A moduli space of Higgs bundles and a character variety II

Hitchin's self-duality equations and non-abelian Hodge theory

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Main Goals

- Non-abelian Hodge theory
- Hitchin's self-duality equations
- Application: Goldman's theorem, real variation of Hodge structure, and Teichmüller components, etc.

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Semi-stable Bundles

- A holomorphic vector bundle E of rank l is said to be semi-stable if for all proper sub-bundles H of E we have

$$\mathrm{slope}(\mathrm{H}) = \frac{\deg \mathrm{H}}{\mathrm{rank}\,\mathrm{H}} \leq \frac{\deg \mathrm{E}}{\mathrm{rank}\,\mathrm{E}} = \mathrm{slope}(\mathrm{E}).$$

  It is said to be a stable bundle if the strict inequality holds.
- Any line bundle is stable.
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- For any line bundle L, E is stable iff $\mathrm{E} \otimes \mathrm{L}$ is stable.

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Higgs bundle

- Let E be a holomorphic vector bundle over a compact Riemann M. Then a Higgs field associated with E is a holomorphic section Φ of $\mathrm{End}(E) \otimes K_M$
- A Higgs bundle is a pair (E, Φ) consisting of a holomorphic vector bundle and a Higgs field.
- A stable Higgs bundle is a Higgs bundle such that for any Φ-invariant proper sub-bundles H of E we have $\mathrm{slope}(H) = \frac{\deg H}{\mathrm{rank}\, H} < \frac{\deg E}{\mathrm{rank}\, E} = \mathrm{slope}(E)$.

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Problem

How to prove

Irreducible Representations

↕ A.Weil

Stable Higgs Bundles $\overset{?}{\longleftrightarrow}$ Irreducible Flat connections

Self-duality equations

- Let M be a compact Riemann surface of genus ≥ 2.

$$\begin{cases} d''_A\Phi & = 0 \quad \text{Holomorphic condition} \\ F(A) & = [\Phi, \Phi^*] \quad \text{Unitary condition.} \end{cases}$$

$A \in \mathcal{A}^1(M; \mathbf{ad}\,\mathrm{P})$ and $\Phi \in \mathcal{A}^{1,0}(M; \mathbf{ad}\,\mathrm{P} \otimes \mathbb{C})$.

- In the case of rank 1,

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Solutions of self-dual equations and Higgs pairs

Theorem (N.Hitchin)

There is a one to one correspondence irreducible solutions of the $\mathbf{SO}(3)$ *self-duality equations modulo unitary gauge transformations and rank 2 stable Higgs pairs modulo complex gauge transformations.*

- An $\mathbf{SO}(3)$-bundle can be thought as an $\mathbf{SU}(2)$ or $\mathbf{U}(2)$-bundle.

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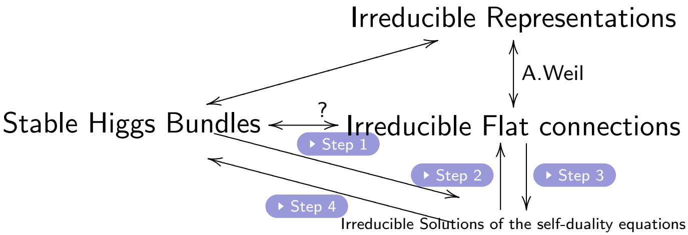
Idea of proof



Step 1 and Step 2

Hitchin (Cf. Donaldson (Narashimhan-Seshadri))

- Given a stable pair (E, Φ), can we find (A, Φ) such that $F(A) + [\Phi, \Phi^*] = 0$ and $d''_A \Phi = 0$?
- Minimizing sequence w.r.t. an adapted metric on the orbit of (d''_E, Φ):

$$\int_M \|F(A_n) + [\Phi_n, \Phi_n^*]\|^2 \text{ with } d''_{A_n} \Phi_n = 0.$$

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- The Uhlenbeck's weak compactness theorem: A limit (A, Φ) exits.
- Stability implies that the limit (A, Φ) is actually in the orbit of (d''_E, Φ) by the complex gauge group.
- Step 2: Once we find (A, Φ), the irreducible flat connection is given by

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Step 3 and Step 4

Donaldson and Corlette

Definition

A $\rho(\pi_1(M))$-equivariant function $\widetilde{H} : \widetilde{M} \to \mathbf{GL}(n, \mathbb{C})/\mathbf{U}(n)$ is called a harmonic metric if it is an extremal of an energy functional

$$\mathcal{E}(\widetilde{H}) = \int_{\widetilde{M}} |d\widetilde{H}|^2$$

- Problem: Given an irreducible flat $\mathbf{PSL}(2, \mathbb{C})$-connection D on $\mathrm{P}^{\mathbb{C}}$, is there a unique decomposition $D = A + \Phi + \Phi^*$ such that (A, Φ) satisfies the self-dual equations?
- $\mathbf{ad}\,\mathrm{P}^{\mathbb{C}} = \mathbf{ad}\,\mathrm{P} \oplus i\mathbf{ad}\,\mathrm{P}$.

