Angle Structures and Hyperbolic Structures

Craig Hodgson

University of Melbourne

Throughout this talk:

$M =$ compact, orientable 3-manifold with
$\partial M =$ incompressible tori, $\partial M \neq \emptyset$.

Theorem (Thurston): M irreducible, atoroidal, acylindrical $\Longrightarrow$ int M has a complete hyperbolic structure of finite volume.

Mostow-Prasad Rigidity: This hyperbolic structure is *unique* up to isometry. (So geometric invariants are *topological invariants*).

Problem: How to *compute* such hyperbolic structures?

Powerful technique (Thurston, Weeks)

Use "ideal triangulations" of M:

- decompose int M topologically into ideal tetrahedra:

- try to find *shapes* for ideal tetrahedra in hyperbolic space $\mathbb{H}^3$ which fit together correctly to give a hyperbolic structure for M.

In practice, this works extremely well
– see Jeff Weeks' program SnapPea!

Open questions:

Assume M complete hyperbolic (as above).

Q1. Is there an ideal triangulation $\mathcal{T}$ of M by positively oriented ideal hyperbolic tetrahedra?

Yes, if we allow some *flat* ideal tetrahedra with vertices contained in $\mathbb{H}^2 \subset \mathbb{H}^3$: subdivide the "canonical" idea cell decomposition [Epstein-Penner], dual to the Ford complex.

Yes, if we replace M by a finite sheeted cover [Luo-Schleimer-Tillmann].

Q1'. If so, construct such a $\mathcal{T}$ algorithmically from a topological description of M.

Q2. Given a topological ideal triangulation $\mathcal{T}$ of M, can $\mathcal{T}$ be realized geometrically (i.e. by postively oriented ideal hyperbolic tetrahedra)?

This implies, for example, that the edges of $\mathcal{T}$ are isotopic to *geodesics* in M.

Today, we describe an approach to Q2 by maximizing volume in a space of *singular* hyperbolic structures on M.

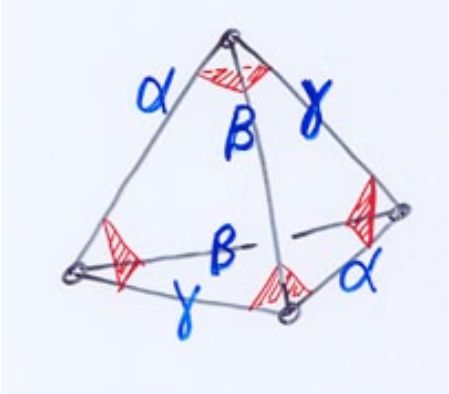
This uses ideas of Rivin [Ann. of Math, 1994] for studying convex ideal polyhedra in $\mathbb{H}^3$ with prescribed dihedral angles.

Shapes of ideal tetrahedra in $\mathbb{H}^3$

are described by 3 dihedral angles α, β, γ such that

$$\alpha + \beta + \gamma = \pi.$$

Opposite edges have the *same* dihedral angle.



(Choosing horospherical cross sections around vertices gives Euclidean triangles with angles α, β, γ.)

Thus, the shapes are parametrized by

$$\Delta := \{(\alpha, \beta, \gamma) : 0 < \alpha, \beta, \gamma < \pi, \alpha + \beta + \gamma = \pi\} \subset \mathbb{R}^3$$

Consider M as above ($\partial M =$ tori) with a topological ideal triangulation $\mathcal{T}$.

Let

$$n = \# \text{ tetrahedra in } \mathcal{T}.$$

Then $n = \#$ edges in $\mathcal{T}$, since $\chi(M) = 0$.

