A moduli space of Higgs bundles and a character variety III

Hitchin system and Krichever-Lax matrices

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Main Goals

- Hitchin system and Hilbert scheme
- Krichever-Tyurin parameters and dynamical system

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Definition

A dynamical system is said to be an Algebraically Completely Integrable system if

- it is a completely integrable system

$$\mathbf{H} : M^{2l} \to \mathbb{C}^l \text{ by } \mathbf{H}(m) = (H_1(m), \ldots, H_l(m))$$

- a generic fiber of $\mathbf{H}$ is an (Zariski) open set of an abelian variety
- each Hamiltonian flow of X_{H_i} is linear on a generic fiber.

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A holomorphic structure on a smooth complex vector bundle E of rank l over a compact Riemann surface $\mathfrak{R}$ is a differential operator d''_A satisfying

$$d''_A(fs) = \overline{\partial} f \otimes s + f d''_A s \text{ where } s \in \mathcal{A}^0(\mathfrak{R}, \mathrm{E}) \text{ and } f \in C^\infty(\mathfrak{R}).$$

- Symplectic form: For $(A, \Phi) \in \mathrm{T}^*_A \mathfrak{A}$

$$\omega_{(A,\Phi)}((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_{\mathfrak{R}} \mathrm{Tr}(\dot{\Phi}_2 \wedge \dot{A}_1 - \dot{\Phi}_1 \wedge \dot{A}_2).$$

- A momentum map $\mu : \mathrm{T}^* \mathfrak{A}^s \to \mathrm{Lie}(\mathcal{G})^*$ induced by the action of gauge group $\mathcal{G}$ is given by

$$\begin{cases} \mu(A, \Phi) & = d''_A \Phi \\ \mu^{-1}(0)/\mathcal{G} & \cong \mathrm{T}^* \mathcal{N} \end{cases}.$$

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Hitchin System

Theorem (Hitchin)

Let $\mathcal{N}$ be a moduli space of stable holomorphic vector bundles over a compact Riemann surface $\mathfrak{R}$ of genus > 1. Then $\mathrm{T}^*\mathcal{N}$ is an algebraically completely integrable system.

Spectral curve

$$\begin{array}{ccc} \lambda_z \in \mathrm{K}_{\mathfrak{R}} & \xrightarrow{\pi^*} & \pi^*\lambda \in \pi^*\,\mathrm{K}_{\mathfrak{R}} \\ \pi\downarrow & & \downarrow \\ z \in \mathfrak{R} & \xleftarrow{\pi} & \lambda_z \in \mathrm{K}_{\mathfrak{R}}\,. \end{array} \tag{1}$$

A spectral curve $\widehat{\mathfrak{R}}$ associated with a Higgs field $\Phi_{[A]} \in \mathrm{T}^*\,\mathcal{N}$ is the zero locus of a section $\pi^*\det(\lambda_z \cdot I_{l\times l} - \Phi_{[A]}(z)) \in (\pi^*\,\mathrm{K}_{\mathfrak{R}})^l$

$$\widehat{\mathfrak{R}} = \{\lambda_z \in \mathrm{K}_{\mathfrak{R}} \mid \pi^*\det(\lambda_z \cdot I_{l\times l} - \Phi_{[A]}(z)) = 0\}.$$

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Abelianization Program

- The Hitchin map

$$\mathbf{H} : T^*\mathcal{N} \to \bigoplus_{i=1}^{l} H^0(\mathfrak{R}, K^i_{\mathfrak{R}}).$$

- $\mathbf{H}^{-1}(q)$ is an open set in $\mathrm{Jac}(\widehat{\mathfrak{R}})$ where $g(\widehat{\mathfrak{R}}) = l^2(g-1)+1$.
- The Hitchin's Abelianization Program.
- $T^*\mathcal{N} \subseteq \mathcal{M}$

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Example for rank 1

- $$\begin{aligned} T^* \mathcal{N} \cong T^* \operatorname{Jac}(\mathfrak{R}) &\cong \operatorname{Jac}(\mathfrak{R}) \times H^{1,0}(\mathfrak{R}) \\ &\cong \operatorname{Jac}(\mathfrak{R}) \times H^0(\mathfrak{R}, K_{\mathfrak{R}}) \end{aligned}$$
- Projection:

$$\begin{aligned} \mathbf{H} : \operatorname{Jac}(\mathfrak{R}) \times H^0(\mathfrak{R}, K_{\mathfrak{R}}) &\longrightarrow H^0(\mathfrak{R}, K_{\mathfrak{R}}) \\ ([A], [\Phi]) &\longmapsto [\Phi]. \end{aligned}$$

- $\widehat{\mathfrak{R}} \cong \mathfrak{R}$.

