ARITHMETIC INVARIANTS OF HYPERBOLIC 3–MANIFOLDS

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These are unfinished notes related to my lectures on arithmetic invariants of hyperbolic manifolds for the NIMS workshop “Hyperbolic geometry: algorithmic, number theoretic and numerical aspects”. They are in part cut and pasted from elsewhere. They are therefore not for publication outside the workshop.

A good general reference is the book by Maclachlan and Reid [8]

1. Notation and terminology for algebraic number theory

1.1. **Number fields.** A *number field* K is a finite extension of $\mathbb{Q}$. That is, K is a field containing $\mathbb{Q}$, and finite-dimensional as a vector space over $\mathbb{Q}$. This dimension d, denoted $d = [K : \mathbb{Q}]$, is the *degree* of the number field. K has exactly d embeddings into the complex numbers,

$$\theta_i \colon K \to \mathbb{C}, \qquad i = 1, \dots, d = r_1 + 2r_2\,,$$

where r_1 is the number of them with real image, and the remaining embeddings come in r_2 complex conjugate pairs. Indeed, the “Theorem of the Primitive Element” implies that K is generated over $\mathbb{Q}$ by a single element, from which if follows that $K \cong \mathbb{Q}[x]/(f(x))$ with $f(x)$ an irreducible polynomial of degree d; the embeddings $K \to \mathbb{C}$ arise by mapping the generator x of K to each of the d zeros in $\mathbb{C}$ of $f(x)$.

A *concrete number field* is a number field K with a chosen embedding into $\mathbb{C}$, i.e., K given as a subfield of $\mathbb{C}$. The union of all concrete number fields is the field of algebraic numbers in $\mathbb{C}$, which is the concrete algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$ of $\mathbb{Q}$.

An *algebraic integer* is a zero of a monic polynomial with rational integer coefficients. The algebraic integers in K form a subring $\mathcal{O}_K \subset K$, the *ring of integers of K*. It is a Dedekind domain, which is to say that any ideal in $\mathcal{O}_K$ factors uniquely as a product of prime ideals. Each prime ideal $\mathfrak{p}$ (or “prime” for short) of $\mathcal{O}_K$ is a divisor of a unique ideal (p) with $p \in \mathbb{Z}$ a rational prime (determined by $|\mathcal{O}_K/\mathfrak{p}| = p^e$ for some $e > 0$). The factorization of (p) as a product $(p) = \mathfrak{p}_1^{f_1} \dots \mathfrak{p}_k^{f_k}$ of primes of $\mathcal{O}_K$ follows patterns which can be found in any text on algebraic number theory. In particular, the exponents f_i are 1 for all but a finite number of primes $\mathfrak{p}$ of $\mathcal{O}_K$, which are called *ramified*.

For the ring $\mathcal{O}_\mathbb{Q} = \mathbb{Z}$ of $\mathbb{Q}$, any ideal is principal, and the factorization of the ideal (n) into a product of ideals (p_i) expresses the familiar unique prime factorization of rational integers. In general $\mathcal{O}_K$ is a unique factorization domain (UFD) if and only if it is a PID (every ideal is principal), which is somewhat rare. It is presumed to happen infinitely often, but this is not proven.

Given a prime $\mathfrak{p}$ of $\mathcal{O}_K$, there is a multiplicative norm $||.||_\mathfrak{p}$ defined for $a \in \mathcal{O}_K$ by $||a||_\mathfrak{p} := c^{-r}$, where $\mathfrak{p}^r$ is the largest power of $\mathfrak{p}$ which “divides” a (i.e., contains a) and c is a positive constant[1]; the norm is then determined for arbitrary elements

