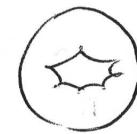
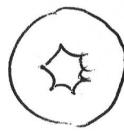


# Rigidity & arithmeticity



1.

$\curvearrowleft \curvearrowright$   $\mathcal{T} = \mathbb{R}^{6g-6}$  different marked hyperbolic metric up to isometry  
surface  $g \geq 2$  or  $\mathcal{T} / M(S)$  different hyp metrics.  
 $M(S)$  = mapping class group up to isotopy to identity

If  $n \geq 3$ , the situation is totally different.

$M = \mathbb{H}^n / P$  finite volume hyperbolic mfld  $n \geq 3$ .

It has such a unique hyp metric.

In algebraic term,

$$P_1 \cong_{\text{isomorphic}} P_2 \text{ lattices in } \text{SO}(n, 1) \Rightarrow P_1 = g P_2 g^{-1} \text{ for some } g \in \text{SO}(n, 1)$$

This statement is true for all semisimple Lie gp except  $SL(2, \mathbb{R})$

This is kind of a global rigidity

$AHM \subseteq R = \{ \rho: \pi_1(M) \rightarrow \text{PSL}(2, \mathbb{C}) = \text{SO}^+(3, 1) \} / \text{conjugacy}$  character variety

discrete faithful = a point.

Long before Mostow rigidity, Weil proved that  $H^1(\Gamma, \mathcal{O}_{\text{Ad}(\bar{\rho})}) = 0$

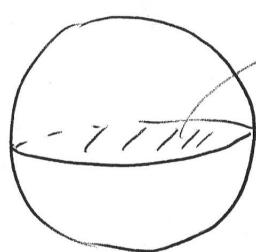
$$\Gamma \xrightarrow{i} G$$

lattice

Zariski tangent space  
of the character variety

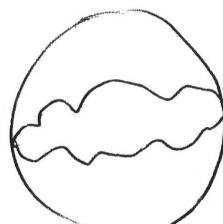
i.e., lattices in semisimple Lie gp except  $SL(2, \mathbb{R})$  is locally rigid.

One can enlarge the target group  $G$  and study the rigidity phenomenon.



$$\pi \subseteq SL(2, \mathbb{R})$$

$\varphi$   
quasi-conformal  
deformation



Fuchsian

$$\rho' = \varphi \rho \varphi^{-1} \subseteq SL(2, \mathbb{C})$$

quasifuchsian grp

$$\Gamma \subseteq \mathrm{SU}(n, 1) \subseteq \mathrm{SU}(m, 1) \quad m > n \geq 1$$

$\Gamma$  is locally rigid in  $\mathrm{SU}(m, 1)$ . (Goldman-Millson)

For example  $\Gamma \subseteq \mathrm{SO}(2, 1) \subseteq \mathrm{SU}(2, 1)$  can be continuously deformed to a bigger group but  $\Gamma \subseteq \mathrm{SU}(1, 1) \subseteq \mathrm{SU}(2, 1)$  cannot be continuously deformed.

More examples. Local rigidity.

$$\Gamma \subseteq \mathrm{SO}(3, 1), \mathrm{SO}(4, 1) \subseteq \mathrm{Sp}(1, 1) \subseteq \mathrm{Sp}(n, 1) \quad (\text{K-pansu})$$

$$\Gamma \subseteq \mathrm{SU}(n, 1) \subseteq \mathrm{Sp}(n, 1) \subseteq \mathrm{SU}(2n, 2) \subseteq \mathrm{SO}(4n, 4) \quad (\text{K-Klingler-Pansu})$$

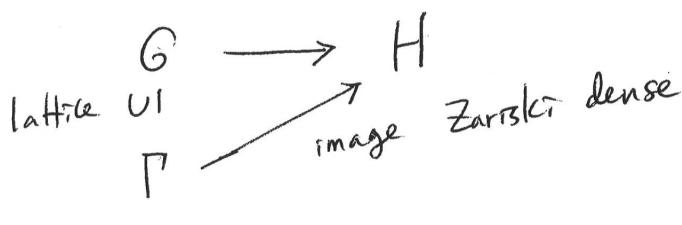
$G$  semisimple R-rank of  $G = \dim$  of maximal flat in  $X$

$G/\Gamma = X$  symmetric space

e.g.  $\mathrm{SO}(n, \mathbb{R}), \mathrm{SU}(n, 1), \mathrm{Sp}(n, 1)$  rk 1 semisimple Lie groups

associated  $H_{\mathbb{R}}^n$   $H_{\mathbb{C}}^n$   $H_{\mathbb{H}}^n$

Margulis superrigidity thm for  $\text{rk} \geq 2$ .



