begin{array}{ccc} H_T^*(M) & \to & H^*(M) \\ \pi^* \uparrow & & \\ H^*(BT) & & \end{array}$$

Remark2: If T is the n-dim torus, then $H^*(BT;\mathcal{R}) = \mathcal{R}[x_1, \ldots, x_n]$ where $\deg x_i = 2$.

Equivariant cohomological rigidity of Hirzebruch surfaces

Let H_k be the Hirzebruch surface. The $H^*(BT)$-algebra type of $H_{T^2}(H_k)$ is given by

$$\begin{array}{rcl} H^*(BT^2) = \mathbf{Z}[x_1, x_2] & \xrightarrow{\pi^*} & H_T^*(H_k) \simeq \mathbf{Z}[\tau_1, \tau_2, \tau_3, \tau_4]/\langle \tau_1\tau_3, \tau_2\tau_4 \rangle \\ x_1 & \longmapsto & \tau_1 - \tau_3 \\ x_2 & \longmapsto & \tau_2 - \tau_4 + k\tau_3 \end{array}$$

Proposition 1. *If $H_{T^2}(H_k) \simeq H_{T^2}(H_{k'})$ as $H^*(BT)$-algebra, then $k = k'$, i.e., $(H_k, T^2) \cong (H_{k'}, T^2)$.*

$$\Downarrow$$

Hirzebruch surfaces satisfy equivariant cohomological rigidity.

Cohomological rigidity problem in Toric Topology

The Hirzebruch surfaces are part of the toric manifolds (M^{2n}, T^n).
Theorem 3 (Masuda). *Let (M, T^n) and (M', T^n) be two toric manifolds.*

$$(M, T^n) \cong (M', T^n) \Longleftrightarrow H_T^*(M) \simeq H_T^*(M'),$$

i.e., toric manifolds satisfy the equivariant cohomological rigidity.

Cohomological Rigidity Problem of Toric manifolds

[Masuda-Suh '06] Let M and M' be two toric manifolds.

$$M \cong M' \overset{?}{\Longleftrightarrow} H^*(M) \simeq H^*(M').$$

This problem is still open but many partial affirmative answers are shown by Choi-Masuda-Suh and so on.

In this talk, we will consider this problem for **the toric hyperKähler manifolds**.

2. Toric hyperKähler manifolds

T^m acts on $\mathbf{H}^m = \mathbf{C}^m \oplus \mathbf{C}^m$ by $(z,w)\cdot t = (zt, wt^{-1})$. Then the hyperKähler moment map $\mu_{\mathbf{R}} \oplus \mu_{\mathbf{C}} : \mathbf{H}^m \to (\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m_{\mathbf{C}})^*$ defined by

$$\mu_{\mathbf{R}}(z,w) = \frac{1}{2}\sum_{i=1}^{m}(|z_i| - |w_i|)\partial_i \in (\mathfrak{t}^m)^*;$$

$$\mu_{\mathbf{C}}(z,w) = 2\sqrt{-1}\sum_{i=1}^{m}(z_i w_i)\partial_i \in (\mathfrak{t}^m_{\mathbf{C}})^*.$$

For a subgroup $K \overset{\iota}{\hookrightarrow} T^m$, we have the hyperKähler moment map

$$\mu_{HK} : \mathbf{H}^m \to \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbf{C}}$$

by $\mu_{HK} = (\iota^* \oplus \iota^*_{\mathbf{C}}) \circ (\mu_{\mathbf{R}} \oplus \mu_{\mathbf{C}})$.

Toric hyperKähler variety: $\mu_{HK}^{-1}(\alpha, 0)/K$ where $\alpha \neq 0 (\in \mathfrak{k}^*)$