[1] The value of c is unimportant for topological considerations but is standardly taken as $c = N(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$

of K by the multiplicative property $||ab||_{\mathfrak{p}} = ||a||_{\mathfrak{p}}||b||_{\mathfrak{p}}$. This norm determines a translation invariant topology on K and the completion of K in this topology is called $K_{\mathfrak{p}}$. The unit ball around 0 in $K_{\mathfrak{p}}$ is its *ring of integers* $\mathcal{O}_{K_{\mathfrak{p}}}$, and the open unit ball is the unique maximal ideal in this ring. The norm $||.||_{\mathfrak{p}}$ is *non-Archimedean*, i.e., it satisfies the strong triangle inequality $||a+b||_{\mathfrak{p}} \le \max(||a||_{\mathfrak{p}}, ||b||_{\mathfrak{p}})$. Up to equivalence (norms are *equivalent* if one is a positive power of the other), the only non-Archimedean multiplicative norms are the ones just described, and the only other multiplicative norms on K are the norms $||a||_\theta := |\theta(a)|$ given by absolute value in $\mathbb{C}$ for an embedding $\theta\colon K \to \mathbb{C}$. The completion of K in the topology induced by one of these is $\mathbb{R}$ or $\mathbb{C}$ according as the image of θ lies in $\mathbb{R}$ or not.

The fields $\mathbb{R}$, $\mathbb{C}$, $K_{\mathfrak{p}}$ arising from completions are *local fields*[2]. The name is geometrically motivated: one thinks of $\mathcal{O}_K$ as a ring of functions on a "space" with a "finite point" for each prime ideal, plus $r_1 + r_2$ "infinite points" corresponding to the embeddings in $\mathbb{R}$ and $\mathbb{C}$; "local" means focusing on an individual point. One therefore refers to an embedding of K into $K_{\mathfrak{p}}$ as a "finite place" and an embedding into $\mathbb{R}$ or $\mathbb{C}$ as an "infinite place," and if an object A associated with K (e.g., an algebra A over K) has corresponding objects associated to each place (e.g., $A \otimes K_{\mathfrak{p}}$, $A \otimes \mathbb{R}$, $A \otimes \mathbb{C}$) then a "property of A at the (finite or infinite) place" means that property for the associated object. We stress that an "infinite place" refers to the embedding of K in C up to conjugation, so there are r_1 real places and just r_2 complex places.

1.2. **Quaternion algebras.** References for this section are [16] and [4]. A *quaternion algebra* over a field K is a simple algebra over K of dimension 4 and with center K. The simplest example is the algebra $M_2(K)$ of 2×2 matrices over K. This is the only quaternion algebra up to isomorphism for $K = \mathbb{C}$. For $K = \mathbb{R}$ there are exactly two, namely $M_2(\mathbb{R})$ and the Hamiltonian quaternions. The situation for the non-Archimedean local fields $K_{\mathfrak{p}}$ is similar: there are exactly two quaternion algebras over each of them, one being the trivial one $M_2(K_{\mathfrak{p}})$ and the other being a division algebra. In each case the trivial quaternion algebra M_2 is called *unramified* and the division algebra is called *ramified*. For a number field K the classification of quaternion algebras over K is as follows:

Theorem 1.1 (Classification). *A quaternion algebra E over K is ramified at only finitely many places (i.e., only finitely many of the $E \otimes K_{\mathfrak{p}}$ and $E \otimes \mathbb{R}$'s are division algebras) and is determined up to isomorphism by the set of these "ramified places." The number of ramified places is always even, and every set of places of K of even size arises as the set of ramified places of a quaternion algebra over K.*

A quaternion algebra E over K can always be given in terms of generators and relations in the form

$$E = K\langle i, j : i^2 = \alpha,\ j^2 = \beta,\ ij = -ji\rangle\,,$$

with $\alpha, \beta \in K^*$. The *Hilbert symbol* notation $\left\{\frac{\alpha,\beta}{K}\right\}$ refers to this quaternion algebra. For example, $\left\{\frac{-1,-1}{\mathbb{R}}\right\}$ is Hamilton's quaternions, and $\left\{\frac{1,\beta}{K}\right\} = M_2(K)$ for any K. The Hilbert symbol for a given quaternion algebra is far from unique, but computing the ramification—and hence the isomorphism class—of a quaternion algebra from

[2]The definition of local field is: non-discrete locally compact topological field. The ones mentioned here are all that exist in characteristic 0.

the Hilbert symbol is not hard, and is described in [16], see also [4] for a description tailored to 3-manifold invariants.

