Bounds for minimal dilatations of pseudo-Anosovs

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Hyperbolic geometry:

algorithmic, number theoretic and numerical aspects.

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Introduction

Question:

Which surface automorphism is the "simplest"?

i.e.

* Which surface automorphism have minimal dilatation?
* Which surface automorphism have minimal volume?

Talk Plan

1. Surface automorphisms – dilatation, entropy and volume
2. Entropy vs Volume
3. On the minimal dilatation

Why surface automorphism?

One reason is very simple.

Topological classification of surface had finished long time ago. So, next object must be maps from surface to itself.

Another reason is it helps to understand 3-manifold.

Conj (Virtually fibered conjecture)**.**

M: closed, irreducible atoroidal 3-manifold with infinite π_1,

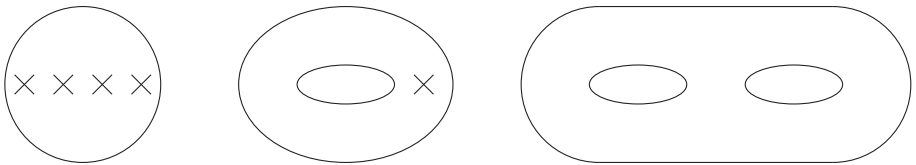
Then M has a finite cover which is a surface bundle over the circle.

1 Surface Automorphisms

Surface

D_n : n-punctured disk

$\Sigma_{g,n}$: n-punctured surface of genus=g



Mapping class group

The set of $f : \Sigma_{g,n} \to \Sigma_{g,n}$ becomes a group with composition as a product. The mapping class group is the quotient of this group by isotopy.

$$\mathcal{M}(\Sigma_{g,n}) = \{f : \Sigma_{g,n} \to \Sigma_{g,n}\}/\sim_{\text{isotopy}}$$

Topological Entropy

Def. *$f : X \to X$, C: open cover of X*

$$\mathrm{ent}(f, C) = \lim_{n\to\infty} \frac{1}{n} \log |C \vee f^{-1}(C) \vee \cdots f^{-n+1}(C)|$$

$$\mathrm{ent}(f) = \sup_{C} \mathrm{ent}(f, C)$$

where $A \vee B$ is a common refinement of A and B.

Classification Theorem

Def.

φ is periodic $\Leftrightarrow \exists n,\ \varphi^n = \mathrm{id}$

(Virtually identity)

φ is reducible $\Leftrightarrow \exists \mathcal{C} = \{C_1, C_2, \cdots, C_k\} \subset \Sigma,\ \varphi(\mathcal{C}) = \mathcal{C}$

(φ is a mapping class on smaller surface $\Sigma \setminus \mathcal{C}$)

Thm (Nielsen, Thurston)**.**

Non periodic irreducible mapping class is pseudo-Anosov.

Anosov map

Example

$f : T^2 \to T^2$, induced by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$

eigen values and eigen vectors are

* $\lambda = \dfrac{2}{3+\sqrt{5}} \approx 2.618$ and $v = \left(-1 + \dfrac{2}{3+\sqrt{5}},\ 1\right)$
* $\lambda^{-1} = \dfrac{2}{3-\sqrt{5}} \approx 0.382$ and $v' = \left(-1 + \dfrac{2}{3-\sqrt{5}},\ 1\right)$

f preserve two foliation $\mathcal{F}^s$ and $\mathcal{F}^u$.

The "width" of the foliation stretch/shrink by λ.

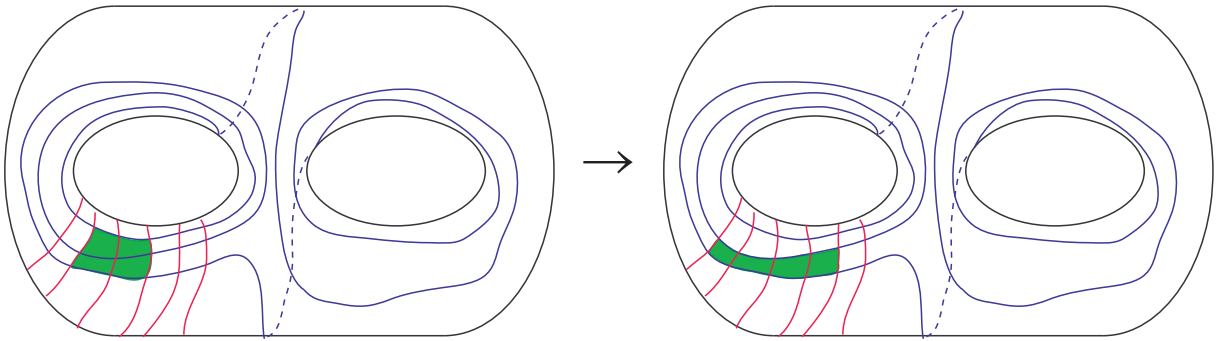
Remark that degree of minimal polynomial of λ is always 2.

Def

A map f is pseudo-Anosov. $\Leftrightarrow$ There are two invariant measured foliations (L^s, μ^s), (L^u, μ^u) on S and a positive constant $\lambda_f > 1$ such that

$$f(L^s, \mu^s) = (L^s, \lambda\mu^s), \quad f(L^u, \mu^u) = (L^u, \lambda^{-1}\mu^u).$$

λ_f is called the dilatation of f.



Def

A mapping class φ is pseudo-Anosov. $\Leftrightarrow$ There is a pseudo-Anosov map $f \in \varphi$.

The dilatation of mapping class φ is $\lambda_\varphi := \lambda_f$.

Fact

Degree of minimal polynomial of λ is bounded by a constant which only depends on the topology of Σ.

We can define minimal dilatation as

$$\lambda_{\min}(g,n) := \min_{\varphi \in \mathcal{M}(\Sigma_{g,n})} \lambda_\varphi$$

Fact

$$\mathrm{ent}(\varphi) = \log(\lambda_\varphi)$$

We can define minimal entropy as

$$\mathrm{ent}_{\min}(g,n) := \log(\lambda_{\min}(g,n))$$

Def

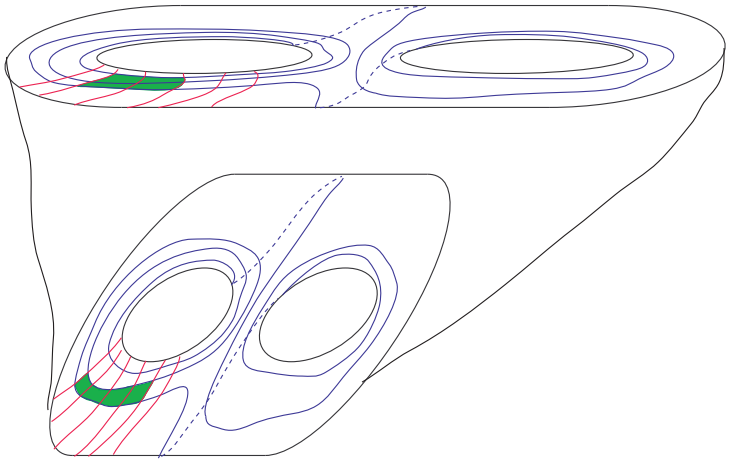
The mapping torus T_f of f is

$$T_f = S \times [0,1] / \sim_f$$

where $(x,0) \sim (f(x),1)$.

For a mapping class φ,

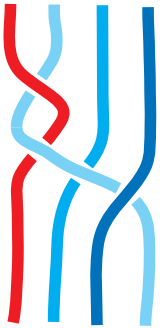
we can define $T_\varphi := T_f \quad (f \in \varphi)$.

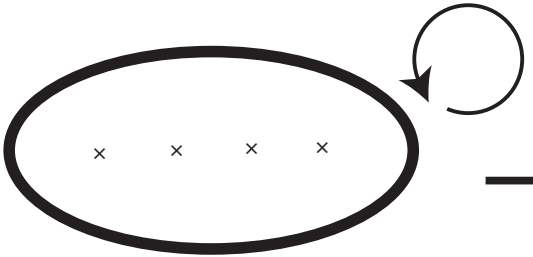


Thm (Thurston)**.**

$$\varphi : \text{pseudo} - \text{Anosov} \Leftrightarrow T_\varphi : \text{hyperbolic}$$

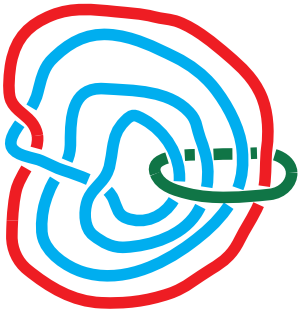
2 Entropy vs hyperbolic volume

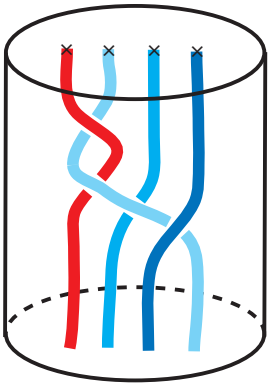




mapping class

→ **Entropy**
= log(dilatation)
(complexity of map)





mapping torus

→ **Hyperbolic Volume**
(complexity of 3-mfd)

Computer Experiment

We have computed the pseudo-Anosovness, the entropy and the volume for all words of mapping classes of punctured disk of limited length using `SnapPea` (Jeff Weeks) and `trains` (Toby Hall).

