A moduli space of Higgs bundles and a character variety I

Rank 1 toy model and Hodge theory

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Main Goals

- Equivalence of deformation theories for rank 1 case.
- Explicit examples about the geometry of moduli spaces.

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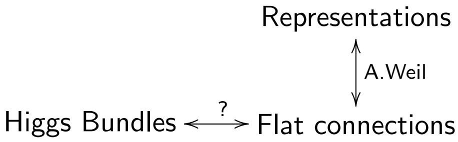
- Explicit examples about the geometry of moduli spaces.

Problem

How to prove



Deformation theory

- A deformation theory (or transformation groupoid) consists of a category $\mathcal{C}$ with a group G action.
- An equivalence of two deformation theories requires three conditions:
- Surjection on $\mathrm{Iso}(\mathcal{C}) \Longrightarrow \mathrm{Iso}(\mathcal{C}')$
- Full and Faithful on $F : \mathcal{C} \to \mathcal{C}'$

$$\mathrm{Mor}(x, y) \Longleftrightarrow \mathrm{Mor}(F(x), F(y))$$

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Betti groupoid

- $$\begin{aligned} \mathrm{Hom}(\pi_1, G) &\hookrightarrow G \times \cdots \times G \\ \rho &\longmapsto (\rho(\alpha_1), \rho(\beta_1), \dots, \rho(\alpha_g), \rho(\beta_g)) \end{aligned}$$
- $[\rho(\alpha_1), \rho(\beta_1)] \cdots [\rho(\alpha_g), \rho(\beta_g)] = \mathrm{id}$
- Take a GIT quotient: $\mathrm{Hom}(\pi_1, G)/G$

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Group cohomology

- $\mathbf{Ad}\rho : \pi_1 \xrightarrow{\rho} G \xrightarrow{\mathbf{Ad}} \mathfrak{g}$
- $\mathrm{H}^0(\pi_1; \mathfrak{g}_{\mathbf{Ad}\rho}) = \{v \in \mathfrak{g} \mid \mathbf{Ad}\rho(\gamma)v = v \text{ for all } \gamma \in \pi_1\}$
- $$\mathrm{H}^1(\pi_1; \mathfrak{g}_{\mathbf{Ad}\rho}) = \frac{\mathcal{Z}^1(\pi_1, \mathfrak{g}_{\mathbf{Ad}\rho})}{\mathcal{B}^1(\pi_1, \mathfrak{g}_{\mathbf{Ad}\rho})}$$
  $$= \frac{\{f : \pi_1 \to \mathfrak{g} \mid f(\gamma_1\gamma_2) = f(\gamma_1) + \mathbf{Ad}\rho(\gamma_1)f(\gamma_2)\}}{\{f_v : \pi_1 \to \mathfrak{g} \mid f_v(\gamma) = \mathbf{ad}\rho(\gamma)v - v \text{ for } v \in \mathfrak{g}\}}$$

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Tangent space

Theorem (A. Weil)

$$T_\rho \operatorname{Hom}(\pi_1, G) = \mathcal{Z}^1(\pi_1, \mathfrak{g}_{\mathbf{Ad}\rho})$$
$$T_{[\rho]} \operatorname{Hom}(\pi_1, G)/G = H^1(\pi_1, \mathfrak{g}_{\mathbf{Ad}\rho})$$

Idea of proof.

- $\rho_t(\gamma) = \exp(tu(\gamma) + O(t^2))\rho(\gamma)$
- $\rho_t(\gamma_1\gamma_2) = \rho_t(\gamma_1)\rho_t(\gamma_2)$

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Obstruction: $\{\xi \in H^1(\pi_1, \mathfrak{g}_{\mathbf{Ad}\rho}) \mid [\xi, \xi] = 0\}$

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Connection and Curvature

Definition

A connection on a smooth complex vector bundle E of rank r over a compact Riemann surface M is a differential operator d_η satisfying

$$d_\eta(fs) = df \otimes s + fd_\eta s \text{ where } s \in \mathcal{A}^0(M, \mathrm{E}) \text{ and } f \in C^\infty(M, \mathbb{C}).$$

After fixing a trivialization, $d_\eta = D_0 + \eta$.