In terms of the above presentation, the map $i \mapsto -i, j \mapsto -j, ij \mapsto -ij$ of a quaternion algebra E to itself is an anti-automorphism called conjugation, and the *norm* of $x = a + ib + jc + ijd \in E$ is defined as $N(x) := x\bar{x} = a^2 + \alpha b^2 + \beta c^2 + \alpha\beta d^2$.

1.3. **Arithmetic subgroups of** $\mathrm{SL}(2,\mathbb{C})$ **and** $\mathrm{PSL}(2,\mathbb{C})$**.** For a quaternion algebra E over K the set $\mathcal{O}_E$ of integers of E (elements which are zeros of monic polynomials with coefficients in $\mathcal{O}_K$) does not form a subring. One considers instead an *order* in E: any subring $\mathcal{O}$ of E, contained in $\mathcal{O}_E$ and containing $\mathcal{O}_K$ and of rank 4 over $\mathcal{O}_K$. E has infinitely many maximal orders; we just pick one of them.

The subset $\mathcal{O}^1 \subset \mathcal{O}$ of elements of norm 1 is a subgroup. At any complex place, E becomes $E \otimes \mathbb{C} = M_2(\mathbb{C})$ and $\mathcal{O}^1$ becomes a subgroup of $\mathrm{SL}(2,\mathbb{C})$, while at an unramified real place E becomes $E \otimes \mathbb{R} = M_2(\mathbb{R})$ and $\mathcal{O}^1$ becomes a subgroup of $\mathrm{SL}(2,\mathbb{R})$. We thus get an embedding of $\Gamma := \mathcal{O}^1/\{\pm 1\}$

$$\Gamma \subset \prod_{i=1}^{r_1^u} \mathrm{PSL}(2,\mathbb{R}) \times \prod_{j=1}^{r_2} \mathrm{PSL}(2,\mathbb{C})\,,$$

where r_1^u is the number of unramified real places of K. This subgroup is a lattice (discrete and of finite covolume).

If $r_1^u = 0$ and $r_2 = 1$ this gives an arithmetic subgroup of $\mathrm{PSL}(2,\mathbb{C})$ (similarly for $r_1^u = 1, r_2 = 0$ and $\mathrm{PSL}(2,\mathbb{R})$). Up to commensurability this group only depends on E and not on the choice of order $\mathcal{O}$. Any subgroup commensurable with an arithmetic subgroup—i.e., sharing a finite index subgroup with it up to conjugation—is, by definition, also arithmetic.

The general definition of an arithmetic group is in terms of the set of $\mathbb{Z}$-points of an algebraic group which is defined over $\mathbb{Q}$. Borel shows in [1] that all arithmetic subgroups of $\mathrm{PSL}(2,\mathbb{C})$ (and $\mathrm{PSL}(2,\mathbb{R})$) can be obtained as described above.

2. Arithmetic invariants of hyperbolic manifolds

2.1. **Invariant trace field and quaternion algebra.** A *Kleinian group* Γ will mean a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ for which $\mathbb{H}^3/\Gamma$ is finite volume (it may be an orbifold). Let $\overline{\Gamma} \subset \mathrm{SL}(2,\mathbb{C})$ be the inverse image of Γ under the projection $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$.

Definition 2.1. The *trace field* of Γ is the field $\mathrm{tr}(\Gamma)$ generated by all traces of elements of $\overline{\Gamma}$.

The *invariant trace field* is the field $k(\Gamma) := \mathrm{tr}(\Gamma^{(2)})$ where $\Gamma^{(2)}$ is the group generated by squares of elements of Γ. It can also be computed as $k(\Gamma) = \mathbb{Q}(\{(\mathrm{tr}(\gamma))^2 \mid \gamma \in \overline{\Gamma}\})$ ([15, 8], see also [11]).

The *invariant quaternion algebra* of Γ is the $k(\Gamma)$-subalgebra of $M_2(\mathbb{C})$ (2×2 matrices over $\mathbb{C}$) generated over $k(\Gamma)$ by the elements of $\overline{\Gamma}^{(2)}$. It is denoted $A(\Gamma)$.