For  $\text{rk} = 1$ , Corlette proved super-rigidity for  $\mathrm{Sp}(n, 1)$ ,  $\mathbb{F}_p^{2n}$ .

$\Rightarrow$  Every lattice in s.s Lie gp w/ super-rigidity

is arithmetic i.e.  $\exists$  algebraic  $\mathbb{Q}$ -gp  $H$  s.t

$$\begin{array}{ccc} H(\mathbb{R}) & \xrightarrow{\phi} & G \\ \text{U1} & \text{cpt kernel} & \text{U1} \\ H(\mathbb{Z}) & & \Gamma \end{array} \quad \phi(H(\mathbb{Z})) \& \Gamma \text{ commensurable.}$$

$$SL(2, \mathbb{Z}) \subseteq PSL(2, \mathbb{R})$$

e.g. example  $d$  pos. square free integer

$$\mathcal{O}_d \subseteq \mathbb{Q}(\sqrt{d})$$

ring of integers  $\mathcal{O}_d = \begin{cases} \mathbb{Z}[\sqrt{d}] & d \equiv 1, 2 \pmod{4} \\ \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & d \equiv 3 \pmod{4} \end{cases}$

Picard modular gp

$SU(2,1) ; \mathcal{O}_d$  matrix of signature  $(2,1)$  w/ entries in  $\mathcal{O}_d$ .  
 $\subseteq SU(2,1) =$  isometry gp of  $H^2_{\mathbb{C}}$ .

non-uniform lattice.

Not known that  $\exists$  non-arithmetic lattice in  $PU(n,1), n \geq 4$

Interesting facts about arithmetic lattices.

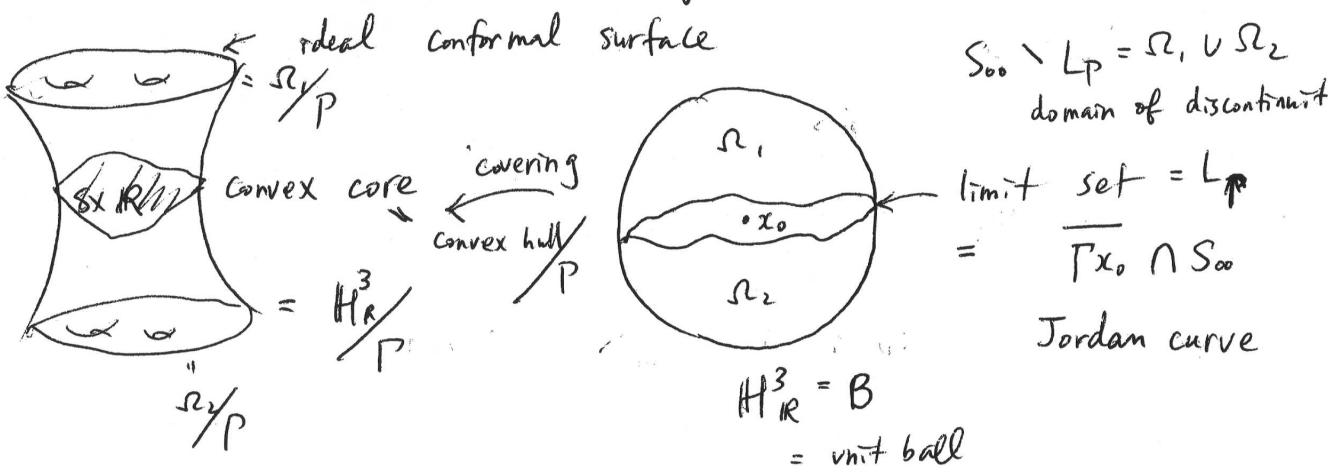
example

$$\lim_{x \rightarrow \infty} \frac{\log (\# \text{conjugacy classes arithmetic lattices} \text{ in } SL_2(\mathbb{R}))}{x \log x} = \frac{1}{2\pi}$$

Contrary to rigidity phenomenon for lattices, there are rich theory for deformation space of infinite volume sfds.

- hyp 3-mfd w/  $\partial$ .