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- Answer: Find a canonical metric to give a unitary decomposition.

Theorem (Corlette,Donaldson)

Let (P, D) be a principal G-bundle with a flat connection. (P, D) admits a harmonic metric if and only if the Zariski closure of the holonomy group of D is a reductive subgroup of G.

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Construction of the moduli space of the solutions of the self-duality equations

- Space: $\mathfrak{A}(M;\mathrm{P}) \times \mathcal{A}^{1,0}(M;\mathbf{ad}\,\mathrm{P} \otimes \mathbb{C})$
- $\mathfrak{A}(M;\mathrm{P}) \Longleftrightarrow \mathcal{A}^{1}(M;\mathbf{ad}\,\mathrm{P}) \Longleftrightarrow \mathcal{A}^{0,1}(M;\mathbf{ad}\,\mathrm{P} \otimes \mathbb{C})$
- In this identification, a Riemannian metric

$$g((\Psi_1,\Phi_1),(\Psi_2,\Phi_2)) = 2i\int_M \mathrm{Tr}(\Psi_1^* \wedge \Psi_2 + \Phi_2 \wedge \Phi_1^*)$$

- $$\begin{cases} I : (\Psi,\Phi) \mapsto (i\Psi, i\Phi) \\ J : (\Psi,\Phi) \mapsto (i\Psi^*, -i\Phi^*) \\ K : (\Psi,\Phi) \mapsto (-\Psi^*, \Phi^*) \end{cases}$$

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Hyperkähler manifold

- Hyperkähler structure:
$$\omega_I(X,Y) = g(IX,Y)$$
$$\omega_J(X,Y) = g(JX,Y)$$
$$\omega_K(X,Y) = g(KX,Y)$$
- Holomorphic symplectic structure w.r.t. I:
$$\Omega_I((\Psi_1,\Phi_1),(\Psi_2,\Phi_2)) = \omega_J + i\omega_K = \int_M \mathrm{Tr}(\Phi_2 \wedge \Psi_1 - \Phi_1 \wedge \Psi_2).$$
- Moment maps:
$$\begin{cases} \mu_{\omega_I}((A,\Phi)) & = F(A) + [\Phi,\Phi^*] \\ \mu_{\Omega_I}((A,\Phi)) & = d''_A\Phi \end{cases}$$

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Hyperkähler reduction

- Hyperkähler reduction $\mathcal{M} = \bigcap_{i=1}^{3} \mu_i^{-1}(0)/\mathcal{G}$

Theorem (Hitchin)

Let $\mathcal{M}$ be the moduli space of irreducible solutions to the self-duality equations on a rank 2 bundle of odd degree and fixed determinant over M with $g(M) \geq 2$. Then

- *All the complex structures of the hyperkähler family other than $\pm I$ are equivalent*
- *Moreover, they are a Stein manifold except $(\mathcal{M}, I)$.*

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Hitchin's $\mathbb{C}^*$-action

- Hitchin's circle action: $(A, \Phi) \mapsto (A, e^{i\theta}\Phi)$.
- Hitchin's $\mathbb{C}^*$-action: $(A, \Phi) \mapsto (A, c\Phi)$
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Dolbeault groupoid and Betti groupoid

- Weil's theorem and the $\mathbf{SO}(3)$ self-duality equations
- $(\mathcal{M}, J)$ is a covering of $\mathrm{Hom}\left(\pi_1, \mathbf{PSL}(2,\mathbb{C})\right)^{odd,irr}/\mathbf{PSL}(2,\mathbb{C})$.