Theorem 2.2. *$k(\Gamma)$ and $A(\Gamma)$ are commensurability invariants of Γ.*

If Γ is arithmetic, then $k(\Gamma)$ and $A(\Gamma)$ equal the defining field and defining quaternion algebra of Γ, so they form a complete commensurability invariant, but this is not so the non-arithmetic case.

It follows that a necessary condition for arithmeticity is that $k(\Gamma)$ have only one non-real complex embedding (it always has at least one). Necessary and sufficient is that, in addition, for each $\gamma \in \overline{\Gamma}$ trace(γ^2) should be an algebraic integer whose absolute value at all real embeddings of k is bounded by 2. Alternatively, all traces should be algebraic integers and $A(\Gamma)$ should be ramified at all real places of k. See [15].

These invariants are already quite powerful invariants of a hyperbolic manifold. For example, if a hyperbolic manifold M is commensurable with an amphichiral manifold N (i.e.,N has an orientation reversing self-homeomorphism) then $k(M) = \overline{k(M)}$ and $A(M) = \overline{A(M)}$ (complex conjugation).

If M has cusps then the invariant quaternion algebra is always unramified, so gives no more information than the invariant trace field, but for closed M unramified invariant quaternion algebras are uncommon; for example among the almost 40 manifolds in the closed census which have invariant trace field $\mathbb{Q}[\sqrt{-1}]$, only two have unramified quaternion algebra.

2.2. **Bloch invariant and related invariants.** For details on what we discuss here see [10, 13, 14] or the expository article [9].

2.2.1. *PSL-fundamental class of a hyperbolic manifold.* The *PSL-fundamental class* of M is a homology class

$$[M]_{PSL} \in H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})\,,$$

where the superscript δ means "with discrete topology".

This class is easily described if M is compact. Write $M = \mathbb{H}^3/\Gamma$ with $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$. The PSL-fundamental class is the image of the fundamental class of M under the map $H_3(M;\mathbb{Z}) = H_3(\Gamma;\mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$, where the first equality is because M is a $K(\Gamma,1)$-space. If M has cusps one obtains first a class in $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta, P;\mathbb{Z})$, where P is a maximal parabolic subgroup of $\mathrm{PSL}(2,\mathbb{C})^\delta$ and one then uses a natural splitting of the map $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta, P;\mathbb{Z})$ to get $[M]_{\mathrm{PSL}}$. This was described in [10] and proved carefully by Zickert in [6], who shows that the class in $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta, P;\mathbb{Z})$ depends on choices of horoballs at the cusps, but the image $[M]_{\mathrm{PSL}} \in H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$ does not.

The group $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ can be conjugated to lie in $\mathrm{PSL}(2,K)$ for a number field K (which can always be chosen to be a quadratic extension of the trace field, but there is generally no canonical choice), so the PSL-fundamental class is then defined in $H_3(\mathrm{PSL}(2,K);\mathbb{Z})$.

Theorem 2.3 ([17])**.** *$[M]_{PSL}$ is actually defined in $H_3(\mathrm{PSL}(2,k);\mathbb{Z})$, where k is the invariant trace field*

This was previously known in the cusped case, but only modulo torsion in the closed case. We sketch the proof. If one chooses a spin structure on M then $[M]_{\mathrm{PSL}}$ lifts to a class $[M]_{\mathrm{SL}} \in H_3(\mathrm{SL}(2;K);\mathbb{Z})$. This group is a $\mathbb{Z}/4$ extension of $H_3(\mathrm{PSL}(2;K);\mathbb{Z})$ (see [7]). In [17] Zickert shows that $H_3(\mathrm{SL}(2;K);\mathbb{Z})$ is naturally isomorphic to $K_3^{ind}(K)$, which is known to satisfy Galois descent. Since $[M]_{\mathrm{SL}}$ is invariant under the action of the Galois group $\mathrm{Gal}(K/k)$, it descends to $H_3(\mathrm{SL}(2;k);\mathbb{Z})$.

The following theorem, which holds also with PSL replaced by SL, summarises results of various people, see [14] and [17] for more details.