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The Hilbert Scheme of points

- Let S be a K3 surface with a line bundle L with $h^0(L) = \widetilde{g} + 1$.
- Define

$$\Phi : \mathrm{Hilb}^{\widetilde{g}} S \to \mathbb{P}(\mathrm{H}^0(S, L)^*) = (\mathbb{P}^{\widetilde{g}})^* \text{ by } \Phi(\xi) = H_\xi.$$

- This is a Beauville-Mukai system of Lagrangian fibration.
- The fibers are open sets in the Jacobi varieties of some curves

Theorem (1995, R. Donagi, R. Lazarsfeld)

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Idea of proof

- $\mathfrak{R} \subset S \subset \mathbb{P}^g$ with $g(\mathfrak{R}) = g$.
- Take $\mathrm{Cone}(S) \subset \mathbb{P}^{g+1}$ and there exists

$$f : \mathrm{BL}_{\mathfrak{R}}\,\mathrm{Cone}(S) \to \mathbb{P}^1 \text{ by } f^{-1}(p) = H_p \cap \mathrm{BL}_{\mathfrak{R}}\,\mathrm{Cone}(S)$$

such that

$$\begin{cases} f^{-1}(p) \cong S & \text{for } p \neq p_0 \\ f^{-1}(p) \cong \overline{\mathrm{K}_{\mathfrak{R}}} & \text{for } p = p_0 \end{cases}$$

- Deformation to a Hitchin system

$$\Phi : \mathrm{Hilb}^{\widetilde{g}}\, f^{-1}(p) \to (\mathbb{P}^{\widetilde{g}})^* = |l\mathfrak{R}| \text{ where } \widetilde{g} = l^2(g-1)+1.$$

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Example

- Take a K3 surface
  $S = \{P(X_0, X_1, X_2, X_3) = 0 \text{ where } \deg P = 4\}$.
- Define
  $$\Phi : \mathrm{Hilb}^3 S \to (\mathbb{P}^3)^* \text{ by } \Phi(\xi) = H_\xi.$$
  Note $\Phi(\xi) = [\det \Phi_0, \det \Phi_1, \det \Phi_2, \det \Phi_3]$ is a Plücker coordinate.
- $\Phi^{-1}(q) \subset \mathrm{Sym}^3(C) \cong \mathrm{Jac}(C)$ where $C = H_\xi \cap S$ and $g(C) = 3$.

Example

- Let $g(\mathfrak{R}) = 2$ and define a Hitchin map

$$\mathbf{H} = \det : \mathrm{T}^*\,\mathcal{N} \to \mathrm{H}^0(\mathfrak{R}, \mathrm{K}^2_{\mathfrak{R}}) \cong \mathbb{C}^3.$$

  Here, $\mathcal{N}$ is the moduli space of stable vector bundles of rank 2 with a fixed determinant bundle.
- $\widehat{\mathfrak{R}} = \{\lambda^2 - q = 0\}$ with $g(\widehat{\mathfrak{R}}) = 5$.
- $\mathbf{H}^{-1}(q) \subset \mathrm{Prym}(\widehat{\mathfrak{R}}) = \mathrm{Jac}(\widehat{\mathfrak{R}}) / \mathrm{Jac}(\widehat{\mathfrak{R}}/\sigma)$.

Theorem (1996, J. Hurtubise)

Let $X = \mathbb{P}(\mathrm{K}_{\mathfrak{R}} \oplus 1)$.

$$\mathrm{Hilb}^{\widetilde{g}}\, X \cong \mathrm{T}^*\,\mathcal{N}, \ \textit{symplectically and birationally.}$$

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Uniquely equipped bundle

Definition

We say that E is a uniquely equipped vector bundle if $\dim_{\mathbb{C}} \mathrm{H}^0(\mathfrak{R}, \mathrm{E}) = l$ and E has an equipment.

- An equipment $\{\eta_1, \dots, \eta_l\}$ generates a fiber E_p for all $p \in \mathfrak{R}$ except lg points γ_i.
- If E is a semi-stable bundle of rank $l > 1$ and degree lg over $\mathfrak{R}$, then $\dim_{\mathbb{C}} \mathrm{H}^0(\mathfrak{R}, \mathrm{E}) = l$.
- (A. N. Tyurin) There is a one-to-one correspondence between semi-stable bundles and uniquely equipped bundles of rank l and degree lg.