Choosing shapes for the ideal tetrahedra gives a point in

$$\Delta^n = \Delta \times \ldots \times \Delta \subset \mathbb{R}^{3n}.$$

The faces (ideal triangles in $\mathbb{H}^3$) can always be glued by isometries to give a hyperbolic structure on

$$M - \{ \text{ edges in } \mathcal{T} \}.$$

For a complete hyperbolic structure *on* M we need:

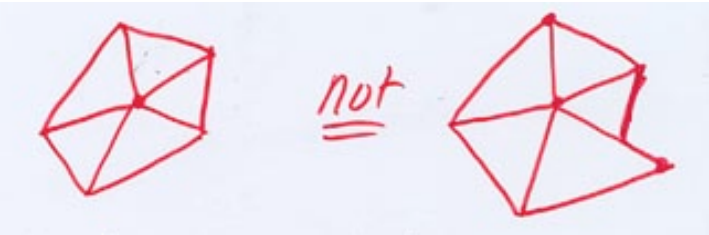
(1) sum of dihedral angles $= 2\pi$ around each edge,

(2) no translational singularities ("shearing") along each edge,

(3) completeness at each "cusp" (i.e. near each torus in ∂M).

Condition (1) is easy to understand.

Condition (2) means we can choose horospherical triangles fitting together around each edge.



Condition (3) is similar: there must be a horospherical torus at each cusp.

To solve these equations, we break the problem into two parts.
Step 1. Solve the linear angle sum equations (1).

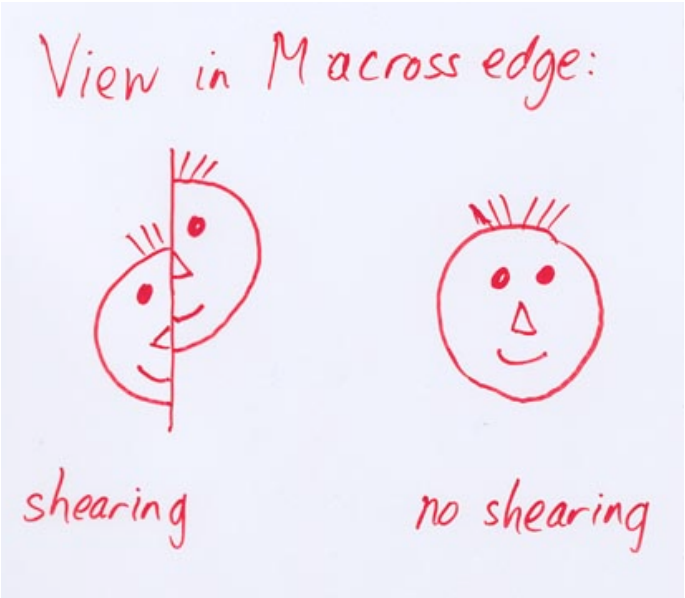
This is a linear programming problem — solve linear equations and inequalities to obtain a convex polytope

$$\begin{aligned} \mathcal{A}(\mathcal{T}) &= \{ \text{ “angle structures” on } \mathcal{T} \} \\ &\subset \Delta^n \subset \mathbb{R}^{3n}, \end{aligned}$$

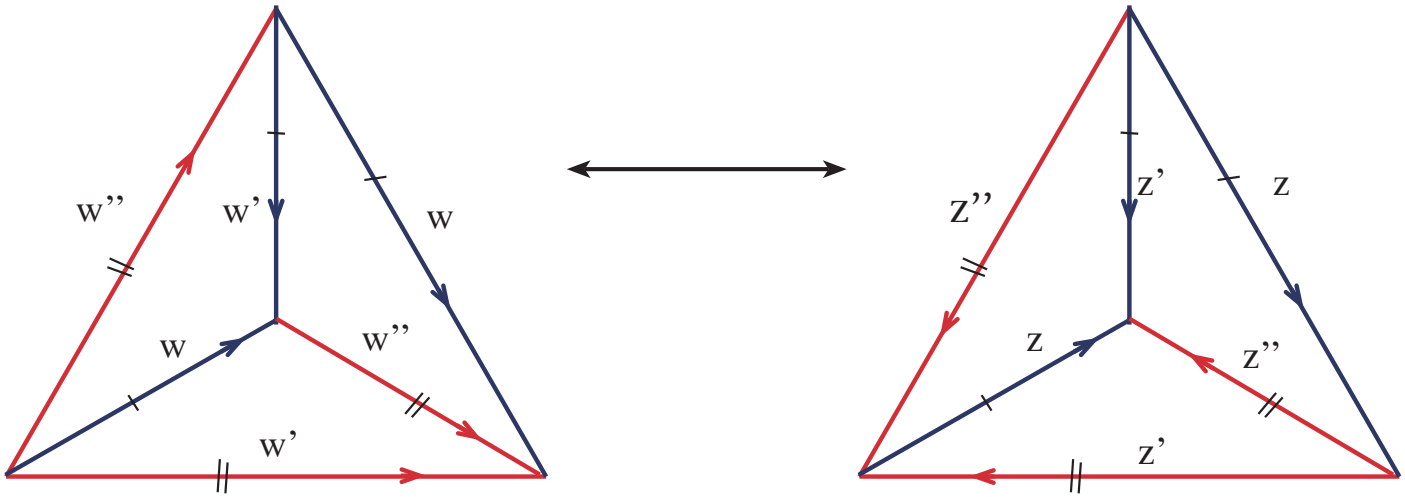
with compact closure $\overline{\mathcal{A}(\mathcal{T})} \subset \overline{\Delta}^n$.