Theorem 2.4. *$H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ is the direct sum of its torsion subgroup, isomorphic to $\mathbb{Q}/\mathbb{Z}$, and an infinite dimensional $\mathbb{Q}$ vector space.*

If $k \subset \mathbb{C}$ is a number field then $H_3(\mathrm{PSL}(2,k);\mathbb{Z})$ is the direct sum of its torsion subgroup and $\mathbb{Z}^{r_2}$, where r_2 is the number of conjugate pairs of complex embeddings of k. Moreover, the map $H_3(\mathrm{PSL}(2,k);\mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ is injective modulo torsion.

Conjecture 2.5 (Rigidity Conjecture)**.** *The* Rigidity Conjecture *conjectures that each the following equivalent statements is true:*

1. $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$ *is countable.*
2. $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}})^\delta;\mathbb{Z}) = H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$
3. $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$ *is the union of the images of the maps* $H_3(\mathrm{PSL}(2,K);\mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$*, as K runs through all concrete number fields.*

2.3. **Invariants of the PSL-fundamental class.** There is a homomorphism

$$\hat{c}\colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \mathbb{C}/\pi^2\mathbb{Z}$$

called the "Cheeger-Simons class" ([2]) whose real and imaginary parts give Chern-Simons invariant and volume:

$$\hat{c}([M]_{PSL}) = \pi^2 \operatorname{cs}(M) + i \operatorname{vol}(M)\,.$$

The Chern-Simons invariant here is the Chern-Simons invariant of the flat connection, which is defined for any complete hyperbolic manifold M of finite volume. If M is closed the Riemannian Chern-Simons invariant $\mathrm{CS}(M) \in \mathbb{R}/2\pi^2$ is also defined; it reduces to $\operatorname{cs}(M) \bmod \pi^2$. See [10] for details.

We denote the homomorphisms given in the obvious way by the real and imaginary parts of $\hat{c}$ by:

$$\operatorname{cs}\colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \mathbb{R}/\pi^2 Z\,, \qquad \operatorname{vol}\colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \mathbb{R}\,.$$

A standard conjecture that appears in many guises in the literature (see [9] for a discussion) is:

Conjecture 2.6. *The Cheeger-Simons class is injective. That is, volume and Chern-Simons invariant determine elements of $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ completely.*

If k is an algebraic number field and $\sigma_1,\dots,\sigma_{r_2}\colon k \to \mathbb{C}$ are its different complex embeddings up to conjugation then denote by vol_j the composition

$$\operatorname{vol}_j = \operatorname{vol}\circ(\sigma_j)_*\colon H_3(\mathrm{PSL}(2,k);\mathbb{Z}) \to \mathbb{R}.$$

The map

$$\mathrm{Borel} := (\operatorname{vol}_1,\dots,\operatorname{vol}_{r_2})\colon H_3(\mathrm{PSL}(2,k);\mathbb{Z}) \to \mathbb{R}^{r_2}$$

is called the *Borel regulator.*

Theorem 2.7. *The Borel regulator maps $H_3(\mathrm{PSL}(2,k);\mathbb{Z})/Torsion$ injectively onto a full sublattice of $\mathbb{R}^{r_2}$.*

The homomorphism cs is injective on the torsion subgroup of $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$, so by Theorems 2.4 and 2.7, $\operatorname{cs}(M) \in \mathbb{R}/\mathbb{Z}$ and $\mathrm{Borel}([M]_{PSL}) \in \mathbb{R}^{r_2(k)}$ determine the PSL-fundamental class $[M]_{PSL} \in H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ completely, where k is the invariant trace field of M. These invariants are computed by the program Snap, see [4] for details.

2.4. **Bloch group and Bloch invariant.** For $\mathbb{C}$ or a subfield K of $\mathbb{C}$ the *Bloch group* $\mathcal{B}(K)$ is a certain quotient of $H_3(\mathrm{PSL}(2,K)^\delta;\mathbb{Z})$ by a torsion subgroup. It has the advantage that the image of $[M]_{\mathrm{PSL}}$ in $\mathcal{B}(\mathbb{C})$ is easily computed from an ideal triangulation. The program "Snap" (see [4]) is able to do this.

