

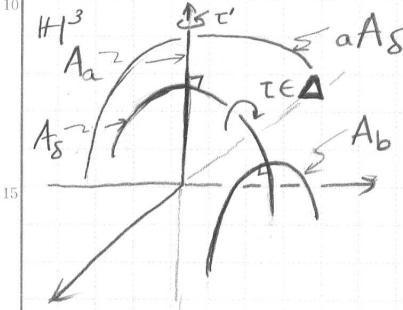
The Geometry and Topology of Arithmetic Hyperbolic

2,3-Manifolds, Orbifolds III

Abundance of Involutions

Let $a, b \in (P)SL_2(\mathbb{C})$ which are hyperbolic and $ab \neq ba$

Assume axes A_a, A_b are disjoint



Jørgensen observed that

\exists involution τ_{ab} such that

$$\tau_{ab} a \tau_{ab}^{-1} = a'$$

$$\tau_{ab} b \tau_{ab}^{-1} = b'$$

$$\tau_{ab} = ab - ba \in PSL_2(\mathbb{C})$$

If $F = \langle a, b \rangle$, τ normalizes F i.e. $\tau F \tau^{-1} = F$

Now assume that $F \subset \Gamma$ and Γ is cocompact

Assume further that Γ is arithmetic

Proposition $\tau = \tau_{ab} \in \Delta$ and Δ is commensurable with Γ .

Proof. Let us assume for convenience that $\Gamma = \Gamma^{(2)}$

The proof will follow from

Claim 1 Let $\mathcal{O} = R_k [1, a, b, ab]$
ring of integers

\mathcal{O} is an order of B

$$\Gamma \subset SL_2(\mathbb{C})$$

$$B = A\Gamma \subset M_2(\mathbb{C})$$

$$k = k\Gamma$$

Claim 2 τ normalizes \mathcal{O} i.e. $\tau \mathcal{O} \tau^{-1} = \mathcal{O}$

From these claims, this constructs $\Delta \ni \tau$.

Why $\mathcal{L} = \text{order of } B$

$N(\mathcal{L}) = \{x \in B^*: x\mathcal{L}x^{-1} = \mathcal{L}\}$ Normalizer of Order

$P(N(\mathcal{L}))$ is commensurable with Γ

Notice If $\alpha \in \mathcal{L}'$, $x \in N(\mathcal{L})$, $x\alpha x^{-1} \in \mathcal{L}$
 $n(x\alpha x^{-1}) = n(\alpha) = 1 \Rightarrow x \in N(\mathcal{L}')$

Since $P\mathcal{L}' = T_{\mathcal{L}}'$

$\Rightarrow P(N(\mathcal{L}')) \supset T_{\mathcal{L}}'$ is discrete and compact.
 $\Rightarrow P(N(\mathcal{L}))$ discrete

Digression

Order $\mathcal{O} \subset B :=$ ring of integers in B that contains R_k ,
 and k -basis for B .

Conjugate on B $x \mapsto \bar{x}$

$\text{tr}_{B/k}(x) = x + \bar{x}$ } x is an integer of B

$n_{B/k}(x) = x\bar{x}$ } when $n_{B/k}(x), \text{tr}_{B/k}(x) \in R_k$

Ex $B = \left(\frac{-1, 3}{\mathbb{Q}} \right)$, $\alpha = j$, $\beta = \frac{3j+4if}{5}$

These are integers, but $\alpha+\beta$, $\alpha\beta$ are not.

Proof of Claim 1

Note $\{1, \alpha, \beta, \alpha\beta\}$ are \mathbb{C} -linearly independent.

($\langle \alpha, \beta \rangle$ is non-elementary)

Need to show products of basis elts are in \mathcal{O} .

Recall If $x \in SL_2(\mathbb{C})$ $x + \bar{x}^T = \text{tr}(x) \cdot I$

If $x \in \mathcal{O}' (\subseteq SL_2(\mathbb{C}))$ $x + \bar{x}^T = (\text{tr } x) I$

$\Rightarrow \bar{x}^T \in \mathcal{O}$

$\Rightarrow \alpha^T, \beta^T, (\alpha\beta)^T \in \mathcal{O}$

Consider $a \cdot a = a^2$.