These angle structures were introduced by Casson and Rivin, and have been used by Lackenby, Guéritaud, Futer and others.

Geometrically, these give hyperbolic structures with "shearing" type singularities along the edges of $\mathcal{T}$.



Example: The figure eight knot complement



The edge equations are:

$$z^2 z' w^2 w' = 1, \quad z'(z'')^2 w'(w'')^2 = 1$$

or

$$\begin{aligned} 2\log(z) + \log(z') + \log(w) + 2\log(w') &= 2\pi i, \\ \log(z') + 2\log(z'') + \log(w') + 2\log(w'') &= 2\pi i. \end{aligned}$$

Taking the imaginary part gives the equations for angle structures.

Write

$$\alpha_1 = \arg(z), \beta_1 = \arg(z'), \gamma_1 = \arg(z''),$$

$$\alpha_2 = \arg(w), \beta_2 = \arg(w'), \gamma_2 = \arg(w'').$$

Then angle structures satisfy the equations

$$\begin{aligned} 2\alpha_1 + \beta_1 + 2\alpha_2 + \beta_2 &= 2\pi \\ \beta_1 + 2\gamma_1 + \beta_2 + 2\gamma_2 &= 2\pi \\ \alpha_1 + \beta_1 + \gamma_1 &= \pi \\ \alpha_2 + \beta_2 + \gamma_2 &= \pi \end{aligned}$$

and inequalities

$$\alpha_i > 0, \beta_1 > 0, \gamma_i > 0 \text{ for } i = 1, 2.$$

Casson, Rivin: Existence of an angle structure on M implies that ∂M consists of tori and Klein bottles, and that M is irreducible and atoroidal.

Duality results for linear programming imply that angle structures can also be characterized in terms of *normal surfaces* in the triangulation $\mathcal{T}$ [Luo-Tillmann].

Angle structures can be computed using Ben Burton's program "Regina".

Step 2: Solve the *non-linear* equations (2) and (3).

