Hyperbolic Invariants and Computing hyperbolic structures on 3-Orbifolds

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Some References

W. Thurston, Geometry and topology of 3-manifolds, Lecture Notes, Princeton University (1979), http://www.msri.org/publications/books/gt3m/

W. Neumann and D. Zagier, Volumes of hyperbolic three-manifolds, Topology 24 (1985), 307–332.

W. D. Neumann and A. W. Reid, Arithmetic of hyperbolic manifolds, Topology '90, de Gruyter, (1992) 273–310.

D. Gabai, R. Meyerhoff and P. Milley, Mom technology and hyperbolic 3-manifolds, Proc. 2008 Ahlfors-Bers Colloquium (AMS Cont. Math. series), arXiv:0910.5043.

D. Heard, Orb: a computer program for finding hyperbolic structures on 3-orbifolds and manifolds, http://www.ms.unimelb.edu.au/∼snap/orb.html

D. Heard, C. Hodgson, B. Martelli and C. Petronio, Hyperbolic graphs of small complexity, Exper. Math. 2010, arXiv:0804.4790.

Invariants for 3-manifolds

The following crucial theorem shows that the geometry of a hyperbolic manifold is a topological invariant in dimensions 3 and above.

Mostow-Prasad Rigidity Theorem

If M_1, M_2 are hyperbolic manifolds of finite volume, of dimension ≥ 3, and $\pi_1(M_1) \cong \pi_1(M_2)$, then M_1 and M_2 are isometric.

Hence all *geometric invariants* of the hyperbolic structure (e.g. volume, the set of lengths of closed geodesics) are actually *topological invariants*.

Analytic invariants such as the set of eigenvalues of the Laplacian are also topological invariants.

Further, if $M = \mathbb{H}^3/\Gamma$ where $\Gamma \subset PSL_2(\mathbb{C})$, then the group Γ is unique up to conjugation in $PSL_2(\mathbb{C})$. Hence *algebraic invariants* of Γ are topological invariants of M, e.g.
the *trace field* $\mathbb{Q}(\{\mathtt{tr}\gamma : \gamma \in \Gamma\})$,
the *invariant trace field* $\mathbb{Q}(\{\mathtt{tr}^2\gamma : \gamma \in \Gamma\})$

In fact, these are both algebraic number fields (i.e. finite degree extensions of $\mathbb{Q}$). The invariant trace field is a commensurability invariant (i.e. invariant under finite sheeted covers).

Hyperbolic volume

The volume is very good way of distinguishing hyperbolic 3-manifolds, and is an excellent measure of the complexity of a manifold. It is not a complete invariant, but there are only finitely many manifolds of any given volume.

Theorem [Thurston and Jørgensen]
Let $\mathcal{H}$ be the set of isometry classes of complete orientable hyperbolic 3-manifolds of finite volume. The volume function $\mathrm{vol} : \mathcal{H} \to \mathbb{R}$ is finite to one, and its image is well-ordered (i.e. each subset has a smallest element), closed, and of order type ω^ω.

Thus, the set of volumes is ordered as follows:

$$0 < v_0 < v_1 < \ldots < v_\omega < v_{\omega+1} < \ldots$$
$$< v_{2\omega} < \ldots < v_{3\omega} < \ldots < v_{\omega^2} < \ldots .$$

The general index is a polynomial in ω with coefficients in $\{0, 1, 2, 3, \ldots\}$.

Here v_0 is the smallest volume of any orientable hyperbolic 3-manifold, v_1 is the next lowest volume, etc. v_ω is the first limit volume, and represents the volume of the smallest cusped hyperbolic 3-manifold M_ω. Performing Dehn filling on M_ω produces a collection of closed hyperbolic manifolds of volumes less than v_ω but whose limit is v_ω.

A few of the lowest volumes are known. Cao and Meyerhoff have shown that $v_\omega = 2.02988\ldots$.. This is the volume of the figure eight knot complement (and another closely related manifold).

Finding v_0 was an outstanding problem for nearly 30 years. Very recently, Gabai-Milley-Meyerhoff have proved that

$$v_0 = 0.9427\ldots,$$

where the right hand side represents the volume of the “Weeks manifold”, obtained by (5,-1), (5,2) surgery on the Whithead link.