The following graphs are mapping classes of

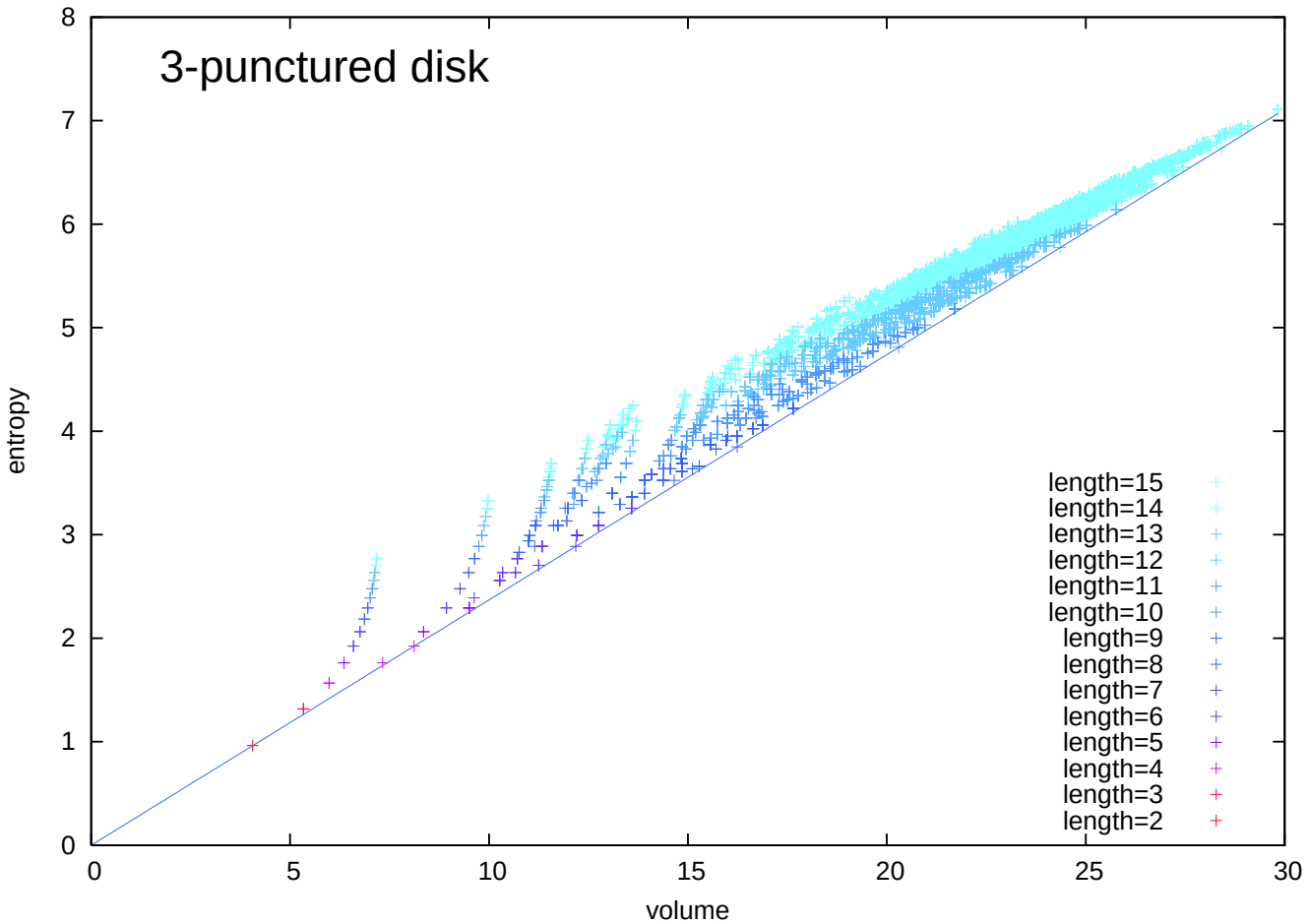
3-punctured disk (3-braids) up to length=15,

4-punctured disk (4-braids) up to length=12,

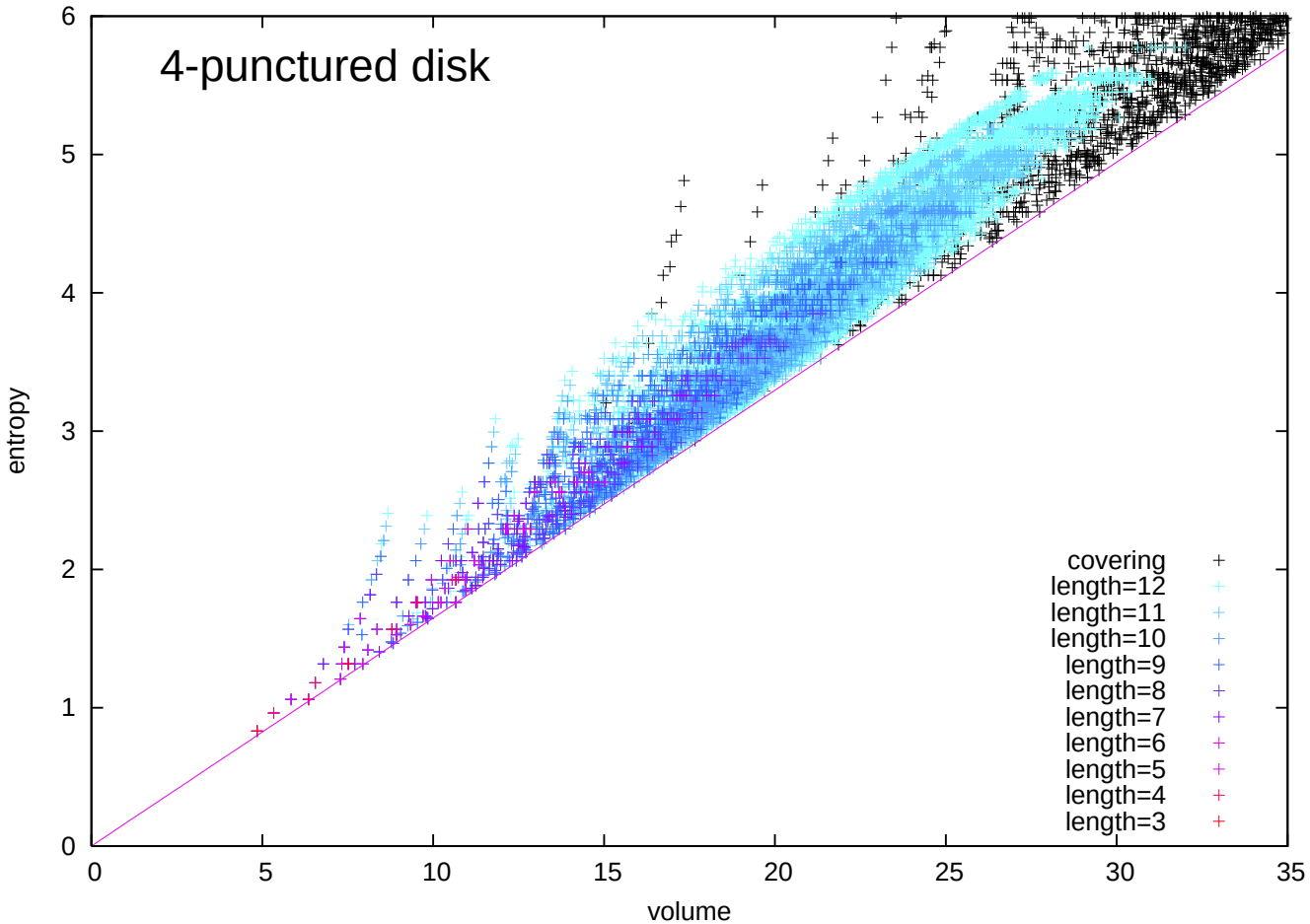
5-punctured disk (5-braids) up to length=10, and

6-punctured disk (6-braids) up to length=9.