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Tyurin parameter

- $\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{i,j}\eta_j(\gamma_i)$ and $\eta_l(\gamma_i) \neq 0$.
- The Tyurin parameters associated with a uniquely equipped bundle E

$$\left\{\gamma_i, \{\alpha_{i,j}\}_{j=1}^{l}\right\}_{i=1}^{lg} \in \mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1}).$$

- The diagonal action of $\mathbf{SL}(l, \mathbb{C})$ on the symmetric power of $\mathbb{P}^{l-1}$

$$\mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})/\mathbf{SL}(l, \mathbb{C}).$$

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Matrix Divisor

Matrix divisor $E = \left[\{\mathrm{E}_i\}\right]$: A normal form is

$$\mathrm{E}_{p_k,i} = \begin{pmatrix} z^{d_{1,k}} & 0 & \cdots & 0 \\ 0 & z^{d_{2,k}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z^{d_{l,k}} \end{pmatrix} \begin{pmatrix} 1 & \alpha_{1,2,k,i}(z) & \cdots & \alpha_{1,l,k,i}(z) \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{l-1,l,k,i}(z) \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Here $\alpha_{r,s,k,i}(z) \in \frac{\mathbb{C}[[z]]}{z^{d_{s,k}-d_{r,k}}\mathbb{C}[[z]]}$ and $D = \sum_{k=1}^{N} m_k p_k$. For an index j where U_j does not contain p_k, a normal form of E_j is defined by

$$\mathrm{E}_j = \mathrm{id}_{l \times l}\,.$$

Matrix Divisor

For an effective divisor $D = \sum_{k=1}^{N} p_k$, $\mathrm{E}_j = \mathrm{id}_{l \times l}$ and

$$\mathrm{E}_{p_k,i} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_{1,l,k,i} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & 1 & \alpha_{l-1,l,k,i} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Here $\alpha_{r,l,k,i} \in \mathbb{C}$.

Matrix Divisor

The set $\{\mathrm{G}_{ij}\}$ of transition functions where

$$\mathrm{E}_i \cdot \mathrm{G}_{ij} = \mathrm{E}_j \text{ on } U_i \cap U_j.$$

Hence,

$$\mathrm{G}_{\gamma_k,ij} = \mathrm{E}^{-1}_{\gamma_k,i} \cdot \mathrm{E}_{\gamma_k,j} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_{1,l,k,j} - \alpha_{1,l,k,i}\frac{z_{k,j}}{z_{k,i}} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & 1 & \alpha_{l-1,l,k,j} - \alpha_{l-1,l,k,i}\frac{z_{k,j}}{z_{k,i}} \\ 0 & \cdots & \cdots & 0 & \frac{z_{k,j}}{z_{k,i}} \end{pmatrix}.$$

Krichever-Lax matrices I

A Krichever-Lax matrix associated to Tyurin parameters (γ, α) and a canonical divisor K of a compact Riemann surface $\mathfrak{R}$ of genus g is a matrix-valued meromorphic function $\mathrm{L}(p; \gamma, \alpha)$ with at most simple poles at γ_i and poles at K satisfying the following conditions: There exist $\boldsymbol{\beta}_j \in \mathbb{C}^l$ and $\kappa_j \in \mathbb{C}$ for $j = 1, \ldots, lg$ such that a local expression in a neighborhood of γ_j is given by

Krichever-Lax matrices II

for $j = 1, \ldots, lg$

$$\mathsf{L}(p;\gamma,\alpha) = \frac{\mathsf{L}_{j,-1}(\gamma,\alpha)}{z(p) - z(\gamma_j)} + \mathsf{L}_{j,0}(\gamma,\alpha) + O\big((z(p) - z(\gamma_j))\big)$$

with the following two constraints

- $\mathsf{L}_{j,-1}(\gamma,\alpha) = \boldsymbol{\beta}_j^T \cdot \boldsymbol{\alpha}_j$, i.e., of rank 1 and it is traceless

$$\operatorname{Tr} \mathsf{L}_{j,-1} = \boldsymbol{\alpha}_j \cdot \boldsymbol{\beta}_j^T = 0.$$