Definition

A curvature $F(d_\eta) \in \mathcal{A}^2(M, \operatorname{End} \mathrm{E})$ is

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Symplectic form

Definition

A symplectic form ω on M^{2n} is a closed non-degenerated skew symmetric form.

Let $\mathfrak{A}(M; \mathrm{E})$ be the space of connections on E. A symplectic form on $\mathfrak{A}(M; \mathrm{E})$ is given by

$$\Omega^{\mathbb{B}}(\eta_1, \eta_2) = \int_M \mathbb{B}_*(\eta_1 \wedge \eta_2)$$

where $\mathbb{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ and $\eta_1, \eta_2 \in \mathcal{A}^1(M; \operatorname{End} \mathrm{E})$.

Theorem (Atiyah-Bott)

A curvature $F : \mathfrak{A}(M; \mathrm{E}) \to \mathcal{A}^2(M; \operatorname{End} \mathrm{E})$ *is a moment map.*

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A moment map $\mu : M^{2n} \to \mathfrak{g}^*$ is a smooth map such that

- $d(\mu(m), X) = \omega_m(X^\sharp, \cdot)$
- $\mu(g \cdot m) = \mathbf{Ad}^*_{g^{-1}}\mu(m)$. Here $X^\sharp$ is the associated vector field on M^{2n} with $X \in \mathfrak{g}$.

- When a moment map exists, we say that the action of G on M^{2n} is Hamiltonian.

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- $F(g \cdot d_\eta) = g^{-1} F(d_\eta) g$
- The vector field $X^\sharp_{d_\eta} = d_\eta X$ where $X \in \mathrm{Lie}(\mathcal{G}_l) = \mathcal{A}^0(M; \mathrm{End}\, \mathrm{E})$ and $\mathcal{G}_l = \mathcal{A}^0(M; \mathrm{Aut}(\mathrm{E}))$.

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Using the Marsden-Weinstein symplectic reduction theorem, we may have a symplectic manifold

$$\mu^{-1}(0)/G = \mathcal{F}_l(M)/\mathcal{G}_l.$$

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De Rham cohomology with local coefficients and Tangent space

- $\mathrm{T}_{d_\eta}\mathfrak{A}(M;\mathrm{E}) = \mathcal{A}^1(M;\mathrm{End}\,\mathrm{E})$
- $$\begin{aligned}\mathrm{T}_{d_\eta}\mathcal{F}_I(M) &= \{\xi \in \mathcal{A}^1(M;\mathrm{End}\,\mathrm{E}) \mid d_\eta\xi = 0\} \\ &= \mathcal{Z}^1(M;\mathrm{End}\,\mathrm{E})\end{aligned}$$
- $\mathrm{T}_{d_\eta}(\frac{\mathcal{F}_I(M)}{\mathcal{G}_I}) = \frac{\mathcal{Z}^1(M;\mathrm{End}\,\mathrm{E})}{\mathcal{B}^1(M;\mathrm{End}\,\mathrm{E})} = \mathrm{H}^1_{d_\eta}(M;\mathrm{End}\,\mathrm{E})$

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Line Bundle

- Flat connection $D_0 + \eta$ iff $d\eta = 0$ for $\eta \in \mathcal{A}^1(M; \mathbb{C})$
- $\mathcal{F}_l(M) = \{\eta \in \mathcal{A}^1(M; \mathbb{C}) \mid d\eta = 0\} = \mathcal{Z}^1(M; \mathbb{C})$
- $\mathcal{G}_l = \mathrm{Map}(M; \mathbb{C}^*)$
- Using the exponential sequence,

$$\mathcal{F}_l(M)/\mathcal{G}_l = \mathcal{G}_l^0 \backslash \mathcal{F}_l(M)/(\mathcal{G}_l/\mathcal{G}_l^0) = \frac{\mathrm{H}^1(M; \mathbb{C})}{\mathrm{H}^1(M; \mathbb{Z})} \cong (\mathbb{C}^*)^{2g}$$

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Arbitrary rank case

Theorem (A. Weil)

There is a one to one correspondence between Betti groupoid and De Rham groupoid.