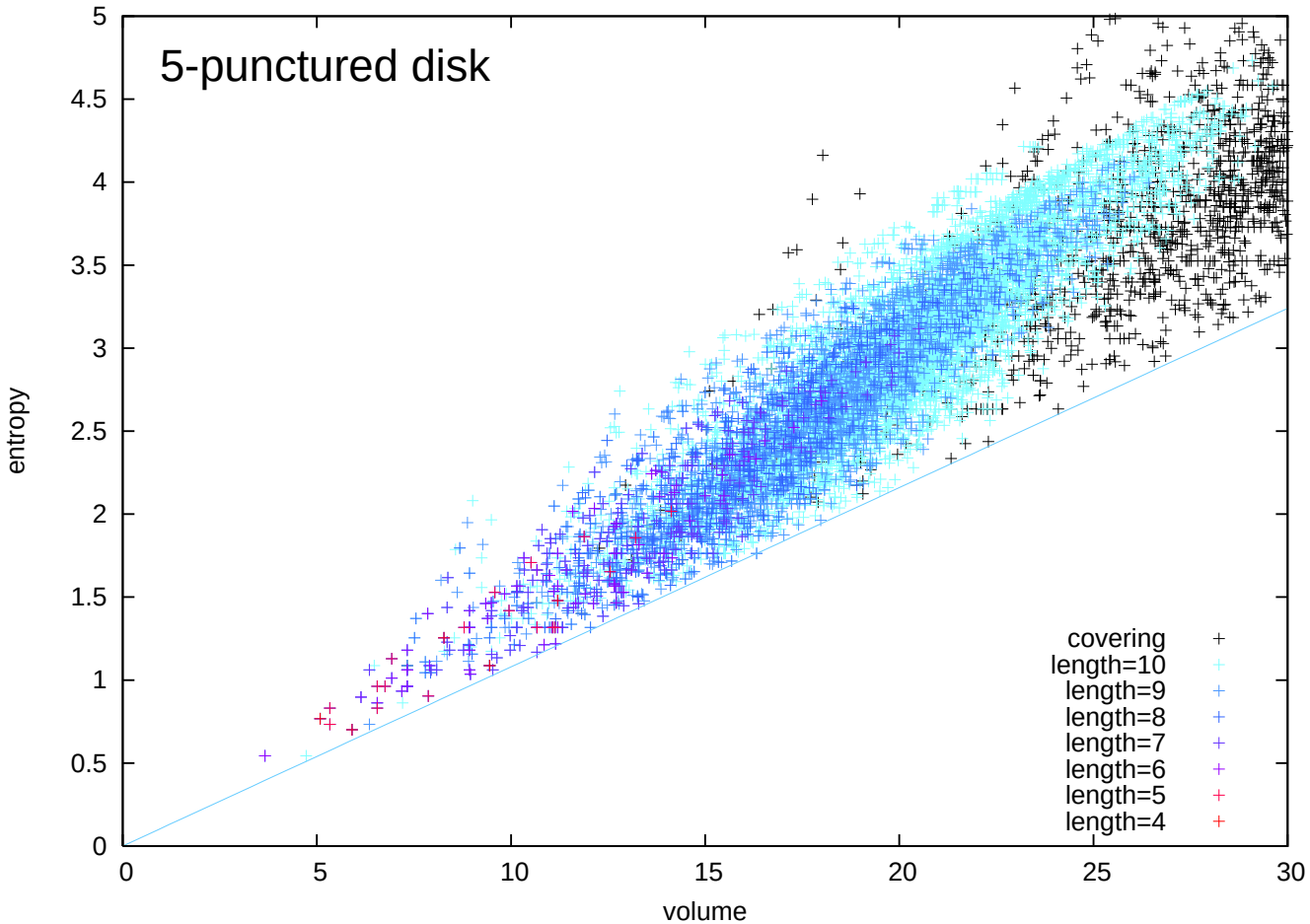
Computer experiment: D_3



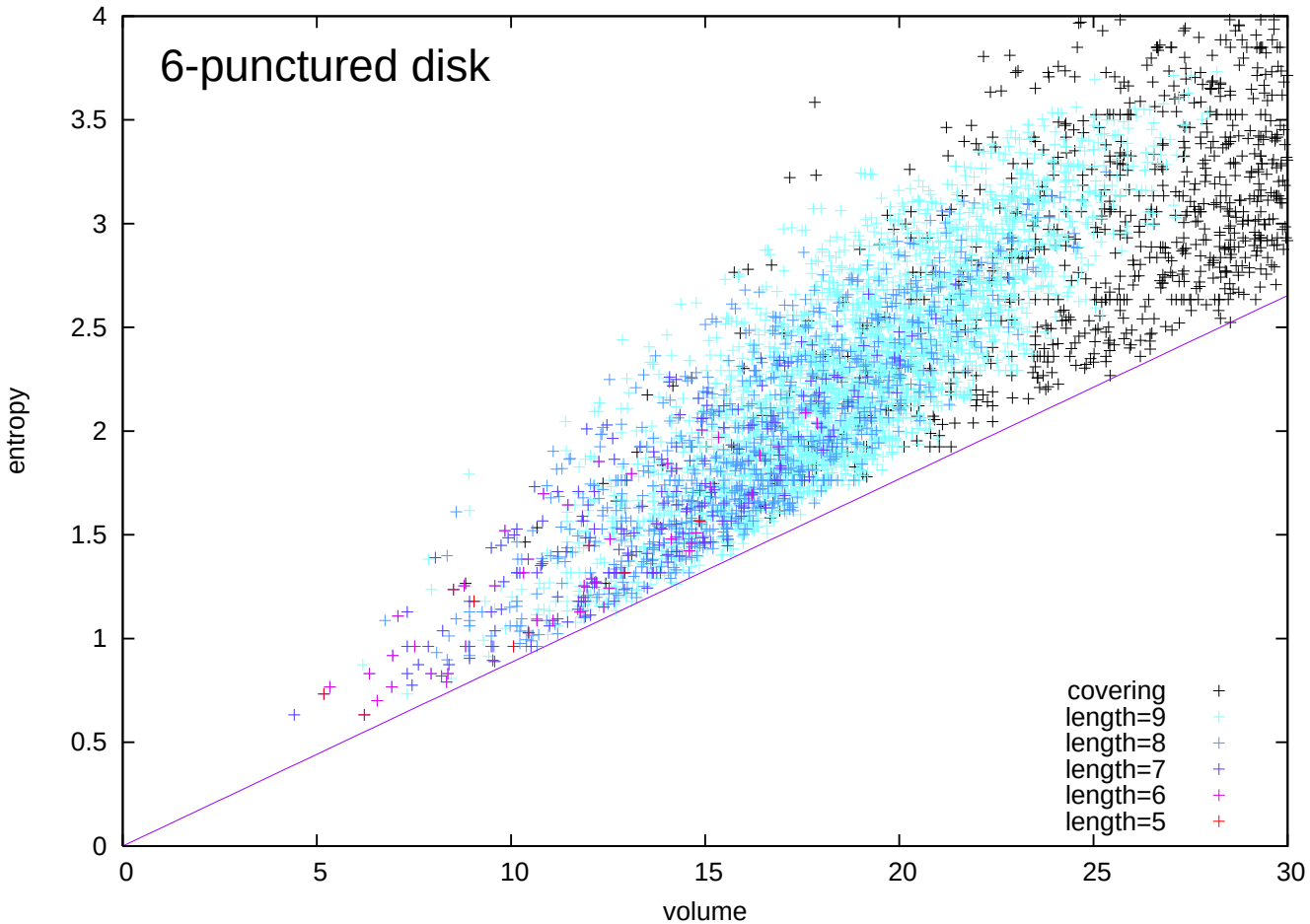
Computer experiment: D_4



Computer experiment: D_5



Computer experiment: D_6



Thm (Kojima - Kin - T).

$\exists c_1 = c_1(\Sigma)$ *s.t. for all pseudo-Anosov* $f : \Sigma \to \Sigma$,

$$\mathrm{ent}(f) \geq c_1 \ \mathrm{vol}(f)$$

This theorem is the direct conclusion of the following two theorems.

Thm (Brock, 2003).

$$\mathrm{vol}(f) < K \cdot \inf_{x \in \mathcal{T}(S)} d_{WP}(x, f(x))$$

Thm (Linch, 1974).

$$d_{WP}(x, y) < \sqrt{\mathrm{area}(\Sigma)} \cdot d_T(x, y)$$

The converse inequality holds under some geometric restriction.

Thm (Kojima - Kin - T)**.**
$\exists c_2 = c_2(\Sigma, \varepsilon)$ *s.t. for any pseudo-Anosov* $f : \Sigma \to \Sigma$ *whose mapping torus* T_f *has no closed geodesics of length* $< \varepsilon$,

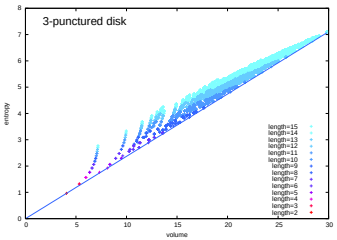
$$\mathrm{ent}(f) \leq c_2 \ \mathrm{vol}(f)$$

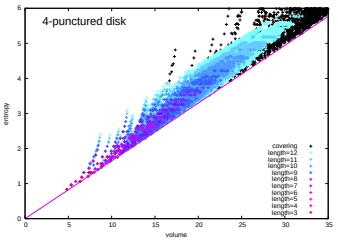
Kojima extend the result for non pseudo-Anosov maps.

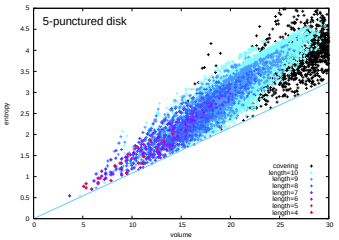
Thm (Kojima)**.**
$\exists c = c(\Sigma, \varepsilon)$ *s.t. for any* $f : \Sigma \to \Sigma$ *with some geometric condition,*

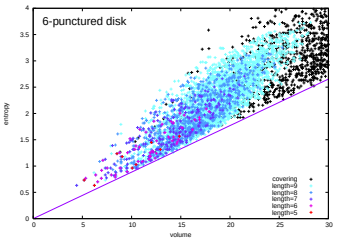
$$c^{-1} \ \mathrm{ent}(f) \leq ||T_f||_{\mathrm{Gr}} \leq c \ \mathrm{ent}(f)$$

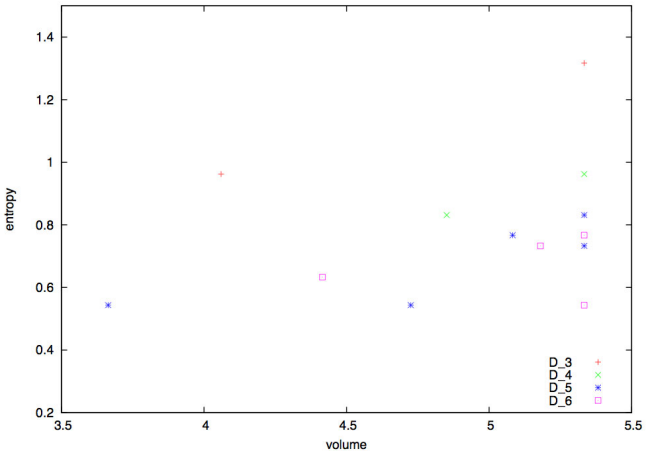
Manifolds with small volume and small entropy