- $\boldsymbol{\alpha}_j$ is a left eigenvector of $\mathsf{L}_{j,0}$

$$\boldsymbol{\alpha}_j \mathsf{L}_{j,0}(\gamma,\alpha) = \kappa_j \boldsymbol{\alpha}_j.$$

Krichever-Lax matrices

- Let us denote the set of Krichever-Lax matrices associated to Tyurin parameters (γ, α) and a canonical divisor K by $\mathcal{L}^K_{\gamma,\alpha}$.
- For $\zeta_{\gamma,\alpha}$ such that around γ_j for $j = 1, \ldots, lg$,

$$\zeta_{\gamma,\alpha}(z) = \frac{c_j \boldsymbol{\alpha}_j}{z - z(\gamma_j)} + O(1) \text{ where } c_j \in \mathbb{C},$$

$$\begin{cases} \zeta_{\gamma,\alpha}(z)\frac{\mathsf{L}_{j,-1}}{z-z_j} = \frac{d_j \boldsymbol{\alpha}_j}{z-z(\gamma_j)} + O(1) \\ \zeta_{\gamma,\alpha}(z)\mathsf{L}_{j,0} = \frac{\kappa_j c_j \boldsymbol{\alpha}_j}{z-z(\gamma_j)} + O(1) \end{cases}.$$

- L is a Higgs field, i.e., a section of End $\mathsf{E}_{\gamma,\alpha} \otimes \mathsf{K}_{\mathfrak{R}}$.

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Correspondence

Lemma

Let $(\gamma, \alpha) \in \mathcal{M}'_0$. *There is a bijective map*

$$\mathsf{L} \mapsto \left\{ \boldsymbol{\alpha}_j, \boldsymbol{\beta}_j, \gamma_j, \kappa_j \right\}_{j=1}^{lg}$$

between $\mathcal{L}^K$ *and a subset* $\mathcal{V}$ *of* $\mathcal{S}^{lg}(\mathbb{P}^{l-1} \times \mathbb{C}^l \times \mathfrak{R} \times \mathbb{C})$ *defined by*

$$\boldsymbol{\alpha}_j \cdot \boldsymbol{\beta}_j^T = 0 \text{ for } j = 1, \ldots, lg \text{ and}$$

$$\sum_{p \in \mathfrak{R}} \mathrm{res}(\mathsf{L} \otimes \omega) = \sum_{j=1}^{lg} \boldsymbol{\beta}_j^T \cdot \boldsymbol{\alpha}_j = 0_{l \times l}. \tag{2}$$

Correspondence

Let $\psi(p,\mu)\mathsf{L}(p) = \mu(p)\psi(p,\mu)$ with $\sum_{i=1}^{l} \psi_i(\widehat{p}) = 1$ and

$$\widehat{\mathfrak{R}} = \left\{ \det\left(\mu \cdot I_{l\times l} - \mathsf{L}(p;\gamma,\alpha)\right) = \mu^l + \sum_{d=1}^{l} h_d(p;\mathsf{L})\mu^{l-d} = 0 \right\}.$$

Theorem (I. Krichever)

Let $[\mathsf{L}] \in \mathcal{L}^K/\mathbf{SL}(l,\mathbb{C})$ be an $\mathbf{SL}(l,\mathbb{C})$-orbit of L in $\mathcal{L}^K$. Then there is a one-to-one correspondence

$$[\mathsf{L}] \longleftrightarrow \left((h_1,\dots,h_l),[\widehat{D}]\right) = \left(\widehat{\mathfrak{R}},[\widehat{D}]\right).$$

$[\widehat{D}]$ is an equivalence class of an effective divisor of degree $\widehat{g}+l-1$ on $\widehat{\mathfrak{R}}$ where $\widehat{g}$ is the genus of $\widehat{\mathfrak{R}}$.

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Correspondence

- Universal symplectic form

$$\Omega = \delta \operatorname{Tr}(L\delta \log \Psi) \Longrightarrow \omega = \sum_{s=1}^{lg} \operatorname{res}_{\gamma_s} \Omega dz$$

- J. Hurtubise, I. Krichever

$$\omega = \sum_{s=1}^{\widetilde{g}} \delta k(\widehat{\gamma}_s) \wedge \delta z(\widehat{\gamma}_s) \text{ on } \operatorname{Hilb}^{\widetilde{g}} K_{\mathfrak{R}} .$$