Idea of proof.

- Monodromy: For a given flat connection $d_\eta = D_0 + \eta$, construct a monodromy matrix

$$\exp\left(\int_{\gamma_i} \eta\right).$$

- Flat bundle: For a given monodromy ρ, construct $(\widetilde{M} \times_\rho \mathbb{C}^n/\pi_1) := \widetilde{M} \times_\rho \mathbb{C}^n$ whose natural flat structure is induced from $\widetilde{D} : \mathcal{A}^k(\widetilde{M}; \widetilde{M} \times \mathbb{C}^n) \to \mathcal{A}^{k+1}(\widetilde{M}; \widetilde{M} \times \mathbb{C}^n)$.

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Higgs Bundle for rank 1

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Definition

A rank 1 Higgs bundle is a pair (d''_η, Φ) of a holomorphic structure d''_η and a holomorphic 1-form Φ with respect to d''_η.

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- After fixing a trivialization, $d''_\eta = D''_0 + \eta''$ where $\eta'' \in \mathcal{A}^{0,1}(M;\mathbb{C})$
- The space $\mathrm{Hol}(M)$ of holomorphic structures on a line bundle is an affine space model with $\mathcal{A}^{0,1}(M;\mathbb{C})$.
- The Dolbeault groupoid, $(\mathrm{Higgs}(M), \mathcal{G}_l)$, is isomorphic to

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Adapted metric

Definition

An adapted metric H to a holomorphic structure d''_η is a Hermitian metric whose unique unitary connection compatible with d''_η is flat.

- $\mathcal{A}^{0,1}(M) = \mathrm{H}^{0,1}(M) \oplus \overline{\partial}\mathcal{A}^0(M) \Longrightarrow \Psi = \Psi_0 + \overline{\partial}\log g$

$$\begin{array}{ccccc}
\mathrm{Hol}(M) & & \mathrm{Hol}_u(M) & & \mathcal{F}_u(M) \\
\uparrow & & \uparrow & & \uparrow \\
\Psi - g\overline{\partial}g^{-1} & \longleftrightarrow & (\Psi_0 - g\overline{\partial}g^{-1}, 1) & \longleftrightarrow & \Psi_0 - \overline{\Psi_0} - gdg^{-1} \\
\uparrow \text{action of } g^{-1} & & \uparrow & & \uparrow \\
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Adapted metric

$$\begin{array}{ccc} \mathrm{Higgs}(M) = \mathrm{Hol}(M) \times \mathrm{H}^{1,0}(M) & \longrightarrow & \mathcal{F}_l(M) \\ \uparrow & & \uparrow \\ (D_0'' + \Psi, \Phi) & \longrightarrow & D_0 + \Psi_0 - \overline{\Psi_0} + \Phi + \overline{\Phi} \end{array}$$

Harmonic metric

Definition

A Hermitian metric $H : M \to \mathbb{R}^+$ on a line bundle is harmonic if $\log H$ is a harmonic function.

- $\Re\eta = \phi_1 + \frac{1}{2} d \log H$ and $\mathcal{H}^1_{\Delta_d}(M) \cong \mathcal{H}^1_{\Delta_{\bar\partial}}(M)$

$$\begin{array}{ccc} \mathcal{F}_l(M) & \longrightarrow & \mathrm{Higgs}(M) = \mathrm{Hol}(M) \times \mathrm{H}^{1,0}(M) \\ \uparrow & & \uparrow \\ D_0 + \eta & \longrightarrow & \left((i\Im\eta + \frac{1}{2} d \log H)^{0,1}, (\Re\eta - \frac{1}{2} d \log H)^{1,0} \right) \end{array}$$

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Higgs coordinates

Pick a harmonic form η.