Properties of toric hyperKähler varieties

• A toric hyperKähler variety $M_\alpha = \mu_{HK}^{-1}(\alpha, 0)/K$ is a $4n$-dimensional orbifold, where $n = m - \dim K$.
• M_α has the $T^n = T^m/K$-action.
• This T^n-action is hyperhamiltonian, i.e., this action preserves the hyperKähler structure and has a hyperKähler moment map $\widetilde{\mu}_{\widehat{\alpha}} = \widetilde{\mu}_{\mathbf{R}} \oplus \widetilde{\mu}_{\mathbf{C}}$ such that

$$\begin{aligned} \widetilde{\mu}_{\mathbf{R}}[z, w] &= \frac{1}{2}\sum_{i=1}^{m}(|z_i| - |w_i|)\partial_i - \widehat{\alpha} \in \ker \iota^* \simeq (\mathfrak{t}^n)^* \subset (\mathfrak{t}^m)^*; \\ \widetilde{\mu}_{\mathbf{C}}[z, w] &= 2\sqrt{-1}\sum_{i=1}^{m}(z_i w_i)\partial_i \in \ker \iota_{\mathbf{C}}^* \simeq (\mathfrak{t}_{\mathbf{C}}^n)^* \subset (\mathfrak{t}_{\mathbf{C}}^m)^*, \end{aligned}$$

where $\widehat{\alpha} \in (\mathfrak{t}^n)^*$ such that $\iota^*(\widehat{\alpha}) = \alpha$.

$$\widetilde{\mu}_{\widehat{\alpha}} : M_\alpha \to (\mathfrak{t}^n)^* \oplus (\mathfrak{t}_{\mathbf{C}}^n)^* \simeq \ker(\iota^* \oplus \iota_{\mathbf{C}}^*) \hookrightarrow (\mathfrak{t}^m)^* \oplus (\mathfrak{t}_{\mathbf{C}}^m)^* \xrightarrow{\iota^* \oplus \iota_{\mathbf{C}}^*} \mathfrak{k}^* \oplus \mathfrak{k}_{\mathbf{C}}^*,$$

Example

Let $K = \Delta$ be the diagonal subgroup in T^{n+1}.

The moment map $\mu_{HK} =: \mathbb{H}^{n+1} \to \mathbf{R} \oplus \mathbf{C}$ is defined by

$$\mu_{HK}(z, w) = \frac{1}{2} \sum_{i=1}^{n+1} (|z_i| - |w_i|) \oplus 2\sqrt{-1} \sum_{i=1}^{n+1} (z_i w_i).$$

Let $\alpha = 1 \in \mathbf{R}$ It is easy to show that

$$\mu_{HK}^{-1}(1, 0)/\Delta = T^* \mathbf{C}P^n$$

with the induced $T^n = T^{n+1}/\Delta$ action on $\mathbf{C}P^n$.

Hyperplane arrangements

To define the toric hyperKähler variety M_α, we need to use the exact sequence

$$(\mathfrak{t}^n)^* \xrightarrow{\rho^*} (\mathfrak{t}^m)^* \xrightarrow{\iota^*} \mathfrak{k}^*,$$

and the non-zero element $\alpha \in \mathfrak{k}^*$.

There is a lift $\widehat{\alpha} \in (\mathfrak{t}^m)^*$ of α, i.e., $\iota^*(\widehat{\alpha}) = \alpha$.

Hyperplane arrangement of M_α: $\mathcal{H}_{\widehat{\alpha}} = \{H_1, \ldots, H_m\}$ such that

$$H_i = \{x \in (\mathfrak{t}^n)^* \mid \langle \rho^*(x) + \widehat{\alpha}, \mathbf{e}_i \rangle = 0\}$$

where $\mathbf{e}_i$ $(i = 1, \ldots, m)$ is the basis of $\mathfrak{t}^m \simeq \mathbf{R}^m$.

Remark: $\rho_*(\mathbf{e}_i) \in \mathfrak{t}^n$ determines the (weighted) normal vector of H_i and $\langle \widehat{\alpha}, \mathbf{e}_i \rangle$ determines the position of H_i.

Example

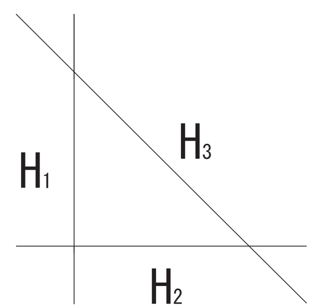
$T^*\mathbf{C}P^2$ is constructed by $\Delta \overset{\iota}{\hookrightarrow} T^3$ and $\alpha = 1 \in \mathfrak{k}^*$. Then

$$\iota^* : (\mathfrak{t}^3)^* \ni (a,b,c) \mapsto a+b+c \in \mathfrak{k}^*$$
$$\rho^* : (\mathfrak{t}^2)^* \ni (x,y) \mapsto (x,y,-x-y) \in (\mathfrak{t}^3)^*.$$

We may take $\widehat{\alpha} = (1,0,0) \in (\mathfrak{t}^3)^*$.