The Thurston-Jørgensen Theorem follows from the hyperbolic Dehn surgery theorem, and the following:

Theorem For any bound $V > 0$ there is a finite collection of hyperbolic 3-manifolds such that every hyperbolic 3-manifold of volume $\leq V$ results from Dehn surgery on a member of this collection.

Making this result more explicit should eventually lead to a reasonable *classification* of hyperbolic 3-manifolds, e.g. a list of the lowest volume manifolds.

Recent work by Gabai-Meyerhoff-Milley, using their “Mom technology” has made great progress in this direction:

Theorem [Gabai, Meyerhoff, Milley]
Let N be a one-cusped orientable hyperbolic 3-manifold with $Vol(N) \leq 2.848$. Then N can be obtained by Dehn filling one of 21 (known) cusped hyperbolic 3-manifolds.

This leads to the proof that the Weeks manifold is the unique closed orientable hyperbolic 3-manifold of lowest volume. (This is an arithmetic manifold!)

More on hyperbolic Dehn Filling

Let M be a 1-cusped hyperbolic 3-manifold, $M = \mathrm{int}\,\overline{M}$ where $\overline{M}$ is a compact 3-manifold with $\partial\overline{M} = T =$ torus.

Fx a basis for $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.
For any pair of relatively prime integers (p, q), let $M(p, q)$ =(p, q)-Dehn filling on M.

Let $\gamma_{p,q}$ denote the *core geodesic* in $M(p, q)$, i.e. the geodesic homotopic to the core circle of the added solid torus.

Hyperbolic Dehn Filling Theorem (Thurston)
$M(p, q)$ is hyperbolic whenever $|p| + |q|$ is sufficently large.
Furthermore, as $|p| + |q| \to \infty$,
(i)The manifolds $M(p, q)$ converge geometrically to M (Gromov-Hausdorff convergence),
(ii) $Vol(M(p, q))$ converges to $Vol(M)$ from below,
(iii) $length(\gamma_{p,q}) \to 0$.

There are also precise estimates on change in volume and length of core geodesic during Dehn filling.

Let M be a 1-cusped hyperbolic 3-manifold,

T a horospherical torus cusp cross-section,

$L(\gamma)$ the length of the Euclidean geodesic on

T homotopic to the surgery curve γ, and

$$\widehat{L} = \widehat{L}(\gamma) = \frac{L(\gamma)}{\sqrt{\text{Area}(T)}}$$

the normalised geodesic length of γ.

Neumann-Zagier (asymptotic behaviour):

As $\widehat{L} \to \infty$,

- the decrease in volume is $\Delta V \sim \frac{\pi^2}{\widehat{L}^2}$

- the geodesic core length $\ell \sim \frac{(2\pi)}{\widehat{L}^2}$.

Hodgson-Kerckhoff give upper and lower bounds.

Question: How do arithmetic invariants behave during Dehn filling?

Here is one result on invariant trace fields:

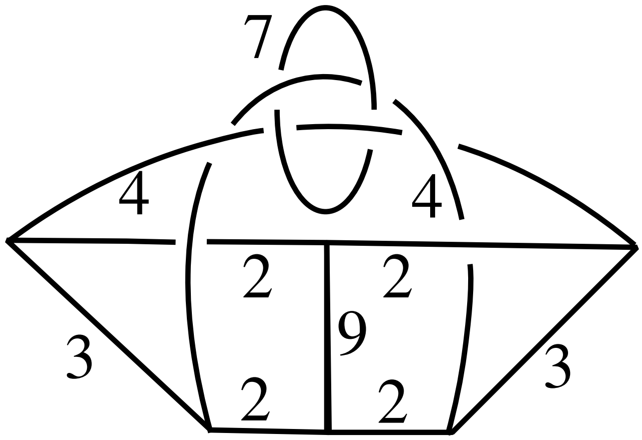
Theorem (H.) Let M be a finite volume 1-cusped hyperbolic 3-manifold, and let $k_{p,q}$ denote the invariant trace field for $M(p,q)$. Then
The degree of $k_{p,q} \to \infty$ as $|p| + |q| \to \infty$.

Computing hyperbolic structures on 3-orbifolds

Recall: A **3-orbifold** is a space locally modelled on $\mathbb{R}^3$ modulo finite groups of diffeomorphisms.