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Lax representation

- Lax Representation

$$\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t].$$

- Isospectral Deformation

$$\frac{d}{dt}\mathrm{Tr}(\mathsf{L}^k) = k\,\mathrm{Tr}\left((\frac{d}{dt}\mathsf{L})\mathsf{L}^{k-1}\right) = k\,\mathrm{Tr}([\mathsf{M},\mathsf{L}]\mathsf{L}^{k-1}) = 0.$$

For a polynomial $\mathsf{P}(\mathsf{L})$ and $[\mathsf{Q},\mathsf{L}] = 0$, we have

$$\begin{aligned}\frac{d}{dt}\mathsf{L} &= [\mathsf{M}+\mathsf{P}(\mathsf{L}), \mathsf{L}] \\ &= [\mathsf{M}+\mathsf{Q}, \mathsf{L}] = [\mathsf{M}, \mathsf{L}].\end{aligned}$$

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Eigenvector mapping

Definition

We shall call (3)

$$\overline{\psi}_t(\gamma(t), \alpha(t)) : \widehat{\mathfrak{R}} \to \mathbb{P}^{l-1} \tag{3}$$

an eigenvector mapping associated to a Lax equation

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- The eigenvector mapping $\overline{\psi}_t$ induces

$$\varphi_{\widehat{\mathfrak{R}}} : \mathcal{L}^K_{\widehat{\mathfrak{R}}}/\mathbf{SL}(l, \mathbb{C}) \to \mathrm{Pic}^{\widehat{g}+l-1}(\widehat{\mathfrak{R}}) \text{ by } \varphi_{\widehat{\mathfrak{R}}}([\mathsf{L}_t]) = \overline{\psi}^*_t(\mathcal{O}_{\mathbb{P}^{l-1}}(1)). \tag{4}$$

We will also call it an eigenvector mapping associated to a spectral curve $\widehat{\mathfrak{R}}$.

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Euler sequence

Pulling back the Euler sequence (5) on $\mathbb{P}^{l-1}$ by $\overline{\psi}_t$ gives the following short exact sequence on $\widehat{\mathfrak{R}}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{l-1}} \longrightarrow \mathbb{C}^l \otimes \mathcal{O}_{\mathbb{P}^{l-1}}(1) \longrightarrow \Theta_{\mathbb{P}^{l-1}} \longrightarrow 0. \tag{5}$$

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From short exact sequence (6), we have a long exact sequence:

$$\mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}^l \otimes \mathsf{L}_t) \longrightarrow \mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\psi}_t^* \Theta_{\mathbb{P}^{l-1}}) \xrightarrow{\delta} \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) \longrightarrow \cdots. \tag{7}$$

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Using $\frac{d}{dt}(\psi \cdot \mathrm{L}) = \mu \cdot \frac{d}{dt}\psi$ and $\frac{d}{dt}\mathrm{L}_t = [\mathrm{M}_t, \mathrm{L}_t]$,

$$\psi_t \mathrm{M}_t + (\frac{d}{dt}\psi_t) = \lambda_t \psi_t. \tag{8}$$

Definition

A residue section $\rho(\mathrm{M}_t) \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)})$ associated to M_t is defined to be the Laurent tail $\{\lambda_{t,i}\}$ of λ_t in Equation (8) at $K = \sum_{i=1}^{2g-2} p_i$.

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This induces a long exact sequence

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Cohomological Interpretation

Theorem

If $[\mathsf{M}_t, \mathsf{L}_t]$ *is tangent to* $\mathcal{L}^K_{\widehat{\mathfrak{R}}}$, *then*

$$\frac{d}{dt}\mathsf{L}_t = \partial\rho(\mathsf{M}_t) \in \mathsf{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) \cong \mathsf{H}^0(\widehat{\mathfrak{R}}, \mathsf{K}_{\widehat{\mathfrak{R}}}).$$

Corollary

L_t *is linear on* $\mathrm{Pic}^{\widehat{g}+l-1}(\widehat{\mathfrak{R}})$ *if and only if*

$$\frac{d}{dt}\rho(\mathsf{M}_t) \equiv 0 \text{ modulo } \operatorname{span}\{\jmath(\mathsf{H}^0(\widehat{\mathfrak{R}}, \pi^*\mathsf{K}^n)), \rho(\mathsf{M}_t)\}.$$