Because $H_i = \{(x,y) \in (\mathfrak{t}^2)^* \mid \langle (x,y,-x-y) + (1,0,0), \mathbf{e}_i \rangle = 0\}$,

$$\begin{aligned} H_1 &= \{(-1,y) \mid y \in \mathbf{R}\}; \\ H_2 &= \{(x,0) \mid x \in \mathbf{R}\}; \\ H_3 &= \{(x,-x) \mid x \in \mathbf{R}\}. \end{aligned}$$



$\mathcal{H}_{(1,0,0)}$ of $T^*\mathbf{C}P^2$

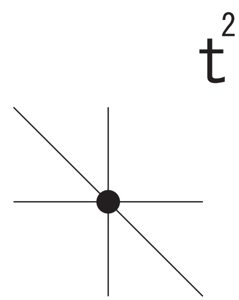
Fundamental Theorem

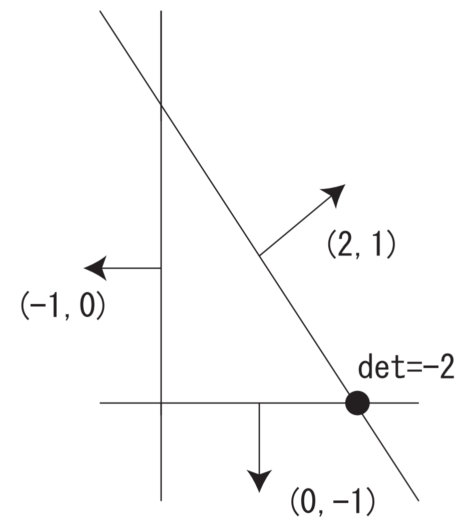
Theorem 4 (Bielawski-Dancer). M *is a smooth manifold* $\overset{\text{iff}}{\Longleftrightarrow}$ *its hyperplane arrangement* $\mathcal{H} = \{H_i\}$ *is smooth, i.e.,*

1. $\dim \cap_{i \in I} H_i = n - \# I$*;*

2. if $\# I = n$ *then* $\{\rho_*(\mathbf{e}_i) \mid i \in I\}$ *spans* $(\mathfrak{t}^n_{\mathbf{Z}})^*$*.*

The right two figures do not occur as the hyperplanes of toric hyperKähler manifolds.





Fundamental fact

There is the following correspondence.

Smooth $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ up to hyperhamiltonian.	$\longrightarrow$ $\longleftarrow$	Smooth $\mathcal{H}_{\widehat{\alpha}}$ up to weighted, cooriented, affine arrangement.

Here, $(M_\alpha, T^n, \mu_{\widehat{\alpha}}) \equiv_w (M'_{\alpha'}, T^n, \mu'_{\widehat{\alpha}'})$ as weak hyperhamiltonian $\overset{\text{def}}{\Longleftrightarrow}$ there is a weak equivariant hyperKähler isometry $f : M_\alpha \to M'_{\alpha'}$ such that $\varphi^* \circ \mu_{\widehat{\alpha}} = f \circ \mu'_{\widehat{\alpha}'}$, where $\varphi : T^n \to T^n$ is the isometry such that $f(x \cdot t) = f(x)\varphi(t)$. If φ is the identity map, then $(M_\alpha, T^n, \mu_{\widehat{\alpha}}) \equiv (M'_{\alpha'}, T^n, \mu'_{\widehat{\alpha}'})$ as hyperhamiltonian.

3. Main Theorem

Recall the definition of toric hyperKähler manifolds $M = \mu_{HK}^{-1}(\alpha, 0)/K$.

$$\mu_{HK} =: \mathbf{C}^m \oplus \mathbf{C}^m \xrightarrow{\mu_{\mathbf{R}} \oplus \mu_{\mathbf{C}}} (\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m_{\mathbf{C}})^* \xrightarrow{\iota^* \oplus \iota^*_{\mathbf{C}}} \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbf{C}} \text{ such that}$$

$$\begin{aligned} \mu_{\mathbf{R}}(z, w) &= \frac{1}{2} \sum_{i=1}^{m} (|z_i| - |w_i|)\partial_i \in (\mathfrak{t}^m)^*; \\ \mu_{\mathbf{C}}(z, w) &= 2\sqrt{-1} \sum_{i=1}^{m} (z_i w_i)\partial_i \in (\mathfrak{t}^m_{\mathbf{C}})^*. \end{aligned}$$

There is the extra S^1-action on the second $\mathbf{C}^m$-factor (w-factor) in $\mu_{HK}^{-1}(\alpha, 0)$ and this S^1-action commutes with K-action.

$$\Downarrow$$

Toric hyperKähler manifold has the $T^n \times S^1$-action.