An orientable 3-orbifold is determined by its *underlying space* Q which is an orientable 3-manifold and *singular locus* Σ which is a trivalent graph (possibly disconnected or empty) with each edge or circle labelled by an integer $n \geq 2$.

For example:



A **hyperbolic structure** on such an orbifold is a singular hyperbolic metric with **cone angles** $2\pi/n$ along each edge labelled n.

At a trivalent vertex we allow:
angle sum $> 2\pi$ giving a finite vertex,
angle sum $= 2\pi$ giving a cusp,
angle sum $< 2\pi$ giving a totally geodesic boundary component.

Method for computing hyperbolic structures

• Decompose the manifold or orbifold into tetrahedra.

• Find geometric shapes for tetrahedra in $\mathbb{H}^3$ (dihedral angles, edge lengths) so that:

1. faces are glued by isometries
2. sum of dihedral angles around each edge is 2π (or the desired cone angle).

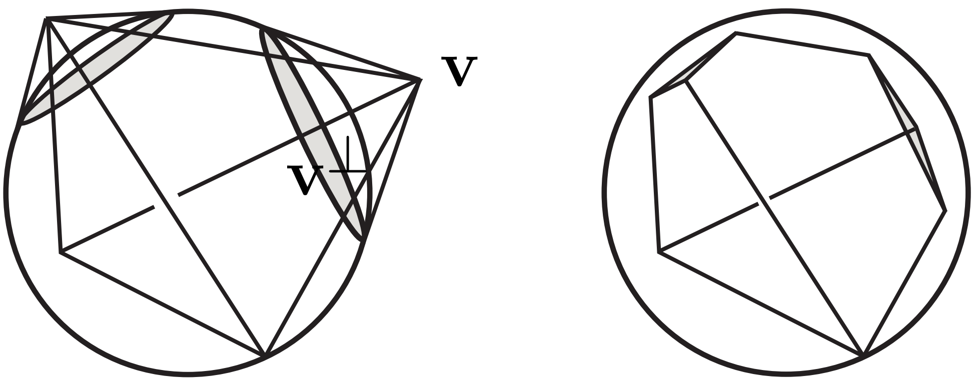
(Also need **completeness** conditions if the space is non-compact.)

Generalized hyperbolic tetrahedra

In hyperbolic geometry can use tetrahedra with

- finite vertices (inside $\mathbb{H}^3$),
- ideal vertices (on the sphere at infinity), or
- hyperinfinite vertices (beyond the sphere at infinity)!

This is easiest to see in the projective model for $\mathbb{H}^3$:



Hyperinfinite vertices are truncated as shown. Interiors of edges must meet $\mathbb{H}^3$.

Orb by Damian Heard

Uses **generalized hyperbolic tetrahedra** with finite, ideal and hyperinfinite vertices. (Can pass continuously between these and allow flat and negatively oriented tetrahedra.)

Can deal with **orbifolds** and **cone-manifolds** where the cone angle around an edge is not necessarily 2π.

Can start with a projection of a graph in S^3 and try to find hyperbolic structures with prescribed cone angles around all the edges

How Orb works

Suppose we have an orbifold in S^3 whose singular locus is a graph Σ with integer labels on the edges. (For this talk, I'll generally assume all vertices are finite.)

Step 1. Finding triangulations

Given a projection of Σ, find a triangulation of S^3 with Σ contained in the 1-skeleton by extending the approach of W. Thurston and J. Weeks. Can also retriangulate to change and simplify the triangulation, using 2-3 and 3-2 moves etc.

Step 2. Finding hyperbolic structures
For the case of tetrahedra with finite vertices, Orb uses one parameter for each edge of the triangulation: $\cosh(length)$.

From these we can calculate the *dihedral angles* of each tetrahedron. Moreover, faces paired by gluing maps will be automatically isometric.

This gives one equation for each edge:
• the sum of dihedral angles around each edge is the desired cone angle.

These can be solved using Newton's method, starting with suitable regular generalized tetrahedra as the initial guess.

By Mostow-Prasad rigidity the hyperbolic structure on the 3-orbifold is unique if it exists. Hence geometric invariants are actually *topological invariants*.

Using Orb we can find: volume (using formulas of A. Ushijima), matrix generators, Dirichlet domains, lengths of closed geodesics, presentations of π_1, homology groups, covering spaces, ...