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Toda Lattices

$$\frac{d}{dt}\begin{pmatrix} a(\xi,t) & b(\xi,t) \\ c(\xi,t) & -a(\xi,t) \end{pmatrix}$$
$$= \left[\begin{pmatrix} 2ab & b^2 \\ bc - \frac{c}{b} & -\frac{a}{b} \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right] = \begin{pmatrix} c & a \\ -\frac{3ac}{b} & -c \end{pmatrix}$$

- Spectral Curve $\widehat{\mathfrak{R}}$

$$0 = \det(\eta I_{2\times 2} - \mathrm{L}(\xi)) = \eta^2 - bc - a^2.$$

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$$\frac{d}{dt}\begin{pmatrix} b_1 & a_1 & a_3\xi^{-1} \\ a_1 & b_2 & a_2 \\ a_3\xi & a_2 & b_3 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & -a_1 & a_3\xi^{-1} \\ a_1 & 0 & -a_2 \\ -a_3\xi & a_2 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & a_1 & a_3\xi^{-1} \\ a_1 & b_2 & a_2 \\ a_3\xi & a_2 & b_3 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 2(a_3^2 - a_1^2) & a_1(b_1 - b_2) & a_3\xi^{-1}(b_3 - b_1) \\ a_1(b_1 - b_2) & 2(a_1^2 - a_2^2) & a_2(b_2 - b_3) \\ a_3\xi(b_3 - b_1) & a_2(b_2 - b_3) & 2(a_2^2 - a_3^2) \end{pmatrix}.$$

In general,

$$\begin{cases} \frac{d}{dt}a_k & = a_k(b_k - b_{k+1}) \text{ with } a_0 = a_n \\ \frac{d}{dt}b_k & = 2(a_{k-1}^2 - a_k^2) \text{ with } b_1 = b_{n+1} \end{cases}.$$

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- (N. Saitoh) Toda $\Longrightarrow$ KDV

$$\frac{\partial}{\partial t}u = 6u\frac{\partial}{\partial x}u + \frac{\partial^3}{\partial x^3}u \text{ as } n \to x.$$

- (B. Dubrovin, I. Krichever, S. Novikov) KDV Equation

$$\frac{\partial}{\partial t}\mathrm{L} = [\mathrm{M}, \mathrm{L}] \text{ where } \begin{cases} \mathrm{L} & = -\partial_x^2 + u(x,t) \\ \mathrm{M} & = -4\partial_x^3 + 6u\partial_x + \frac{\partial}{\partial x}u \end{cases}.$$

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- Spectral Curve $\widehat{\mathfrak{R}}$

$$
\begin{aligned}
0 &= \det(\eta I_{3\times 3} - \mathrm{L}(\xi)) \\
&= \eta^3 + (b_1 b_2 + b_2 b_3 + b_1 b_3 - \sum_{i=1}^{3} a_i^2)\eta \\
&+ a_1^2 b_3 + a_2^2 b_1 + a_3^2 b_2 - b_1 b_2 b_3 - a_1 a_2 a_3(\xi + \xi^{-1}).
\end{aligned}
$$

- $\widehat{\mathfrak{R}}$ is a hyper-elliptic curve ($g(\mathcal{S}) = 2$)).
- 3 first integrals

$$
\mathrm{Tr}\,\mathrm{L}(\xi) = b_1 + b_2 + b_3, \mathrm{Tr}\,\mathrm{L}^2, \text{ and } \mathrm{Tr}\,\mathrm{L}^3.
$$

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$$\begin{pmatrix} b_1 & a_1 & a_3\xi^{-1} \\ a_1 & b_2 & a_2 \\ a_3\xi & a_2 & b_3 \end{pmatrix} \begin{pmatrix} 1 \\ f \\ g \end{pmatrix} = \eta \begin{pmatrix} 1 \\ f \\ g \end{pmatrix}.$$

- Residue section

$$\begin{pmatrix} 0 & -a_1 & a_3\xi^{-1} \\ a_1 & 0 & -a_2 \\ -a_3\xi & a_2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ f \\ g \end{pmatrix} - \frac{d}{dt} \begin{pmatrix} 1 \\ f \\ g \end{pmatrix} = \lambda(t) \begin{pmatrix} 1 \\ f \\ g \end{pmatrix}.$$

- For $\eta^{-1}(\infty) = p + q$,

$$\begin{cases} \lambda(t) & = -\eta + \text{ holomorphic terms at } q \\ \lambda(t) & = \eta + \text{ holomorphic terms at } p. \end{cases}$$

Consequently,

$$\frac{d}{dt}\rho(\mathrm{M}) \equiv 0.$$