We define a volume function

$$V : \mathcal{A}(\mathcal{T}) \to \mathbb{R}$$

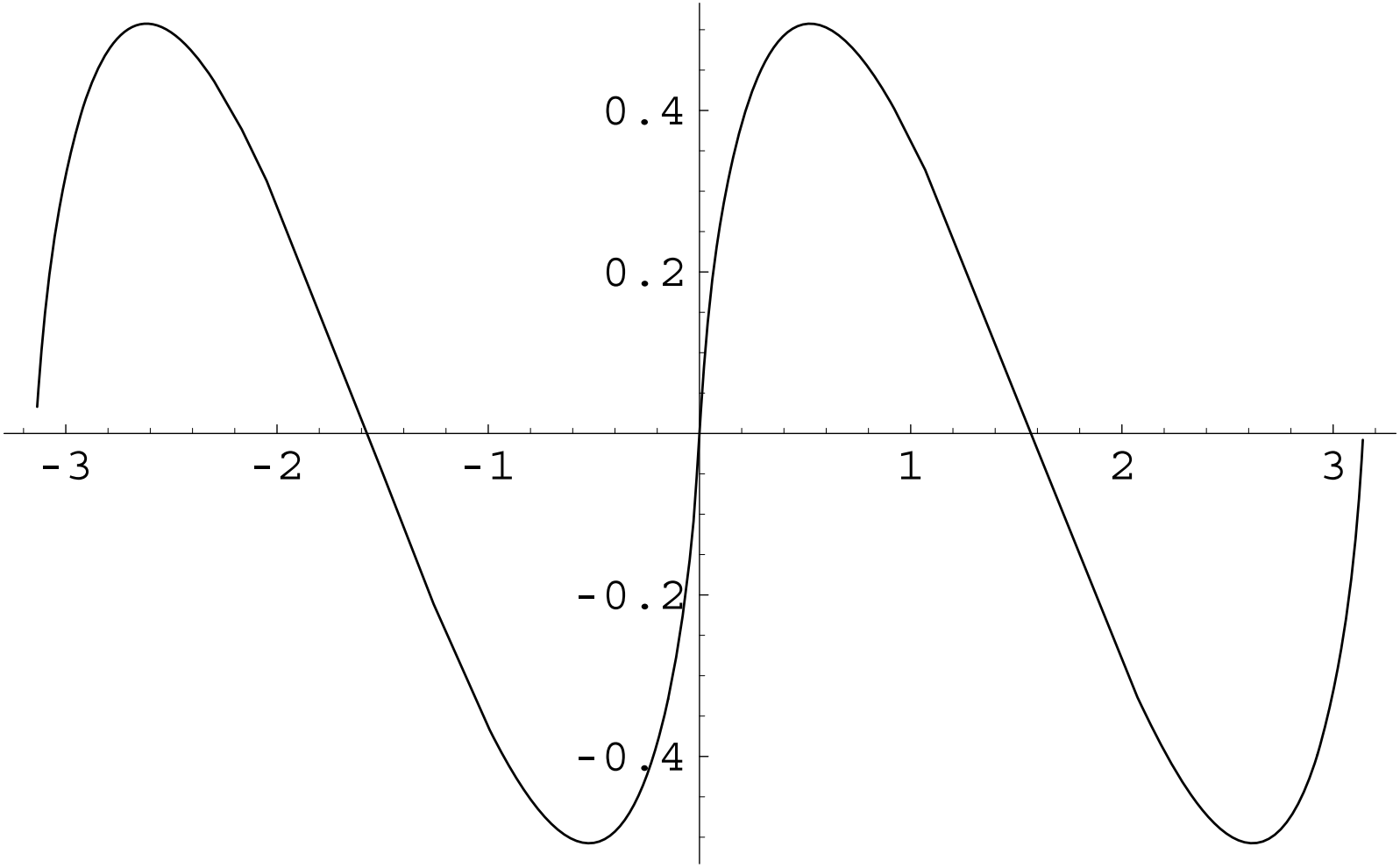
by adding up the *hyperbolic volumes* of the ideal tetrahedra in $\mathbb{H}^3$ given by a point in $\mathcal{A}(\mathcal{T})$.
For an ideal tetrahedron in $\mathbb{H}^3$ with dihedral angles α, β, γ we have:

$$\mathrm{Vol} = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where

$$\Lambda(\alpha) = -\int_0^\alpha \log(2 \sin t)\, dt$$

is the "Lobachevsky function". (See [Milnor].)



Graph of Lobachevsky function for $-\pi \leq \alpha \leq \pi$

Exercise: Prove that the regular ideal tetrahedron is the unique tetrahedra in $\mathbb{H}^3$ of maximal volume ($\approx 1.0149...$).

Hint: Need to maximize

$$\Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

subject to the constraint

$$\alpha + \beta + \gamma = \pi.$$

Then $V : \mathcal{A}(\mathcal{T}) \to \mathbb{R}$ satisfies:

1. V is a continuous function on the compact set $\overline{\mathcal{A}(\mathcal{T})}$, so attains a maximum on $\overline{\mathcal{A}(\mathcal{T})}$.
2. V is strictly concave down on $\mathcal{A}(\mathcal{T})$ [Rivin], so this maximum is unique if it occurs in $\mathcal{A}(\mathcal{T})$.
3. **Theorem** [Rivin, Chan-Hodgson]
   If V attains its maximum at a point in $\mathcal{A}(\mathcal{T})$, then this gives the (unique) complete hyperbolic structure on M.