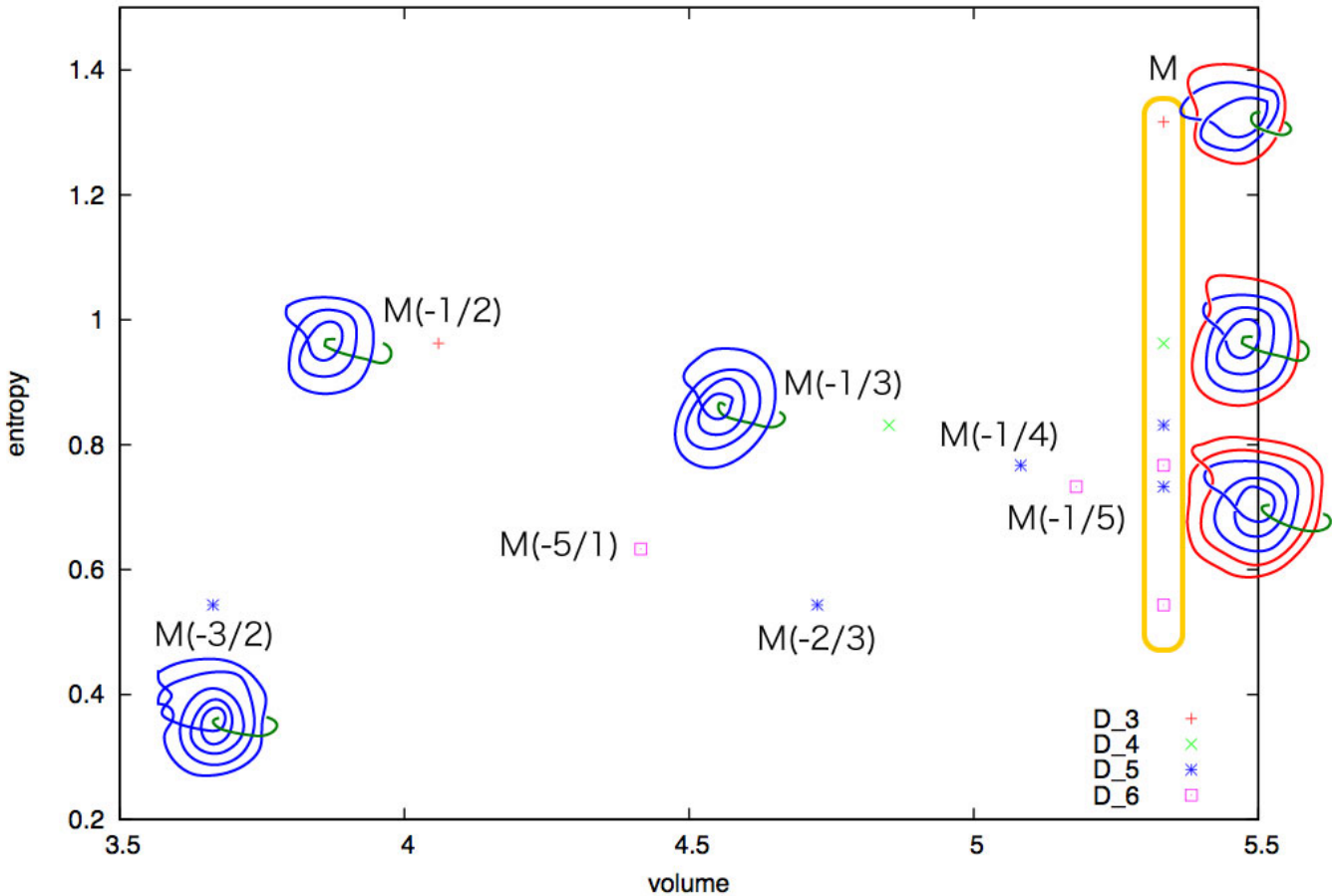








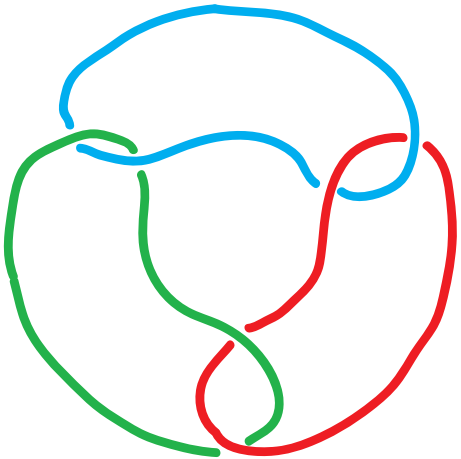
Manifolds with small volume (examples of fiber bundle structure)



The magic manifold M

$M = S^3 - C_3$

$M(p, q)$: (p, q)-Dehn surgery along one component of M



3-chain link C_3

3 Minimal entropy

known facts

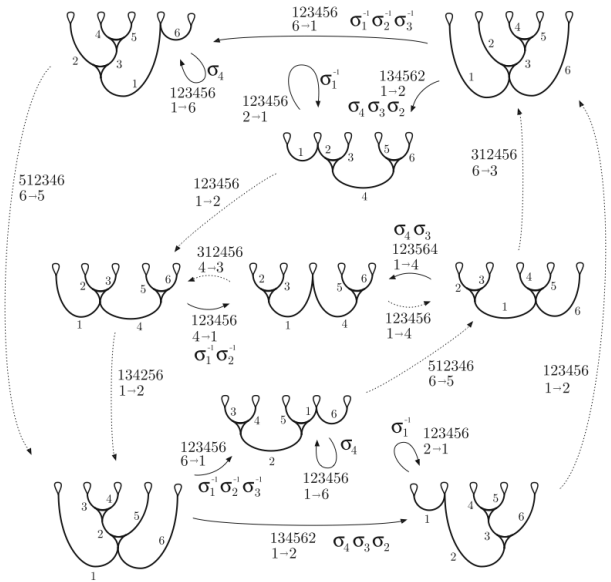
It is difficult to determine the minimal dilatations $\lambda_{\min}(g,n)$. Known minimal dilatations are for $\Sigma_{0,4}, \Sigma_{1,0}, \Sigma_{1,1}, \Sigma_{0,5}$, and $\Sigma_{2,0}$.

Thm (Ham-Song, 2007).

$$\lambda_{\min}(5,0) \approx 1.72208$$

which is the largest zero of

$$x^4 - x^3 - x^2 - x + 1.$$



Upper bounds

There are many bounds for minimal dilatation/entropy. For example by Penner, Hironaka-Kin, Minakawa, and Hironaka.

Thm (Hironaka). *Let $r(k,l)$ be the largest real root of the following polynomial*

$$t^{2k} - t^{k+l} - t^k - t^{k-l} + 1 = 0.$$

- $\lambda_{\min}(g,0) \leq r(g+1,3) \quad (g \equiv 0,1,3,4 \pmod 6)$
- $\lambda_{\min}(g,0) \leq r(g+1,1) \quad (g \equiv 2,5 \pmod 6)$

Thm (Hironaka).

$$\limsup_{g\to\infty} \chi(\Sigma_{g,0})\ \mathrm{ent}_{\min}(g,0) \leq 2\log\left(\frac{3+\sqrt{5}}{2}\right)$$

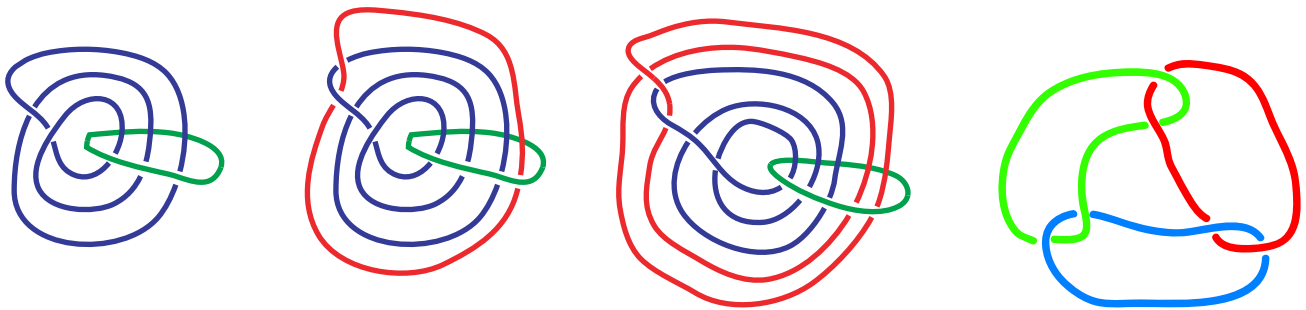
To find better bounds

Our strategy is very simple.

Compute entropies which "live" in the magic 3-manifold M.

i.e.

Find $f : \Sigma g, n \to \Sigma g, n$ s.t. T_f is isomorphic to M or some Dehn filling $M\left(\frac{p_1}{q_1}\right)$, $M\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$, $M\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$



Remark. *All of known minimal entropies "live" in M.*

Hironaka's theorem comes from $M\left(\frac{-1}{2}\right)$.