Equivariant cohomological rigidity theorem

Theorem 5. $(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv_w (M'_{\alpha'}, T, \mu'_{\widehat{\alpha}'}) \overset{\text{iff}}{\Longleftrightarrow}$ *there is a weak algebra isomorphism* $f_T^* : H_T^*(M_\alpha; \mathbf{Z}) \to H_T^*(M'_{\alpha'}; \mathbf{Z})$ *such that* $f_T^*(\widehat{\alpha}) = \widehat{\alpha}'$.

Theorem 6. $(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv (M'_{\alpha'}, T, \mu'_{\widehat{\alpha}'}) \overset{\text{iff}}{\Longleftrightarrow}$ *For the extra* S^1*-actions, there is an algebra isomorphism* $f_{T\times S^1}^* : H_{T\times S^1}^*(M_\alpha; \mathbf{Z}) \to H_{T\times S^1}^*(M'_{\alpha'}; \mathbf{Z})$ *such that* $f_T^*(\widehat{\alpha}) = \widehat{\alpha}'$.

Corollary 1. *Toric hyperKähler manifolds satisfy the weak equivariant cohomological rigidity for* T^n*-action and the equivariant cohomological rigidity for* $T^n \times S^1$*-action.*

Cohomological rigidity theorem

Theorem 7. *Two toric hyperKähler manifolds are diffeomorphic* $\overset{\text{iff}}{\Longleftrightarrow}$ *their cohomology rings are isomorphic and their dimensions are same.*

Theorem 8 (Bielawsky)**.** *Let $\mathcal{M}_n$ be the set of all complete, connected, $4n$-dimensional, hyperKähler manifolds with effective, hyperhamiltonian T^n-actions. Then all elements in $\mathcal{M}_n$ are diffeomorphic to toric hyperKähler manifolds, and vice versa.*

Corollary 2. *$\mathcal{M}_n$ satisfies the cohomological rigidity.*

Remark of the cohomological rigidity theorem

$T^*\mathbf{C}P^n$ and $T^*\mathbf{C}P^n \times \mathbf{H}^\ell$ are examples of toric hyperKähler manifolds.

If $\ell \neq 0$, it is sasy to show that

$$H^*(T^*\mathbf{C}P^n) \simeq H^*(T^*\mathbf{C}P^n \times \mathbf{H}^\ell).$$

but

$$T^*\mathbf{C}P^n \not\cong T^*\mathbf{C}P^n \times \mathbf{H}^\ell.$$

Therefore, we need the condition of the dimension in the cohomological rigidity theorem of toric hyperKähler manifolds.

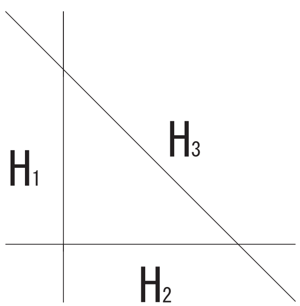
4. Outline of proof

Theorem 9 (Konno). *Let* (M, T) *be a toric hyperKähler manifold and* $\mathcal{H} = \{H_1, \ldots, H_m\}$ *be its hyperplane arrangement. . Then*

$$H_T^*(M; \mathbf{Z}) \simeq \mathbf{Z}[\tau_1, \ldots, \tau_m]/\mathcal{I}$$

where $\deg \tau_i = 2$, *and the ideal* $\mathcal{I}$ *is generated by* $\prod_{j \in J} \tau_j$ *such that* $\cap_{j \in J} H_j = \emptyset$.

For example, recall $(T^*\mathbf{C}P^2, T^2)$ has the following arrangement.