So if we can find an ideal triangulation $\mathcal{T}$ of M such that

1. $\mathcal{A}(\mathcal{T}) \neq \emptyset$, and
2. V has no maximum on the boundary $\partial\mathcal{A}(\mathcal{T}) \subset \overline{\mathcal{A}(\mathcal{T})}$,

then M has a complete hyperbolic structure where $\mathcal{T}$ consists of positively oriented ideal hyperbolic tetrahedra.

This program has been carried out very successfully for once-punctured torus bundles and for 2-bridge knot complements in remarkable work of Guéritaud and Futer.

Conjecture (Rivin?) M irreducible, atoroidal, acylindrical $\Longrightarrow$ there exists $\mathcal{T}$ such that $\mathcal{A}(\mathcal{T}) \neq \emptyset$.

Outline of Proof of Theorem:

Write $\mathcal{A} = \mathcal{A}(\mathcal{T})$, and let $n = \#$ tetrahedra $= \#$ edges in $\mathcal{T}$.

Angle sums around edges give a linear function

$$g : \Delta^n \subset \mathbb{R}^{3n} \to \mathbb{R}^n$$

such that $\mathcal{A} = g^{-1}(*)$, $* = (2\pi, \ldots, 2\pi)$.

Notation:

- Identify each tangent space $T_x\Delta^n$ with a $2n$-dimensional subspace W of $\mathbb{R}^{3n}$ with its induced Euclidean inner product.
- Regard each derivative dg_x as a linear map

  $$g_* : W \to \mathbb{R}^n$$

  with adjoint $g^* : \mathbb{R}^n \to W$.

Step 1: Use Lagrange multipliers.
Hyperbolic volume gives $V : \Delta^n \to \mathbb{R}$.
The restriction $V\big|\mathcal{A}$ has a critical point at x

$$\begin{aligned} \Longrightarrow \operatorname{grad} V &\perp \mathcal{A} \text{ at } x \\ \Longrightarrow \operatorname{grad} V &\in (T_x\mathcal{A})^\perp \\ &= (\ker g_*)^\perp = \operatorname{Im} g^*. \end{aligned}$$

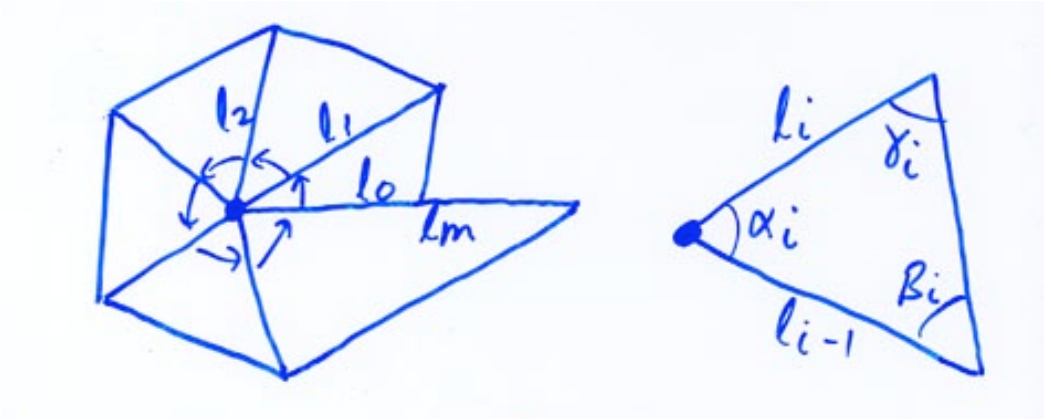
So

$$\operatorname{grad} V = g^*(\lambda)$$

for some $\lambda \in \mathbb{R}^n$. (These are the Lagrange multipliers!)

Step 2: Use the *sine law* to rewrite the no translation conditions in terms of dihedral angles, then in terms of grad V.