There are different definitions of the Bloch group in the literature which differ at most by torsion and agree with each other for algebraically closed fields. We use the following.

Definition 2.8. Let K be a field. The *pre-Bloch group* $\mathcal{P}(K)$ is the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}(K-\{0,1\})$ by all instances of the following relation:

$$[x]-[y]+[\frac{y}{x}]-[\frac{1-x^{-1}}{1-y^{-1}}]+[\frac{1-x}{1-y}]=0,$$

called the *five term relation*. The *Bloch group* $\mathcal{B}(K)$ is the kernel of the map

$$\mathcal{P}(k)\to K^*\wedge_{\mathbb{Z}}K^*,\quad [z]\mapsto 2(z\wedge(1-z)).$$

Suppose we have an ideal triangulation of a hyperbolic 3-manifold M using ideal hyperbolic simplices with cross ratio parameters $z_1,\dots,z_n$. This ideal triangulation can be a genuine ideal triangulation of a cusped 3-manifold, or a deformation of such a one as used by Snap and SnapPea to study Dehn filled manifolds, but it may be more generally any "degree one triangulation"; see [14].

Definition 2.9. The *Bloch invariant* $\beta(M)$ is the element $\sum_1^n[z_j]\in\mathcal{P}(\mathbb{C})$. If the z_j's all belong to a subfield $K\subset\mathbb{C}$, we may consider $\beta(M)$ as an element of $\mathcal{P}(K)$. By [14] it actually lies in $\mathcal{B}(K)\subset\mathcal{P}(K)$ and is independent of triangulation.

Considering $\beta(M)$ as the image of $[M]_{\mathrm{PSL}}$ under a map from $H_3(\mathrm{PSL}(2,K)^\delta;\mathbb{Z})\to\mathcal{B}(K)$, it follows by [17] that $\beta(M)$ can be defined in $\mathcal{B}(k)$, where $k=k(M)$ is the invariant trace field of M.

There are caveats to the definition as a sum of ideal simplex parameters. The cross-ratio parameter of an ideal simplex depends on a chosen ordering of the vertices, and if the orderings do not agree on faces of ideal simplices which are identified in M then the appropriate sum $\sum_1^n[z_j]$ may differ from $\beta(M)$ by a torsion element (of order dividing 12; this torsion issue does not arise in $\mathcal{B}(\mathbb{C})$, which is torsion-free). Not every triangulation has compatible vertex-orderings for the simplices (although there always are degree 1 triangulations which do), and if it does, these orderings induce orientations on the simplices which may be incompatible with the orientation of M, so $\beta(M)$ must then be defined as $\sum_1^n\pm[z_j]$, where the signs reflect orientations.

The Borel regulator $\mathrm{Borel}(M)$ can also be thought of as an invariant of the Bloch invariant $\beta(M)$. It is computed from the simplex parameters as follows. The invariant trace field k of M is contained in the field K generated by the simplex parameters z_i, $i=1,\dots,n$. The j-th component $\mathrm{vol}_j([M]_{PSL})$ of $\mathrm{Borel}(M)$ is

$$\mathrm{Borel}(M)_j=\sum_{i=1}^n\pm D_2(\tau_j(z_i)),$$

where $\tau_j\colon K\to\mathbb{C}$ is any complex embedding that extends $\sigma_j\colon k\to\mathbb{C}$. Here the signs are as above, and D_2 is the "Wigner dilogarithm function"

$$D_2(z)=\mathrm{Im}\ln_2(z)+\log|z|\arg(1-z),\quad z\in\mathbb{C}-\{0,1\},$$

where $\ln_2(z)$ is the classical dilogarithm function. $D_2(z)$ can also be defined as the volume of the ideal simplex with parameter z.

The component with largest absolute value in the Borel regulator is $\pm\operatorname{vol}(M)$ (see [14]). This restricts which elements of $\mathcal{B}(k)$ can be the Bloch invariant of a hyperbolic 3-manifold. A related (and possibly equivalent) restriction is in terms of the Gromov norm, which is defined on $\mathcal{B}(k)$ (see [14]); the Bloch invariants of hyperbolic manifolds are constrained to lie in the cone over a single face of the norm ball.