For hyperbolic manifolds with geodesic boundary we can also compute the **canonical cell decomposition** (defined by Kojima). This allows us to decide if such manifolds are homeomorphic and compute their symmetry groups.

Application 1: Enumeration and classification of knotted graphs in S^3

(Hodgson, Heard; J. Saunderson, N. Sheridan, M. Chiodo)

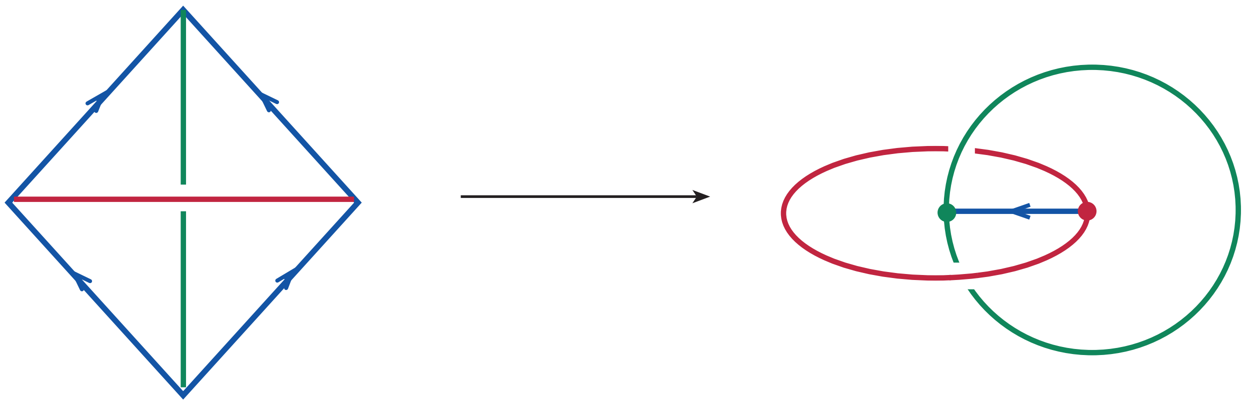
Much work in knot theory has been motivated by attempts to build up knot tables (e.g. Tait, Conway, Hoste-Thistlethwaite-Weeks). A very natural generalization is to study **knotted graphs** in S^3, say up to isotopy. There has been much less work on the tabulation of knotted graphs. In 1989, Rick Litherland produced a table of 90 prime knotted **theta curves** up to 7 crossings, using an Alexander polynomial invariant to distinguish graphs.