Theorem

$\mathrm{Hom}\left(\Gamma, \mathbf{SL}(2,\mathbb{C})\right)^{odd,irr}/\mathbf{SL}(2,\mathbb{C})$ *is smooth, connected, and simply-connected.*

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Generalizations

Theorem (Corlette,Donaldson,Hitchin,Simpson)

- *There is a one to one correspondence between stable Higgs pairs over M and irreducible representations $\rho : \pi_1(M) \to \mathbf{GL}(n, \mathbb{C})$.*
- *There is a one to one correspondence between poly-stable Higgs pairs over M and reductive representations $\rho : \pi_1(M) \to \mathbf{GL}(n, \mathbb{C})$.*

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Real structure

Let $(\mathcal{M}, I)$ be the moduli space of irreducible solutions to the self-duality equations on a rank 2 vector bundle of odd degree and fixed determinant. Define an involution

$$\iota_U : (A, \Phi) \mapsto (A, -\Phi).$$

- The fixed points of S^1-action $\subseteq$ The fixed points of ι_U
- $\mu(A, \Phi) = 2i \int_M \mathrm{Tr}(\Phi \wedge \Phi^*) = \|\Phi\|^2_{L^2}$.
- $\mu^{-1}((d - \frac{1}{2})\pi) = \mathcal{M}_{2d-1}$ for $g - 1 \geq d > 0$.

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Theorem (Hitchin)

The fixed points of ι_U consist of complex submanifolds $\mathcal{M}_0, \mathcal{M}_{2d-1}(1 \leq d \leq g-1)$ each of dimension $3g-3$ where

- *$\mathcal{M}_0$ is isomorphic to the moduli space of stable rank 2 bundles of fixed determinant and odd degree.*
- *$\mathcal{M}_{2d-1}$ is a holomorphic vector bundle of rank $g-2+2d$ over a 2^{2g}-fold covering of $\mathcal{S}^{2g-2d-1}M$.*

- Critical manifold: Normal bundle structure.
- ι_U on $\mathrm{Hom}\left(\pi_1, \mathbf{PSL}(2,\mathbb{C})\right)/\mathbf{PSL}(2,\mathbb{C})$.
- $\mathcal{M}_k/\mathbb{Z}_2^{2g}$ is a holomorphic vector bundle of rank $g-1+k$ over $\mathcal{S}^{2g-2-k}M$ for $0 \leq k \leq 2g-2$.

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The fixed points of ι_U consist of complex submanifolds $\mathcal{M}_0, \mathcal{M}_{2d-1}(1 \leq d \leq g-1)$ each of dimension $3g-3$ where

- *$\mathcal{M}_0$ is isomorphic to the moduli space of stable rank 2 bundles of fixed determinant and odd degree.*
- *$\mathcal{M}_{2d-1}$ is a holomorphic vector bundle of rank $g-2+2d$ over a 2^{2g}-fold covering of $\mathcal{S}^{2g-2d-1}M$.*

- Critical manifold: Normal bundle structure.
- ι_U on $\text{Hom}\left(\pi_1, \mathbf{PSL}(2,\mathbb{C})\right)/\mathbf{PSL}(2,\mathbb{C})$.
- $\mathcal{M}_k/\mathbb{Z}_2^{2g}$ is a holomorphic vector bundle of rank $g-1+k$ over $\mathcal{S}^{2g-2-k}M$ for $0 \leq k \leq 2g-2$.

Rank 1 case

Character variety

Theorem (Narashimhan-Seshadri)

There is a one to one correspondence between stable holomorphic vector bundles and irreducible unitary representations.

Theorem (Hitchin)

Let $\pi_1(M)$ be the fundamental group of a compact Riemann surface of genus $g \geq 2$, and let $\mathrm{Hom}(\pi_1, \mathbf{PSL}(2,\mathbb{R}))^k$ denote the space of homomorphisms of π_1 to $\mathbf{PSL}(2,\mathbb{R})$ whose associated $\mathbb{R}P^1$-bundle has Euler class k. Then $\mathrm{Hom}(\pi_1, \mathbf{PSL}(2,\mathbb{R}))^k/\mathbf{PSL}(2,\mathbb{R})$ is a smooth manifold of dimension $(6g-6)$ which is diffeomorphic to a complex vector bundle of rank $g-1+k$ over $S^{2g-2-k}M$.

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Character variety

Corollary (Milnor,Wood)

The Euler class k of any flat $\mathbf{PSL}(2,\mathbb{R})$-bundle satisfies $|k| \leq 2g-2$.