$$a + \bar{a}^T = (\text{tr } a) I \Rightarrow a^2 + I = (\text{tr } a) a$$

$\Rightarrow \tilde{a}^2$ is a sum of basis elements

$$a \cdot ab = \tilde{a}^2 b$$

From above $a^2 + I = (\text{tr } a) a$

$$\tilde{a}^2 b + b = (\text{tr } a) ab$$

$\Rightarrow \tilde{a}^2 b = (\text{tr } a) ab - b$

aba

$$ab + b^{-1}a^{-1} = ab + (ab)^{-1} = \text{tr}(ab)$$

$$aba + b^{-1} = (\text{tr } ab) a$$

$\Rightarrow aba = \text{sum of basis elements}$

Ex ba

$$\textcircled{2} \quad \tau a \tau^{-1} = a^{-1} \in \mathcal{O}$$

$$\tau b \tau^{-1} = b^{-1} \in \mathcal{O}$$

$$\tau ab \tau^{-1} = a^{-1}b^{-1} = (ba)^{-1} \in \mathcal{O}$$

$$\Rightarrow \tau \in N(\mathcal{O}) \quad \square$$

Going Further

$\tau \in \Delta$, Δ is commensurable with T'

Since Δ is compact, \exists hyperbolic element $w \in \Delta$

such that w has the same axis as τ .

T', Δ commensurable $\Rightarrow \exists \delta = w^n \in T'$

Use " a " to translate A_S

Now we can repeat the previous argument with the order $R_k[1, \delta, a\delta a^{-1}, \delta a \delta a^{-1}]$

$\Rightarrow \exists \tau'$ with the same axis as a

Conclusion Every axis of a hyperbolic element is the axis of an involution.

Characterizing Arithmeticity.

Let $M = \mathbb{H}^3/\Gamma'$ be a closed hyperbolic 3-manifold

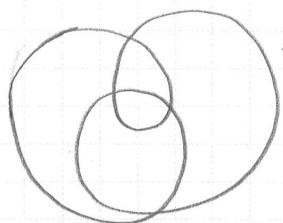
Then M is arithmetic

\Leftrightarrow the following condition holds :

$M_g \ni \tau$ involution Fix $\tau \supset \Delta$ a component of
(or preserving) the preserving of g

↓ finite cover

M



closed geodesic g in M .

Proof. $\tau \in \Gamma'$ be hyperbolic whose axis projects to g .

From above \exists an involution $\tau \in \Delta$ comm. with Γ'

with axis of $\tau = A_\tau$

$$\begin{array}{ccc} \Gamma' & \Delta \ni \tau & \\ \text{finite} \swarrow & \searrow \text{finite} & \\ \Gamma' \cap \Delta & & \end{array}$$

τ normalizes C
and $M_g = \mathbb{H}^3/C$.

$$\begin{array}{c} | \text{finite} \\ C = \text{Core}_\Delta(\Gamma' \cap \Delta) \end{array}$$

Extra Information.

Note that $[\tau, \tau'] = 1$

$$\Rightarrow V = \langle \tau, \tau' \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

V \subset arithmetic Kleinian group
comm. with Γ'

Take $\mathcal{O} = R_k[1, a, \delta, a\delta]$
 $\tau, \tau' \in N(\mathcal{O}) \Rightarrow \Delta = P(N(\mathcal{O})) \quad \square$

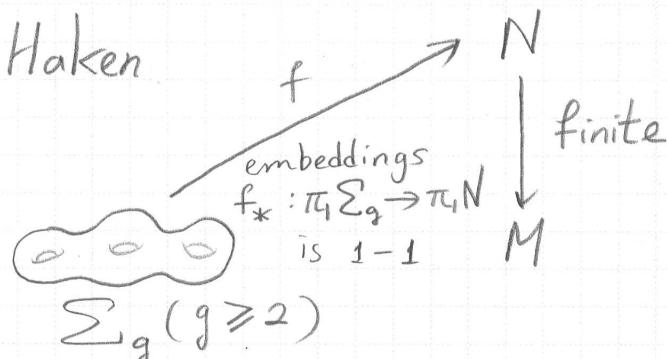
Theorem Every Kleinian group is commensurable with a Δ such that $\Delta \supset \Delta'$.

3-manifolds

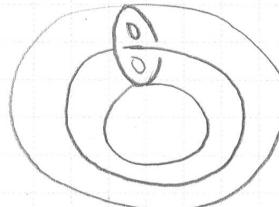
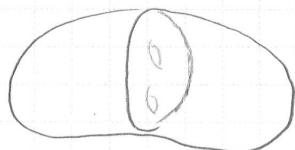
$M = H^3/\Gamma$ closed hyperbolic.