The developing map around an edge gives:



Sine Law $\Rightarrow \dfrac{\ell_i}{\ell_{i-1}} = \dfrac{\sin \beta_i}{\sin \gamma_i}$

So the translation distance along the edge is

$$\begin{aligned} t &= \log\left(\frac{\ell_m}{\ell_0}\right) \\ &= \sum_i \log(\ell_i) - \log(\ell_{i-1}) \\ &= \sum_{i \text{ around edge}} \log(2\sin\beta_i) - \log(2\sin\gamma_i) \\ &= \sum_{i \text{ around edge}} \left(-\frac{\partial V}{\partial \beta_i} + \frac{\partial V}{\partial \gamma_i}\right), \end{aligned}$$

since the volume V is a sum of Lobachevsky functions

$$\Lambda(\theta) = -\int_0^\theta \log(2\sin t)\, dt$$

with derivative

$$\Lambda'(\theta) = -\log(2\sin\theta).$$

Conclusion: The translational holonomies around the edges are

$$g_* J(\text{grad } V) \in \mathbb{R}^n$$

where $J : W = T_x\Delta^n \to W$ satisfies $J^2 = -3I$.

Explicitly,

$$J = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

restricted to each $T_x\Delta \subset \mathbb{R}^3$.

Step 3: Use the results of Neumann and Zagier [Topology '85] on combinatorics of ideal triangulations to conclude that the "no translation" conditions hold.

From steps 1 and 2, the translational holonomies at the edges are

$$g_* J(\text{grad } V) = g_* J g^*(\lambda) = 0,$$

since

$$g_* J g^* = 0$$

by Neumann-Zagier.

The completeness conditions are proved similarly. $\square$

Remarks:

1. This argument gives an elementary proof of a version of Mostow-Prasad Rigidity, provided M has a positively oriented ideal hyperbolic triangulation $\mathcal{T}$.

2. There's also a converse:
   Theorem (Schlenker) If an angle structure $A \in \mathcal{A}(\mathcal{T})$ realizes the complete hyperbolic structure on M, then A is a critical point of V on $\mathcal{A}(\mathcal{T})$.

3. Hyperbolic structures for *Dehn fillings* on M can be obtained by a similar volume maximization procedure: maximize V on the slice of $\mathcal{A}$ where the rotational part of the holonomy is 2π for each surgery curve [Chan-Hodgson].

Some Open Problems

1. Find topological conditions on an ideal triangulation guaranteeing existence of a hyperbolic structure with all the tetrahedra positively oriented (or flat).

The canonical cell decomposition of a cusped hyperbolic 3-manifold gives this.

Do minimal triangulations always have this property? If not, add extra conditions.

2. Find a practical algorithm for testing commensurability and finding commensurators of closed hyperbolic 3-manifolds and orbifolds.

Number theory gives powerful invariants (e.g. invariant trace field, quaternion algebras), but not complete except for arithmetic manifolds.

Snap has algorithms for cusped manifolds, using canonical cell decompositions. [Goodman, Heard, Hodgson, Exper. Math. 17 (2008)]

Naive algorithms (finding all coverings up to some index) not practical.

3. Show that every closed hyperbolic 3-manifold contains infinitely many simple closed geodesics.

S. Kuhlmann showed this is true for orientable cusped hyperbolic 3-manifolds [Alg. Geom Top. 6 (2006)]

It is also is known for many closed manifolds (e.g. Kuhlmann [Geom. Dedicata 131 (2008)], Chinburg-Reid).

Some References

D. Epstein and R. Penner, Euclidean decomposition of noncompact hyperbolic manifolds. J. Diff. Geom. 27 (1988) 67–80.
J. Milnor, Hyperbolic geometry: The first 150 years, Bull. Amer. Math. Soc. 6 (1982) 9–24.
I. Rivin, Euclidean structures on simplicial surfaces and hyperbolic volume, Annals of Math. 139 (1994) 553–580.
I. Rivin, Combinatorial optimization in geometry, Adv. Applied Math. 31 (2003) 242–271.
K. Chan, Constructing hyperbolic 3–manifolds from ideal triangulations, Honours thesis, Univ. of Melbourne, 2002.
F. Luo, S. Tillmann, Angle structures and normal surfaces, Trans. Amer. Math. Soc. 360 (2008) 2849–2866.
B. Burton, Regina: a normal surface theory calculator, http://regina.sourceforge.net/.
W. Neumann and D. Zagier, Volumes of hyperbolic three-manifolds, Topology 24 (1985), 307–332.