Nevertheless, it is plausible that the Bloch group can be generated by Bloch invariants of 3-manifolds. No obstructions to this are known, and there is mild experimental evidence for it for low degree fields that appear as invariant trace fields of manifolds in the cusped and closed censuses. Some computations related to this are given in [4].

3. Scissors Congruence

The *scissors congruence group* $\mathcal{P}(\mathbb{H}^3)$ is the abelian group generated by congruence classes of hyperbolic polyhedra of finite volume modulo all relations of the form: $P = P_1 + \cdots + P_n$ if the polyhedra $P_1, \dots, P_n$ can be glued along faces to create the polyhedron P. Dupont and Sah showed that one obtains the same group whether one allows ideal polyhedra or not ([5]; for an exposition and references for the material of this section see [9]).

The *Dehn invariant* is the map

$$\delta\colon\ \mathcal{P}(\mathbb{H}^3) \to \mathbb{R} \otimes \mathbb{R}/\pi$$

defined on generators of $\mathcal{P}(\mathbb{H}^3)$ as follows. If P is a compact polyhedron then $\delta(P) = \sum_E l(E) \otimes \theta(E)$ where the sum is over the edges E of P and $l(E)$ and $\theta(E)$ are length and dihedral angle. For an ideal polyhedron one first truncates the ideal vertices by horocycles and then uses the same definition, summing only over edges that do not bound one of the horocycle faces of the truncated polyhedron. The kernel of the Dehn invariant will be denoted

$$\mathcal{D}(\mathbb{H}^3) := \ker(\delta\colon\ \mathcal{P}(\mathbb{H}^3) \to \mathbb{R} \otimes \mathbb{R}/\pi).$$

If one subdivides an hyperbolic 3-manifold M into polyhedra then the sum of these polyhedra defines an element $\beta_0(M)$ in the scissors congruence group $\mathcal{P}(\mathbb{H}^3)$ and it is an easy exercise to see that in fact $\beta_0(M)$ is in $\mathcal{D}(\mathbb{H}^3)$.

This group $\mathcal{D}(\mathbb{H}^3)$ is closely related to the Bloch group. Since $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space, it splits as the direct sum

$$\mathcal{B}(\mathbb{C}) = \mathcal{B}_+(\mathbb{C}) \oplus \mathcal{B}_-(\mathbb{C})$$

of its $+1$ and -1 eigenspaces under the action of conjugation. Dupont and Sah [5] showed:

Theorem 3.1. *The Dehn invariant kernel $\mathcal{D}(\mathbb{H}^3)$ is naturally isomorphic to $\mathcal{B}_-(\mathbb{C})$. In fact the natural map of the pre-Bloch group $\mathcal{P}(\mathbb{C})$ to $\mathcal{P}(\mathbb{H}^3)$, defined by mapping a class $[z]$ to the ideal simplex with parameter z, induces a surjection $\mathcal{B}(\mathbb{C}) \to \mathcal{D}(\mathbb{H}^3)$ with kernel $\mathcal{B}_+(\mathbb{C})$. The Bloch invariant $\beta(M)$ is taken to the scissors congruence class $\beta_0(M)$ by this map.*

In particular, this implies that the scissors congruence class $\beta_0(M)$ is orientation-insensitive. In fact, it was first pointed out by Gerling in a letter to Gauss that any polyhedron is scissors congruent to its mirror image. The paper [9] discusses to what extent one may think of the Bloch group as giving an orientation-sensitive version of scissors congruence, and in [14] an explicit interpretation in terms of scissors congruence allowing only cut-and-paste along ideal triangles is described. However, the geometric interpretation of this for $\beta(M)$ needs care — for instance the manifold vol 3 discussed earlier appears to have no subdivision into ideal tetrahedra at all.

Note that if two manifolds have the same scissors congruence class, say $\beta_0(M_1) = \beta_0(M_2)$, this means *a priori* only that M_1 and M_2 are *stably* scissors congruent; that is, there is some polyhedron Q such that $M_1 + Q$ can be cut-and-pasted to form $M_2 + Q$. However, one can show that if M_1 and M_2 are either both compact or both non-compact then adding Q is unnecessary: M_1 can be cut into polyhedra that can be reassembled to form M_2.