H. Moriuchi has recently verified these tables by using Conway's approach and the Yamada polynomial invariant.

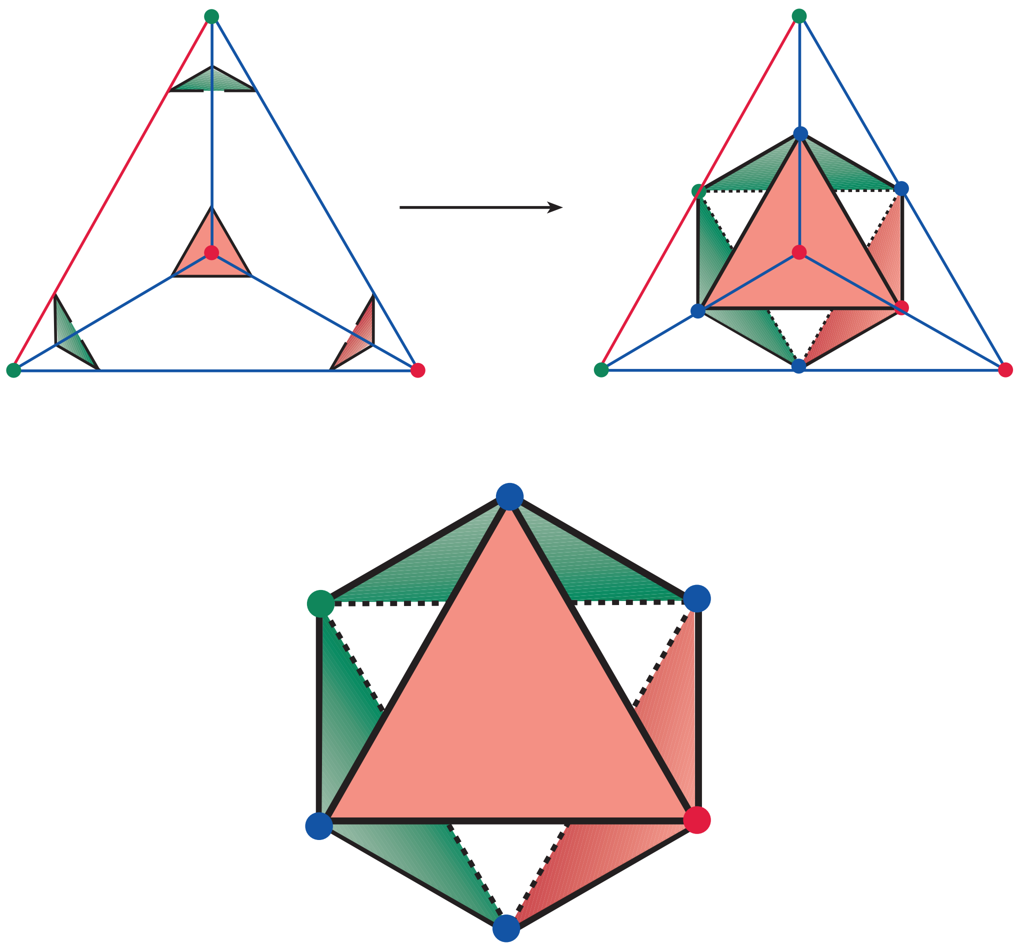
We have shown that these knotted graphs can be distinguished by hyperbolic invariants computed using Orb. In fact there is a complete invariant: We compute the hyperbolic structure with geodesic boundary consisting of 3-punctured spheres, such that all meridian curves are parabolic. (This is a limit of hyperbolic orbifolds where all labels $\to \infty$, i.e. all cone angles $\to 0$). Kojima's canonical decomposition then determines the graph completely. This also allows us to determine the symmetry group of all these graphs.

Example:

The simplest hyperbolic handcuff graph can be obtained from one tetrahedron with the two front faces folded together and the two back faces folded together giving triangulation of S^3 with the graph contained in the 1-skeleton:



A hyperbolic structure with parabolic meridians is obtained by taking a limit of truncated hyperbolic tetrahedra as edge lengths $\to$ 0. The result is a *regular ideal octahedron*!

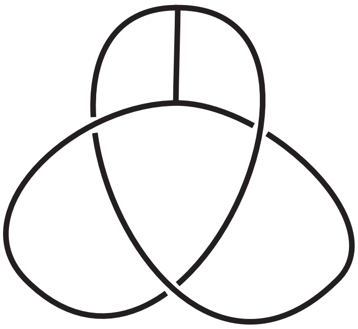


(4 faces are glued in pairs, the other 4 free faces form two totally geodesic 3-punctured spheres.)

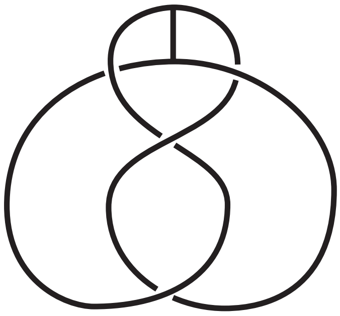
This graph has hyperbolic volume 3.663862377..., and is the *smallest* volume for trivalent graphs by the work of [Miyamoto-Kojima].

Start of Litherland's table of θ graphs

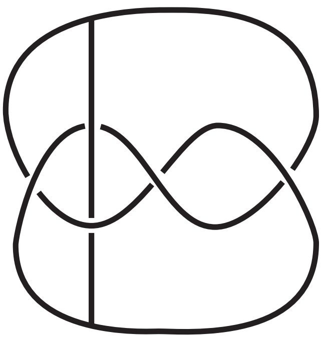
For each graph we give volume of hyperbolic structure with meridians parabolic, symmetry group, reversibility.