Corollary (Goldman)

If $g=2$ and $k=1$, then

$$\mathrm{Hom}(\pi_1, \mathbf{PSL}(2,\mathbb{R}))^1/\mathbf{PSL}(2,\mathbb{R}) \cong M \times \mathbb{R}^4.$$

When $k=2g-2$, then the Teichmüller space

$$\mathbb{C}^{3g-3}.$$

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Variation of Hodge structure

Definition

A real variation of Hodge structure of a weight k over a Riemann surface M is a flat real vector bundle (E, D) together with a smooth direct sum decomposition

$$\mathrm{E}_{\mathbb{C}} = \mathrm{E} \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathrm{E}^{p,q}$$

satisfying

- $\mathrm{F}^r = \bigoplus_{p \geq r} \mathrm{E}^{p,q}$ is a holomorphic subbundle of $\mathrm{E}_{\mathbb{C}}$ relative to the holomorphic structure D''
- $Ds \in \mathcal{F}^{r-1} \otimes \Omega^1_M$ for $s \in \mathcal{F}^r$
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Period map

By taking the Hodge structures of a smooth family $\{X_m\}$ of algebraic varieties over M, we may define a map $\widetilde{\Pi} : M \to \mathfrak{D}$ where $\mathfrak{D}$ is a period domain. But $\widetilde{\Pi}$ is not well-defined. Up to monodromy $\rho : \pi_1(M) \to \Gamma \subset \mathrm{Aut}(\mathrm{H}_{\mathbb{Z}})$, we may define a period mapping

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Ubiquity of variations of Hodge structure

Theorem (Simpson)

Let G be a reductive group. The fixed points of S^1-action on $\mathrm{Hom}(\pi_1, G)$ are the monodromy representations arising from variations of Hodge structure over M.

The ubiquity of variation of Hodge structure by C. Simpson.

Corollary (Simpson)

Let $\rho \in \mathrm{Hom}(\pi_1, G)$. Then for $z \in \mathbb{C}^*$, $\lim_{z\to 0} z\rho$ exists in $\mathrm{Hom}(\pi_1, G)$, i.e., any homomorphism $\pi_1 \to G$ can be deformed to the monodromy representation of a variation of Hodge structure.

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Hitchin-Teichmüller components

- The number of components of $\text{Hom}(\pi_1, \mathbf{PSL}(2,\mathbb{R}))/\mathbf{PSL}(2,\mathbb{R})$ is $4g-3$.
- One of them is homeomorphic to $\mathbb{R}^{6g-6}$, Teichmüller space.

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Let M be a compact oriented surface of genus ≥ 2 and let $G^{\mathbb{R}}$ be the adjoint group of the split real form of a complex simple Lie group $G^{\mathbb{C}}$. Let $\text{Hom}^+(\pi_1, G^{\mathbb{R}})$ denote the space of representations which act completely reducibly on the Lie algebra of $G^{\mathbb{R}}$. Then $\text{Hom}(\pi_1, G^{\mathbb{R}})^+/G^{\mathbb{R}}$ has a connected component homeomorphic to a Euclidean space of dimension $(2g-2)\dim G^{\mathbb{R}}$.

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Convex $\mathbb{RP}^2$-structure

Definition

The above component is called a Hitchin-Teichmüller component.

Theorem (Choi,Goldman)

The Hitchin-Teichmüller component of $\mathrm{Hom}(\pi_1, \mathbf{PSL}(3,\mathbb{R}))^+/\mathbf{PSL}(3,\mathbb{R})$ *is the deformation space of marked convex* $\mathbb{RP}^2$*-structures.*

- When $n \geq 4$ for $\mathrm{Hom}(\pi_1, \mathbf{PSL}(n,\mathbb{R}))^+/\mathbf{PSL}(n,\mathbb{R})$, widely open (?)

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Rank 1 case

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- The fixed point set of $\iota_{\mathbb{R}}$ is

$$\begin{array}{ccccc} J & \rightsquigarrow & \mathrm{Hom}(\pi_1, \mathbb{C}^*) & \hookleftarrow & \mathrm{Hom}(\pi_1, \mathbb{R}^*) \\ & & & & \Updownarrow \\ I & \rightsquigarrow & \mathrm{T}^* \,\mathrm{Jac}(M) & \hookleftarrow & \mathrm{Jac}_2(M) \times \mathrm{H}^{1,0}(M) \end{array}$$

- $\mathrm{Hom}(\pi_1, \mathbb{R}^+) \cong \mathbb{R}^{2g}$ is the identity component of $\mathrm{Jac}_2(M) \times \mathrm{H}^{1,0}(M)$.