Theorem 3.2. *Suppose M_1 and M_2 both have invariant trace field contained in the field K. The following are equivalent:*

1. M_1 and M_2 are stably scissors congruent, that is $\beta_0(M_1) = \beta_0(M_2)$.

2. $\operatorname{Borel}(M_1) + \operatorname{Borel}(-M_1) = \operatorname{Borel}(M_2) + \operatorname{Borel}(-M_2)$ *(this must be computed over a field containing K and $\overline{K}$).*

3. $\operatorname{Borel}(M_1) - \operatorname{Borel}(M_2)$ *is proportional to some* $\operatorname{Borel}(x)$ *with $x \in \mathcal{B}(K \cap \mathbb{R})$.*

Proof. The equivalence of the first two conditions follows because $\beta(-M) = -\overline{\beta}(M)$ and the map $x \mapsto \frac{1}{2}(x - \overline{x})$ defines the projection $\mathcal{B}(\mathbb{C}) \to \mathcal{B}_-(\mathbb{C})$.

Denote $\mathcal{B}(K)_{\mathbb{Q}}$ the image of $\mathcal{B}(K) \otimes \mathbb{Q}$ in $\mathcal{B}(\mathbb{C}) \otimes \mathbb{Q} = \mathcal{B}(\mathbb{C})$ (recall $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space). In [13] it is shown that the $\mathcal{B}(K)_{\mathbb{Q}} \cap \mathcal{B}_+(\mathbb{C}) = \mathcal{B}(K \cap \mathbb{R})_{\mathbb{Q}}$. This is thus the kernel of the map $\mathcal{B}(K) \to \mathcal{P}(\mathbb{H}^3)$, proving equivalence of the third condition. $\square$

The following conjecture has been made by many people. It is, as discussed in [9], also a consequence of Conjecture 2.6 and hence of the Ramakrishnan conjecture.

Conjecture 3.3. *The map* $\operatorname{vol}\colon \mathcal{D}(\mathbb{H}^3) \to \mathbb{R}$ *is injective.*

Snap provides many examples which give evidence for this conjecture.

4. Realizing invariants

Let k be a number field, E a quaternion algebra over k, $\mathcal{O}$ an order in E, Γ a torsion free subgroup of finite index in $\mathcal{O}^*$. Then each complex embedding of k induces a map $\Gamma \to \mathrm{PSL}(2,\mathbb{C})$ and each real embedding at which E is unramified induces a map $\Gamma \to \mathrm{PSL}(2,\mathbb{R})$. Via these maps, Γ acts discretely with finite covolume on a product $\mathbb{X}$ of copies of $\mathbb{H}^3$ and $\mathbb{H}^2$ with one copy of $\mathbb{H}^3$ for each complex place of k and one copy of $\mathbb{H}^2$ for each real place of k at which E is unramified. Denote $Y = \mathbb{X}/\Gamma$. Each projection of $\mathbb{X}$ to one of the $\mathbb{H}^3$ factors gives a codimension 3 foliation on $\mathbb{X}$ which is preserved by the Γ–action, so Y inherits codimension 3 foliations from these projections. This is a transversally hyperbolic foliation: there is a metric on the normal bundle of the foliation which induces a hyperbolic metric on any local transverse section. Similarly, the projections to $\mathbb{H}^2$ give codimension 2 transversally hyperbolic foliations.

Pick one of the codimension 3 foliations $\mathcal{F}$. Let $M^3 \to Y$ be an immersion of a 3-manifold to Y that is everywhere transverse to $\mathcal{F}$. So M^3 has an induced hyperbolic metric. If M^3 is compact this metric is, of course, complete of finite volume. We are interested also in the case that M^3 is not compact, but we require then that the metric is complete of finite volume.

Theorem 4.1. *The invariant trace field and quaternion algebra for M^3 embed in k resp. E (as concrete field and quaternion algebra).*

Conversely, up to commensurability, every finite volume hyperbolic 3–manifold with integral traces occurs this way.

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