Conjecture

① M is virtually Haken



②



$b_1 > 0$

M has a finite sheeted cover with $b_1 > 0$

③ Virtual $b_1 := v b_1$

$v b_1(M) = \sup \{ b_1(X) : X \xrightarrow{\text{finite}} M \}$

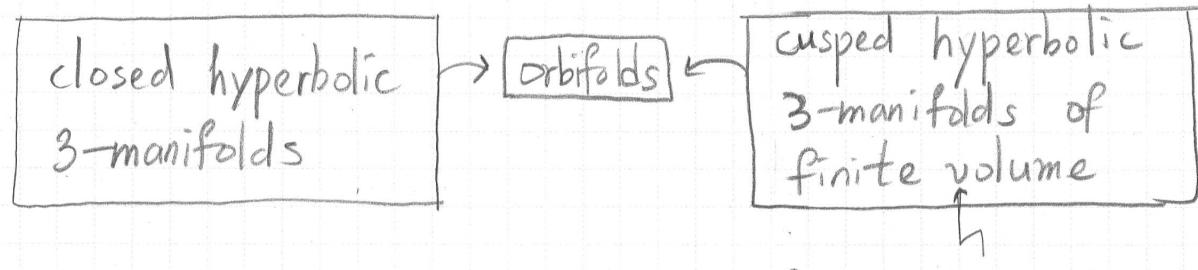
conjecture $v b_1(M) = \infty$

④ Γ is large i.e. Γ

finite

$\Delta \rightarrow F_2$

Manifolds commensurable with orbifolds seem more tractible



Theorem [Lackenby-Long-R]

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with $\text{vb}_1(M) \geq 4$. Assume Γ is commensurable with Δ and $\Delta \supset \mathbb{Z}/2 \times \mathbb{Z}/2$. Then Γ is large.

Cor. If M is arithmetic and $\text{vb}_1 \geq 4$.

$\Rightarrow \pi_1 M$ large \square

Theorem [Cooper-Long-R]

If M is arithmetic and $\text{vb}_1(M) > 0$

$\Rightarrow \pi_1 M$ large

Current state of Affairs (for $\text{vb}_1(M) > 0$ for arithmetic manifolds)

② is still open

(1) If (general) $M \supset$ totally geodesic surface

(i.e. $\pi_1 M \supset$ Fuchsian surface group)

[Long] $\text{vb}_1(M) > 0$ ($\pi_1 M$ large)

Most arithmetic M don't have such a surface
(see lecture II)

M is arithmetic, $M \supset$ totally geodesic surface
 $\Leftrightarrow A(M) = \left(\frac{a-b}{k}\right)$ where $[k : k \cap \mathbb{R}] = 2$
and $a, b \in k \cap \mathbb{R}$

(2) If M is arithmetic and $[k(M) : k(M) \cap \mathbb{R}] = 2$
 $\Rightarrow \nu b_1(M) > 0$ [Li-Millson, Labesse-Schwermer, Lubotzky]

(3) $k(M) \longrightarrow E$ $E = \text{Gal}(L/\mathbb{Q})$
totally real $\rightarrow L$ $\xrightarrow{\text{Galois group}} g(L/\mathbb{Q})$ is solvable
 $\xrightarrow{\mathbb{Q}}$ $\Rightarrow \nu b_1(M) > 0$ [Rajan]

(or If $[k(M) : \mathbb{Q}] \leq 4 \Rightarrow \nu b_1(M) > 0$

(4) [Clozel] Let k have 1 complex place.
 B/k quaternion algebra ramified at all real places
Suppose that for any finite place v that ramifies B
we have $k_v \not\cong$ quadratic extension of \mathbb{Q}_p , $v|p$
Let $\mathcal{O} \subset B$ an order in B then $\nu b_1(H^3/\Gamma_0) > 0$ \square

Ex Don't know what to do!

Let $p(x) = x^5 - x^3 - 2x^2 + 1$. and θ be a complex root.

$k = \mathbb{Q}(\theta)$, $[k : \mathbb{Q}] = 5$, $\text{Gal}(k/\mathbb{Q}) = S_5$.

B/k quaternion algebra \rightarrow arithmetic Kleinian group.

Lecture II \Rightarrow no Fuchsian

v a place where $v|11$, $Nv = 11^2$

$\Rightarrow [k_v : \mathbb{Q}_{11}] = 2$ violate Clozel

B/k ramified at 3 real places + v \square