Fibrations in M

Thurston (semi) norm

Thurston norm $|| \cdot || : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ is defined as follows.

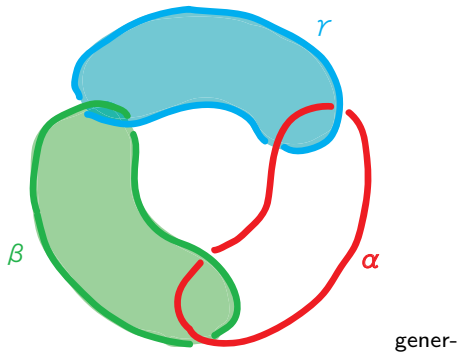
- For integral homology classes,
  $|| \sigma || = \min_S \{\chi_-(S) \mid \sigma = [S]\}$
  $\chi_-(S) = \sum_i \max\{-\chi(S_i), 0\}$ (S_i is a connected component of S)
- Extend linearly for rational homology classes.
- There are unique extension for real homology classes.

Today you can think $|| \cdot ||$ just as negative of Euler characteristic.

How many fiber bundle structures does M admit?

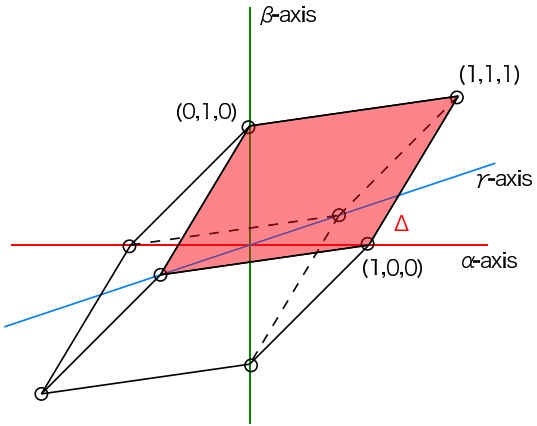
Cor (Thurston). *$H_2(M, \partial M) = \mathbb{Z}^3$ and the unit ball is shown in below. Every 2-cell is a fiber face.*

(Δ: fiber face $\Rightarrow$ lattice point in the cone of Δ is a fiber surface)



gener-

ators of H_2

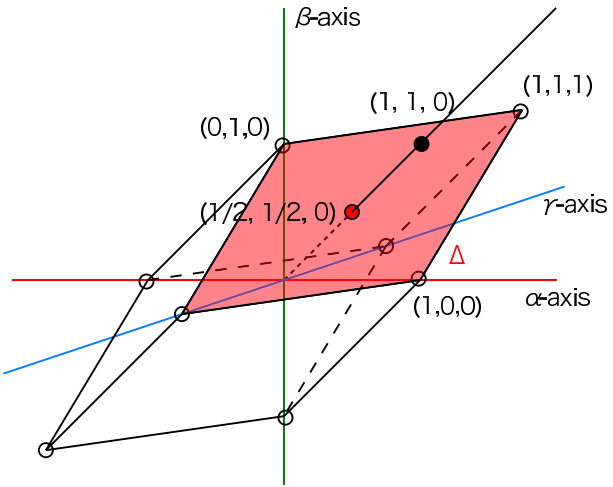


unit ball of Thurston norm

We can concentrate on $\Delta = \{(a, b, c) | a + b - c = 1,\ a > 0, b > 0, a > c, b > c\}$ because of the symmetries of the link.

In other words

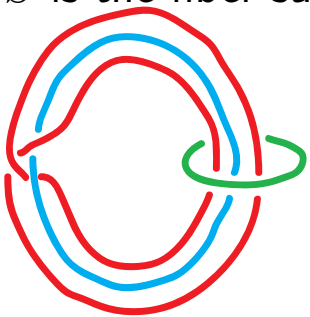
For each rational point $\delta = \dfrac{a\alpha + b\beta + c\gamma}{k}$ in the face Δ, there is a fiber bundle whose fiber surface S has homology class $[S] = k\delta = a\alpha + b\beta + c\gamma$.



red: $\dfrac{\alpha + \beta}{2} \in \Delta$

black: $[S] = \alpha + \beta$

S is the fiber surface of



Compute the topology of a fiber surface

Prop. *Let $\sigma = a\alpha + b\beta + c\gamma$ be an integral homology class in the cone of Δ and S be the realizing surface for σ.*

- $\chi_-(S) = a + b - c$
- $n = |\partial S| = \gcd(a, b+c) + \gcd(b, c+a) + \gcd(c, a+b)$
- *boundary slopes of S are* $\left(-\frac{b+c}{a}, -\frac{c+a}{b}, -\frac{a+b}{c}\right)$
- *genus* $g(S) = (\chi_-(S) + 2 - n)/2$.

Proof. (1) Thurston norm on $\Delta = \{(a,b,c) | a + b - c = 1,\ a > 0, b > 0, a > c, b > c\}$ is 1.

(2), (3) Using the boundary map $\partial : H_2(M, \partial M) \to H_1(\partial M)$. □

Entropy function

For a homology class σ, we denote by $\mathrm{ent}(\sigma)$ the entropy $\mathrm{ent}(f)$ of the map $f : S \to S$ where S is the realizing surface of σ.

Thm (Fried, Matsumoto, McMullen)**.** *Entropy function* $\mathrm{ent}(\delta) : \mathrm{cone}(\Delta) \to \mathbb{R}$ *has following properties.*

- *For integral homology classes,* $\dfrac{1}{\mathrm{ent}(n\delta)} = n\dfrac{1}{\mathrm{ent}(\delta)}$.
- *One can extend* $\dfrac{1}{\mathrm{ent}}$ *to rational classes and then real classes.*
- $\dfrac{1}{\mathrm{ent}}$ *is strictly concave.*
- $\chi(\delta)ent(\delta)$ *is constant on the ray through origin.*
  (*You can get every information of the function only on* Δ.)
- $\chi(\delta)ent(\delta)$ *has unique minimal on* Δ.

How to compute the entropy?

For fixed fiber face Δ, there is a very powerful way to compute the entropy.

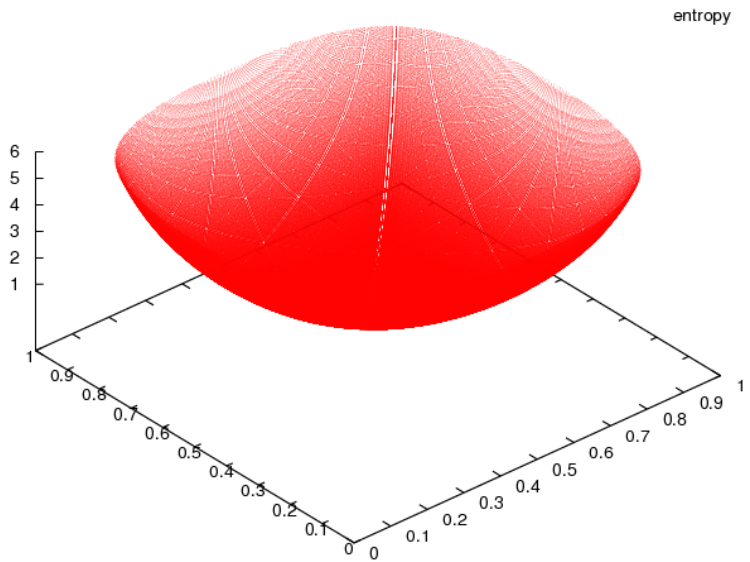
Thm (Fried, Oertel)**.** *For any two point σ, τ in the cone of a fiber surface Δ, their suspended invariant laminations are isotopic.*

Using Oertel's recipe, we have the following polynomial to compute the entropy.

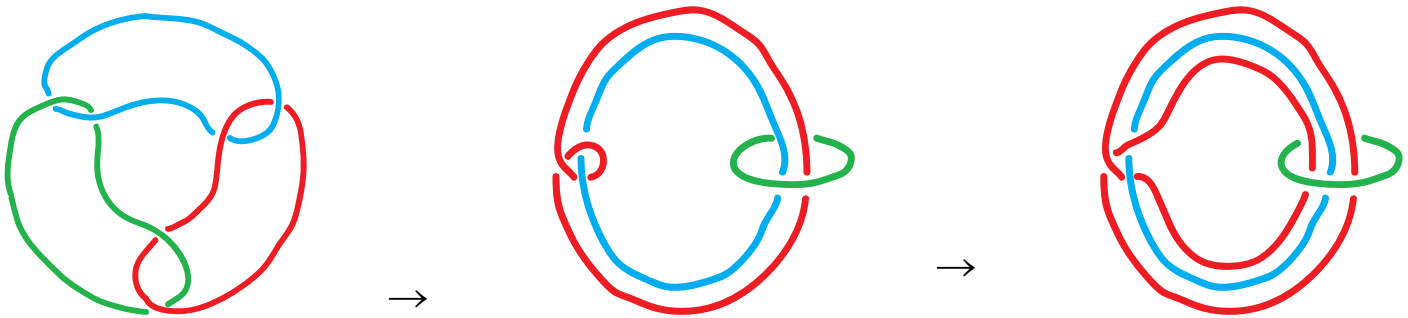
Prop. *The dilatation λ of $\sigma = x\alpha + y\beta + z\gamma$ is the largest real root of the following polynomial.*