$M = S \times I$        $\overset{\circ}{M} = S \times (0, 1)$  admits geometrically finite convex cocompact hyp metrics  
 $S$ : closed  
genus  $\geq 2$       = quasi-fuchsian space =  $\text{QF}(S)$

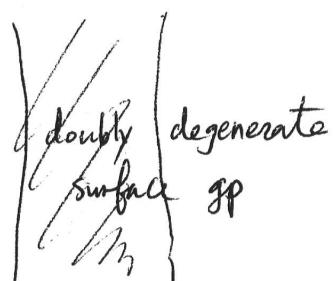
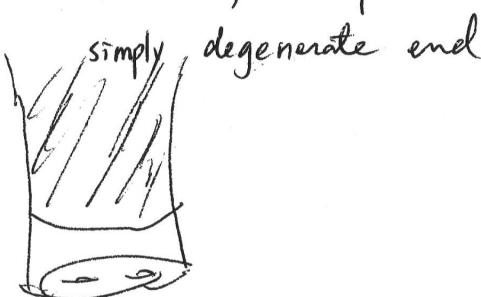


Bers uniformization thm

$$\text{QF}(S) \cong \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \subseteq \text{AH}(S) = \left\{ \varphi : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C}) \right\} / \sim$$

(X, Y)      open

How does  $\partial \text{QF}(S)$  look like? very complicated.  
inside  $\text{AH}(S)$



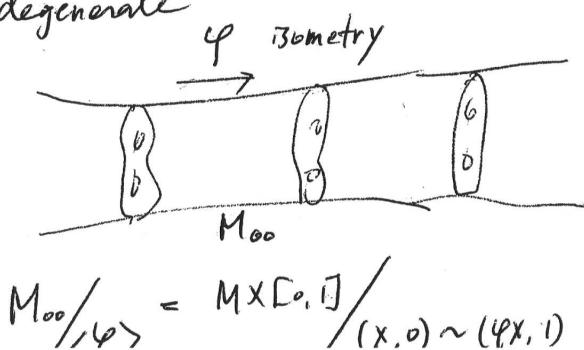
$\varphi$ : pseudo-Anosov

Indeed

$(\varphi^n X, \varphi^n \bar{X})$   $\xrightarrow{\text{converge algebraically in } \text{AH}(S)}$  doubly degenerate

converge  
algebraically  
in  $\text{AH}(S)$

(Thurston's double)  
limit thm



$M =$  hyp. 3-mbd w/ at least one compressible boundary  $S$

i.e.,  $\pi_1(S) \rightarrow \pi_1(M)$   
not injective

If  $\rho_i$  convex cocompact w/ conformal structure on  $S$ ,  $t_i \in \mathcal{T}(S)$

$$t_i \rightarrow [\chi] \quad \chi \text{ doubly incompressible i.e., } i(\chi, \partial A) > 0$$

Thurston's compactification of  $\mathcal{T}(S)$  for any compressing disk or essential annulus  $A$

$\Rightarrow$  subsequence of  $\rho_i$  converges. (K-Lecuire-Oshika)

~~•~~ Rich deformation theory about surface gp in semisimple Lie gps.

$\rho: \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$   $\mathrm{AH}(S)$ , quasifuchsian gps etc.

$\rho: \pi_1(S) \rightarrow \mathrm{PSU}(n, 1)$   $H_q^n = \mathrm{PSU}(n, 1) / \mathrm{SU}(n)$  = complex hyperbolic space

$H_{12}^2 \xrightarrow{f_p} (H_q^n, \omega)$  Kähler form  
p-equiv map  $f_p^* \omega$  descends to  $S$   
gives rise to Toledo invariant

$$\frac{1}{2\pi} \int_S f_p^* \omega \in \mathbb{Z}$$

Recently Burger-Iozzi-Wienhard

$$[\chi(S), -\chi(S)]$$

$\rho: \pi_1(S) \rightarrow \mathrm{Iso}$  (Hermitian symmetric space)

w/ maximal Toledo invariant

↑  
maximal  
Toledo  
invariant

$\Rightarrow \overline{\rho(\pi_1(S))}$  Zariski closure semisimple Lie gp associated  
to a tube type Hermitian domain.

kind of converse

$\rightsquigarrow \rho: \pi_1(S) \rightarrow G$  semisimple Lie gp

If  $\rho$  cannot be deformed to a Zariski dense ref

then  $\overline{\rho(\pi_1(S))}^2$  tube type Hermitian domain. (K-Pansu)