$\mathcal{H}_{(1,0,0)}$ of $T^*\mathbf{C}P^2$

$$\begin{aligned} H_i \cap H_j &\neq \emptyset; \\ H_1 \cap H_2 \cap H_3 &= \emptyset. \end{aligned}$$

Therefore, $H_T^*(T^*\mathbf{C}P^2) \simeq \mathbf{Z}[\tau_1, \tau_2, \tau_3]/\langle \tau_1 \tau_2 \tau_3 \rangle$.

Outline of proof (Equivariant cohomological rigidity)

1. By using the Konno's theorem, we can define the hyperplane arrangement in $H_T^*(M)$.

$$\Downarrow$$

2. If (M, T) is a toric hyperKähler manifold, its hyperplane arrangement and the hyperplane arrangement in $H_T^*(M)$ are equivalent (i.e., same arrangement).

$$\Downarrow$$

3. For the generator $\tau \in H_T^*(M)$, we can define $Z(\tau)$ called the zero length of τ by the number of $\tau|_p = 0$ for $p \in M^T$.

$$\Downarrow$$

$$\Downarrow$$

4. If $Z(\tau) = 0$, then $M = M' \times \mathbf{H}$ for the unique toric hyperKähler $(4n-4)$-dimensional manifold. Hence, we may regard $Z(\tau) \neq 0$.

$$\Downarrow$$

5. If $f : H_T^*(M_\alpha) \simeq H_T^*(M_{\alpha'})$ as weak $H^*(BT)$-algebra, then $f : \{\tau_1, \ldots, \tau_m\} \to \{\tau'_1, \ldots, \tau'_m\}$ up to sign. Therefore, their hyperplane arrangemets are equivalent up to coorinetations.

It follows that $(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv_w (M'_{\alpha'}, T, \mu_{\widehat{\alpha}'})$ by the fundamental fact of toric hyperKähler manifolds.

Remark: The extra S^1-action determines the coorientation of hyperplanes. Hence, it determines the weighted, cooriented hyperplane arrangements.

Outline of proof (Cohomological rigidity)

Theorem 10 (Bielawski-Dancer). *The diffeomorphism type of toric hyperKähler manifolds does not depend on the combinatorial structure of their hyperplane arrangements.*

Therefore, by using Theorem 4 (smoothness of M), the diffeomorphism types of toric hyperKähler manifolds are the following two cases:

$$M_1(k_1, \ldots, k_n);$$
$$M_2(k_0, k_1, \ldots, k_n),$$

where k_i is the number of hyperplanes which are orthogonal with $\mathbf{e}_i$ $(i = 1, \ldots, n)$ and k_0 is the number of hyperplanes which are orthogonal with $\mathbf{e}_1 + \cdots + \mathbf{e}_n$.

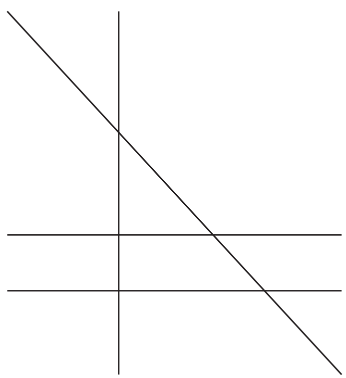
Examples of $M_1(k_1, k_1, \ldots, k_n)$ and $M_2(k_0, k_1, \ldots, k_n)$

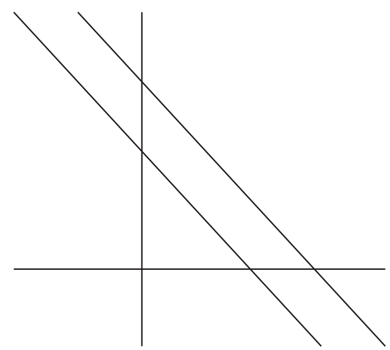
The following left is $M_1(3, 2)$ and the right is $M_2(1, 2, 1)$:

Final step of the proof

If $f : H^*(M_1(k_1,\ldots,k_n)) \simeq H^*(M_1(k'_1,\ldots,k'_n))$, then $(k_1,\ldots,k_n) \equiv (k'_1,\ldots,k'_n)$ up to permutation by comparing $Ann(\tau)$ and $Ann(f(\tau))$. (By the same argument, we can also prove for the case of M_2)

For example, the following $M_2(1,2,1)$ and $M_2(2,1,1)$ are diffeomorphic:





Therefore, by Theorem 10 (Bielawski-Dancer), we can easy to construct the diffeomorpshism.