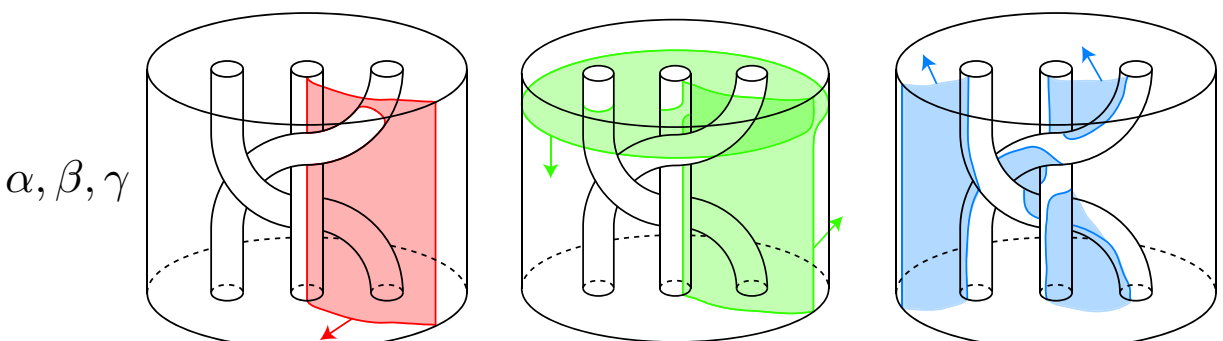
$$f_\Delta(x, y, z) = -\lambda^x - \lambda^y + \lambda^{x+y} + \lambda^z - \lambda^{x+z} - \lambda^{y+z} = 0$$

Entropy function on Δ

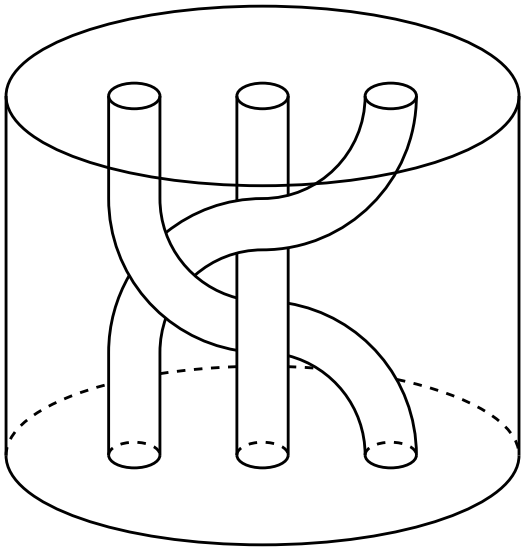


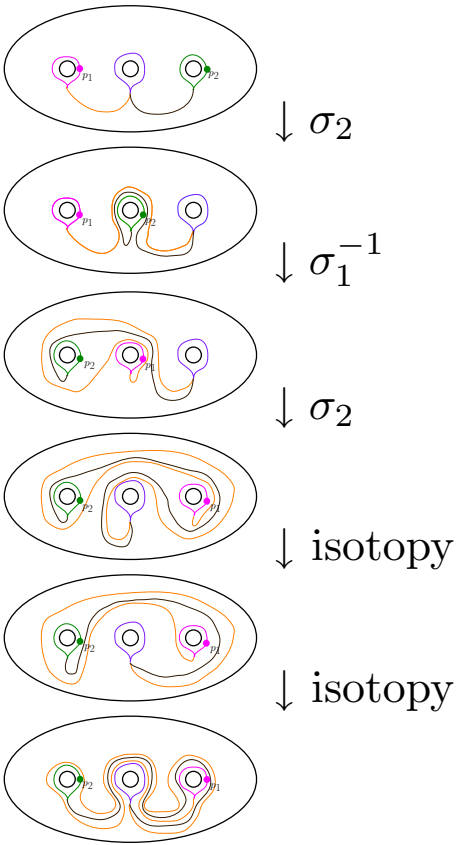
Compute the entropy function (1/4)



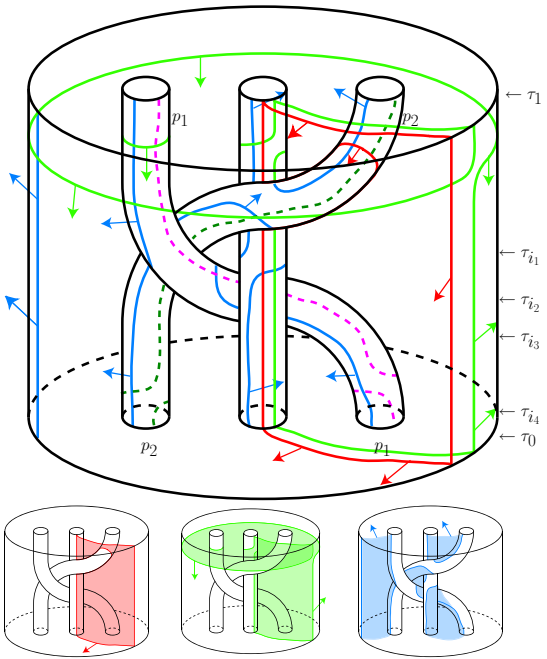


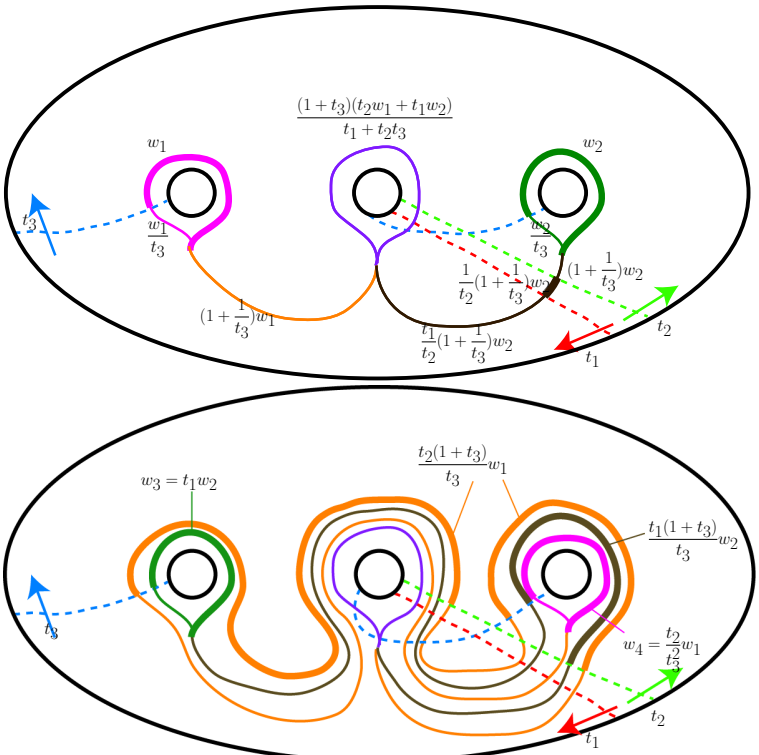
Compute the entropy function (2/4)





Compute the entropy function (3/4)





Compute the entropy function (4/4)

We have the following equation for the measure of the stable lamination.

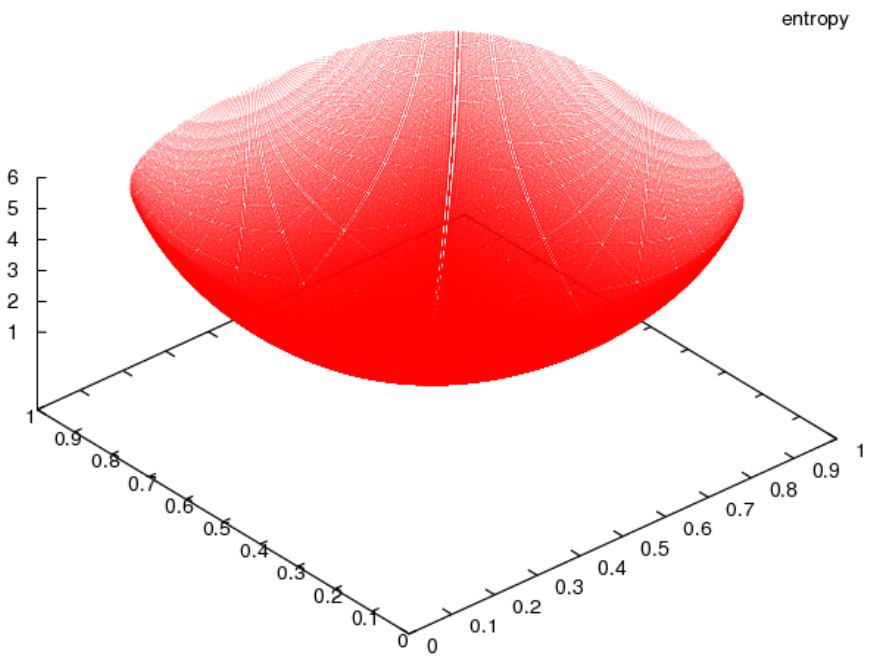
$$\begin{pmatrix} t_1 & \frac{t_2(1+t_3)}{t_3} - 1 \\ \frac{t_1(1+t_3)}{t_3} - 1 & \frac{t_2}{t_3^2} + \frac{t_2(1+t_3)}{t_3} \end{pmatrix} \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{-t_1 - t_2 + t_3 + t_1t_2 - t_1t_3 - t_2t_3}{-t_3} = 0$$

Substituting $t_1 = \lambda^a,\ t_2 = \lambda^b,\ t_3 = \lambda^c$, we have,

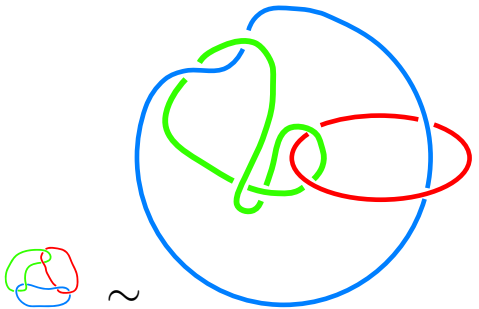
$P(a,b,c)(\lambda) = -\lambda^a - \lambda^b + \lambda^{a+b} + \lambda^c - \lambda^{a+c} - \lambda^{b+c} = 0.$

Entropy function on Δ



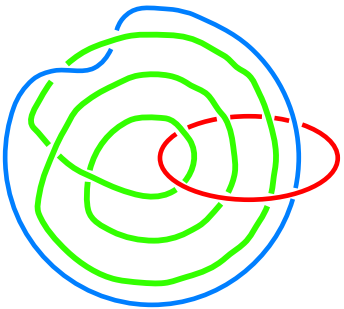
How to compute the entropy after Dehn filling.