$$\begin{array}{ccc}
\mathcal{F}_l(M)/\mathcal{G}_l & \longrightarrow & \mathrm{Higgs}(M)/\mathcal{G}_l \\
\uparrow & & \uparrow \\
D_0 + [\eta] & \longrightarrow & \left[\left((i\Im\eta)^{0,1}, (\Re\eta)^{1,0}\right)\right]
\end{array}$$

Hyperkähler structure

Definition

A hyperkähler manifold is a Riemannian manifold with three co-variant constant orthogonal automorphisms I, J and K of the tangent bundle which satisfy the quaternionic identities

$$I^2 = J^2 = K^2 = IJK = -\,\mathrm{id}\,.$$

- $$\omega_I(X,Y) = g(IX,Y)$$ $$\omega_J(X,Y) = g(JX,Y)$$ $$\omega_K(X,Y) = g(KX,Y)$$
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Hyperkähler structure

- Riemannian metric

$$g(\eta_1, \eta_2) = \int_M \eta_1 \wedge *\overline{\eta_2}.$$

- $\begin{cases} I : \eta \mapsto -*\overline{\eta} \\ I : (\Psi, \Phi) \mapsto (i\Psi, i\Phi) \end{cases}$
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Explicit example for the equivalences

- $\mathrm{Hom}(\pi_1, \mathbb{C}^*)/\mathbb{C}^* = (\mathbb{C}^*)^{2g}$
- $\mathcal{F}_l(M)/\mathcal{G}_l = \mathcal{G}_l^0 \backslash \mathcal{F}_l(M)/(\mathcal{G}_l/\mathcal{G}_l^0) = \frac{\mathrm{H}^1(M;\mathbb{C})}{\mathrm{H}^1(M;\mathbb{Z})} \simeq (\mathbb{C}^*)^{2g}$
- $$\begin{aligned}\mathrm{Higgs}(M)/\mathcal{G}_l &= \left(\mathrm{H}^{0,1}(M)/\,\mathrm{H}^1(M;\mathbb{Z})\right) \times \mathrm{H}^{1,0}(M) \\ &= \mathrm{Jac}(M) \times \mathrm{H}^{1,0}(M) = \mathrm{T}^*\,\mathrm{Jac}\,M \\ &= \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \times \mathbb{C}^g\end{aligned}$$

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Example for hyperkähler structure

- $\mathrm{Hom}(\pi_1, \mathbb{C}^*)/\mathbb{C}^* \cong \mathcal{F}_l(M)/\mathcal{G}_l \cong (\mathbb{C}^*)^{2g}$ is a Stein manifold.
- $\mathrm{Higgs}(M)/\mathcal{G}_l \cong \mathrm{T}^*\,\mathrm{Jac}\,M \cong \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \times \mathbb{C}^g$ is not Stein.

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Real structures

- $\begin{cases} \iota_U : \eta \mapsto -\overline{\eta} \Longleftrightarrow \iota_U : (\Psi, \Phi) \mapsto (\Psi, -\Phi) \\ \iota_{\mathbb{R}} : \eta \mapsto \overline{\eta} \Longleftrightarrow \iota_{\mathbb{R}} : (\Psi, \Phi) \mapsto (-\Psi, \Phi) \end{cases}$
- The fixed point set of ι_U is

$$\begin{array}{ccccc} J & \rightsquigarrow & \mathrm{Hom}(\pi_1, \mathbb{C}^*) & \hookleftarrow & \mathrm{Hom}(\pi_1, \mathbf{U}(1)) \\ & & & & \Updownarrow \\ I & \rightsquigarrow & \mathrm{T}^* \,\mathrm{Jac}(M) & \hookleftarrow & \mathrm{Jac}(M) \end{array}$$

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Real structures

- The fixed point set of $\iota_{\mathbb{R}}$ is

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- The fixed point set of $\iota_U \circ \iota_{\mathbb{R}}$ is

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- $\mathrm{Hom}(\pi_1, \mathbb{R}^+)$ is the set of Higgs fields on a holomorphically trivial line bundle.

Real structures

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Real structures

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