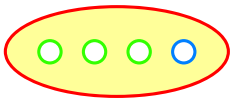
Example: $S^3 - L_{6a2}$



$\sim$

twist along blue disk $\rightarrow$



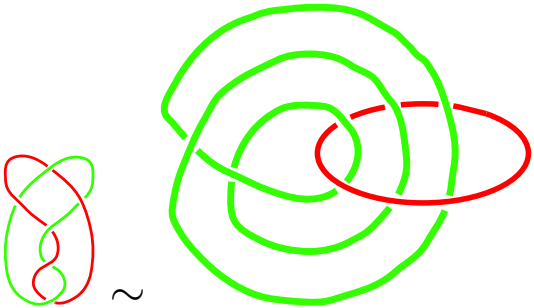


$S^\circ = 2\alpha + \gamma$

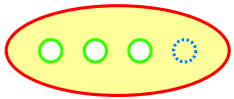
$\lambda = \dfrac{3+\sqrt{5}}{2}$

fill blue component $\downarrow$ along $(-1/2)$

$\downarrow$ fill blue component along $(1/0)$



$\sim$

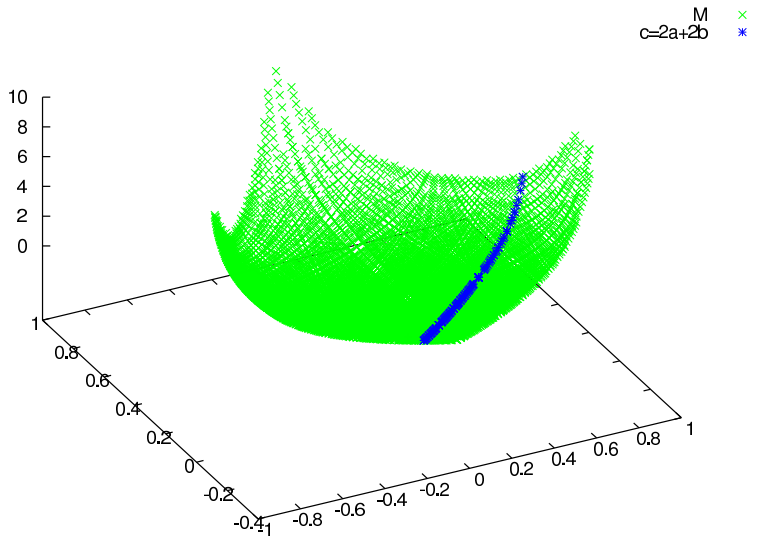


$S^\bullet = \sigma + \tau$

$\lambda = \dfrac{3+\sqrt{5}}{2}$

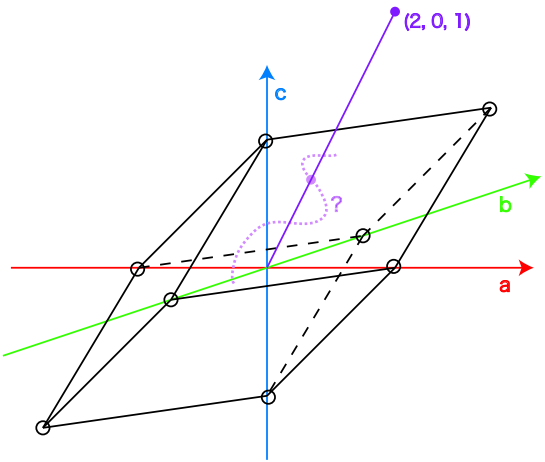
Entropy function on $M(p/q)$

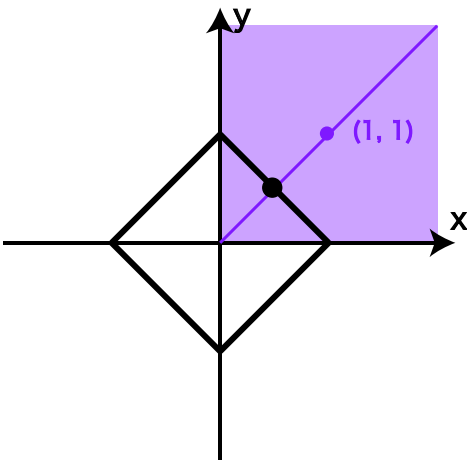
$ent(S^\circ) = ent(S^\bullet)$ except when the invariant foliation of S have a 1-prong singularity on the filled puncture. In that case $ent(S^\circ) > ent(S^\bullet)$.

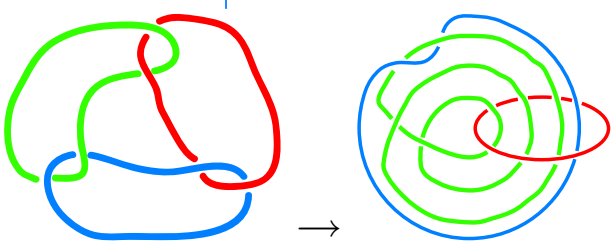


We can compute when the invariant foliation have 1-prong singularity. Namely, $p_1/q_1 \in \mathbb{Z}$ or $p_2/q_2 \in \mathbb{Z}$ or $p_3/q_3 \in \{-2 - 1/n \mid n \in \mathbb{N}\}$.

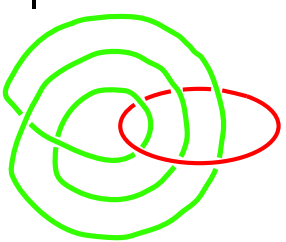
Thurston polytope of M and $M_{(p/q)}$







$S^\circ = 2\alpha + \gamma$ Dehn filling along p/q $S^\bullet = \sigma + \tau$



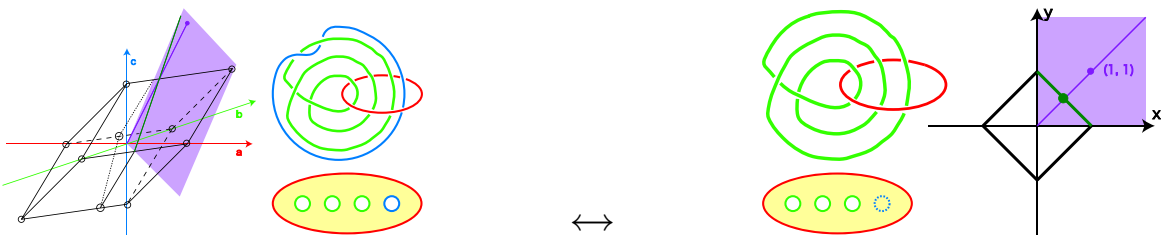
If original fiber surface S° of M has a boundary slope p/q, filled surface $S^\bullet$ is also fiber surface of $M(p, q)$.

Fiber surfaces whose boundary slope is (p/q)

Recall that the slope of $a\alpha + b\beta + c\gamma$ is $\left(-\frac{b+c}{a}, -\frac{c+a}{b}, -\frac{a+b}{c}\right)$.

$$(p,q) = \left(-\frac{a+b}{\gcd(a+b,c)}, \frac{c}{\gcd(a+b,c)}\right) \Rightarrow q(a+b) = -pc$$

$$H_2(M, \partial M; \mathbb{R}) \cap \{(a,b,c) \mid -pc = qa+qb\} \leftrightarrow H_2(M(p/q), \partial M(p/q); \mathbb{R})$$



$$\begin{aligned} ||S^\bullet||_{M(p/q)} &= ||S^\circ||_M - \#\{\text{filled punctures}\} \\ &= ||S^\circ||_M - \gcd(a+b,c) = ||S^\circ||_M - \frac{c}{q} \end{aligned}$$

Experiment

Let $\sigma = x\alpha + y\beta + z\gamma$ where $x, y \in [0, 100]$ and $z \in [-100, \max(x, y))$ s.t. $\gcd(x, y, z) = 1$

- Compute the boundary slopes $\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$ of σ
- Using Theorem of Martelli-Petronio, compute hyperbolicity for M and every Dehn filling $M\left(\frac{p_1}{q_1}\right)$, $M\left(\frac{p_2}{q_2}\right)$, $\cdots$, $M\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$, $\cdots$, $M\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$
- If they are hyperbolic, compute
  * $\chi(\sigma)\ \mathrm{ent}(\sigma)$ and
  * topology of the fiber surface.

Example:

Let $\sigma = 2\alpha + 2\beta + \gamma$.

- Boundary slopes are $(-3/2, -3/2, -4)$.
- $M(-3/2)$, $M(-4)$, $M(-3/2, -3/2)$, $M(-3/2, -4)$ are hyperbolic and $M(-3/2, -3/2, -4)$ is not hyperbolic.
- The fiber of $M(-3/2, -3/2)$ is $\Sigma_{1,1}$ and $\chi(\sigma)\mathrm{ent}(\sigma) = \dfrac{3+\sqrt{5}}{2}$.

...

Table: upper bounds for minimal dilatation

For each g and n, find the homology class σ which attains the smallest entropy among genus $= g$ and $\#$ of boundary $= n$.

Observation

Roughly speaking, the smallest entropy is attained by

- for $\Sigma_{0,n}$, $\sigma = x\alpha + y\beta$ where $x \approx \frac{n}{2}$ and $y \approx \frac{n}{2}$ with $\gcd(x, y) = 1$.
- for $\Sigma_{1,n}$, $M\left(\frac{1}{1}\right)$.
- for $\Sigma_{g,0}$, $M\left(\frac{-1}{2}, *, *\right)$ or $M\left(\frac{-3}{2}, *, *\right)$.
- for $\Sigma_{g,n}$ $n >> g$, $\sigma = x\alpha + y\beta + z\gamma$ where $x \approx n$, $y \approx \frac{n+2g}{2}$, $z \approx \frac{n-2g}{2}$ with $\gcd(x, y, z) = 1$.
- for $\Sigma_{g,n}$ $n < g$, we don't guess anything yet

(with some exceptions).

Remark. *All known minimal entropies appear in this table.*

For $\Sigma_{0,n}$

Thm (Kin-T). *For each $n \geq 4$, the minimum among the dilatations of $f \in \mathcal{M}(\Sigma_{0,n})$ s.t $T(f) \simeq M$ is realized by*

- $n = 4, 6, 8$: *exceptional case.*
- $n = 2k + 1, \ \sigma = k\alpha + (k-1)\beta$
- $n = 4k + 2, \ \sigma = (2k+1)\alpha + (2k-1)\beta$
- $n = 8k + 4, \ \sigma = (4k+3)\alpha + (4k-1)\beta$
- $n = 8k, \ \sigma = (4k+1)\alpha + (4k-3)\beta$

For $\Sigma_{g,0}$

Recall that Hironaka's theorem which comes from $M(-1/2)$.

Thm (Hironaka). *Let $r(k,l)$ be the largest real root of the following polynomial*

$$t^{2k} - t^{k+l} - t^k - t^{k-l} + 1 = 0.$$

- $\lambda_{\min}(g,0) \leq r(g+1,3) \quad (g \equiv 0,1,3,4 \pmod 6)$
- $\lambda_{\min}(g,0) \leq r(g+1,1) \quad (g \equiv 2,5 \pmod 6)$

Thm (Hironaka). $\limsup_{g\to\infty} \chi(\Sigma_{g,0})\mathrm{ent}_{\min}(g,0) \leq 2\log\left(\frac{3+\sqrt{5}}{2}\right).$

For some cases, we have better bounds from $M(-3/2)$.

Thm (K-T). *Let $r(k,l)$ be the largest real root of the following polynomial*

$$t^{2k} - t^{k+l} - t^k - t^{k-l} + 1 = 0.$$

- $\lambda_{\min}(g,0) \leq r(g+2,1) \quad (g \equiv 0,1,5,6 \pmod{10})$
- $\lambda_{\min}(g,0) \leq r(g+2,2) \quad (g \equiv 7,9 \pmod{10})$

Idea of proof

Construct a series of homology classes whose realizing surface is $\Sigma(g,n)$ and their entropy is obtained by the largest root of the polynomial.

- $\sigma = (2g+5)\alpha + (2g+6)\beta + (g+4)\gamma \quad (g \equiv 0,1,5,6 \pmod{10})$
- $\sigma = (2g+6)\alpha + (2g+8)\beta + (g+6)\gamma \quad (g \equiv 7,9 \pmod{10})$

Conjectures

Let $f : \Sigma_{g,0} \to \Sigma_{g,0}$,

1) If mapping torus $T(f)$ is isomorphic to M or a Dehn filling of M, (Hironaka's bound + Our's) is best possible and

$$\limsup_{g\to\infty} \chi(\Sigma_{g,0})\mathrm{ent}_{\min}(g,0) = 2\log\left(\frac{3+\sqrt{5}}{2}\right)$$

2) Without any condition (Hironaka's bound + Our's) is best possible and

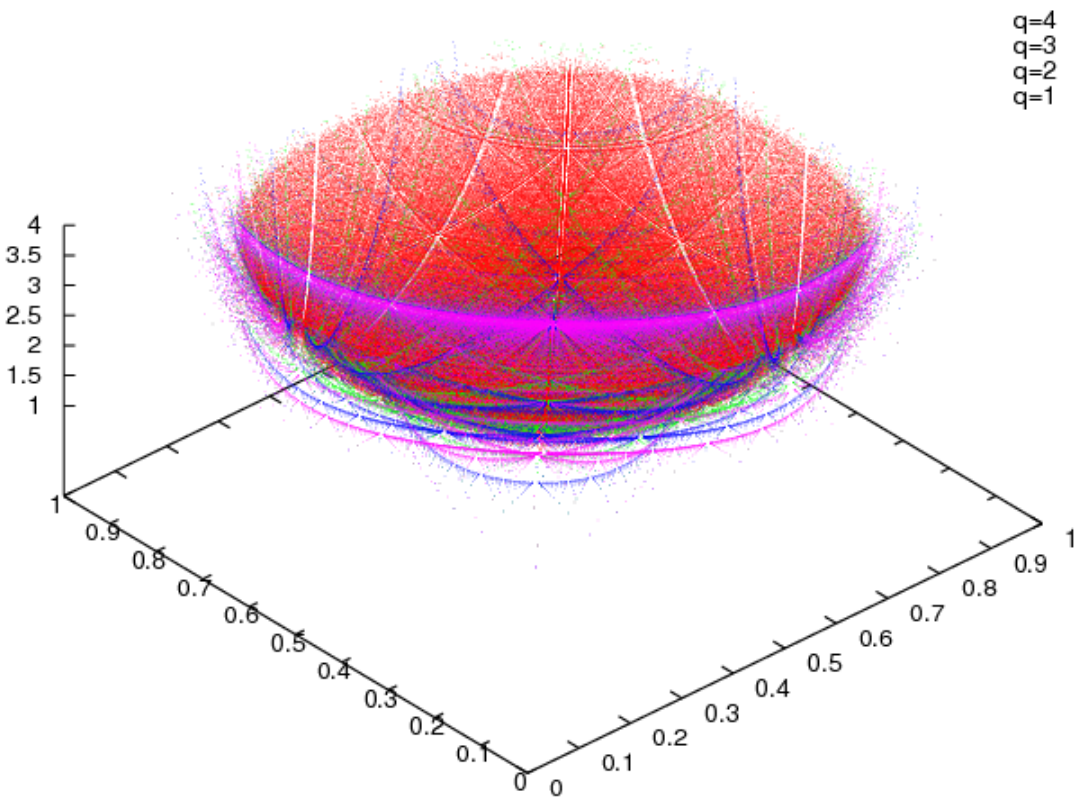
$$\limsup_{g\to\infty} \chi(\Sigma_{g,0})\mathrm{ent}_{\min}(g,0) = 2\log\left(\frac{3+\sqrt{5}}{2}\right)$$

Evidences for conjecture 1

$$
\begin{aligned}
&||S^\bullet||_{M(p_1/q_1,\ p_2/q_2,\ p_3/q_3)} \\
&\quad = ||S^\circ||_M - \#\{\text{filled punctures}\} \\
&\quad = ||S^\circ||_M - \gcd(a+b,c) - \gcd(b+c,a) - \gcd(c+a,b) \\
&\quad = ||S^\circ||_M - \frac{c}{q_1} - \frac{a}{q_2} - \frac{b}{q_3} \\
&\quad = a+b-c - \frac{c}{q_1} - \frac{a}{q_2} - \frac{b}{q_3}
\end{aligned}
$$

Plot the normalized entropy $\chi(\sigma)\mathrm{ent}(\sigma)$ for $\sigma(*,*,*)$.

It gives an upper bounds for $\mathrm{ent}_{\min}(g,0)$.


q=4
q=3
q=2
q=1
4
3.5
3
2.5
2
1.5
1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Evidences for conjecture 2

Thm (Kojima-Kin-T). $\exists c = c(\Sigma_{g,n})$ *s.t.*

$$\mathrm{vol}(T_f) \leq c \cdot \mathrm{ent}(f)$$

Conj. *M is the smallest volume 3-cusped hyperbolic manifold.*

Thm (Farb-Leininger-Margalit). *For any $P > 0$, there exists finitely many 3-manifolds $\{M_1, M_2, \cdots M_n\}$ such that, all pseudo-Anosovs $f : \Sigma_{g,0} \to \Sigma_{g,0}$ whose $\chi(\Sigma_{g,0})\mathrm{ent}(f)$ are less than $\log(P)$ live in one of the Dehn filling $M_i(*)$.*