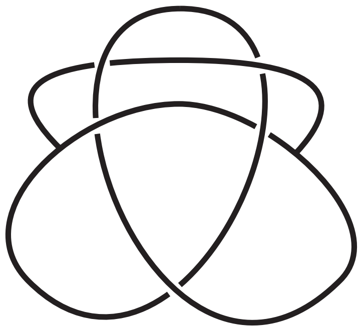
5.333489566898
D_2 r



7.706911802810
D_2 r



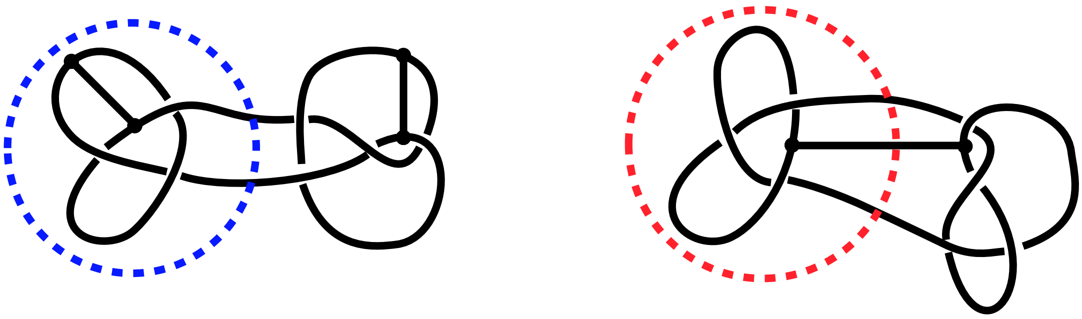
10.396867320885
D_3 n



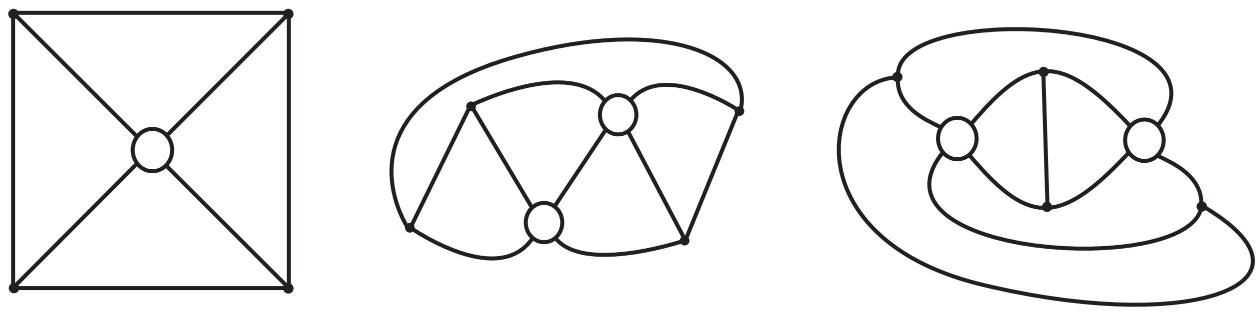
8.929317823097
D_2 r

Building up tables of knotted graphs

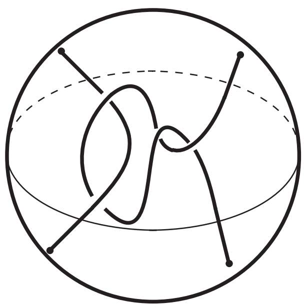
We have also extended these tables to enumerate and classify all prime knotted trivalent graphs in S^3 with 2 or 4 trivalent vertices, and up to 7 crossings. Here **prime** means there is no 2-sphere meeting the graph in at most 3 points dividing the graph into non-trivial pieces.



Our method is based on Conway's approach: First we enumerate **basic prime polyhedra** with vertices of degree 3 and 4, using the program **plantri** of B. McKay and G. Brinkmann.



Then replace degree 4 vertices by **algebraic tangles** to obtain projections of knotted graphs.

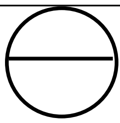
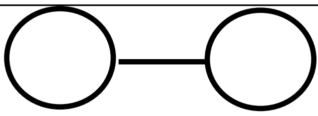
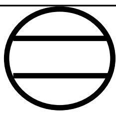
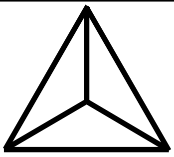
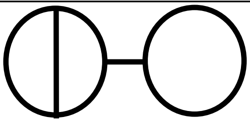
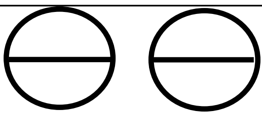
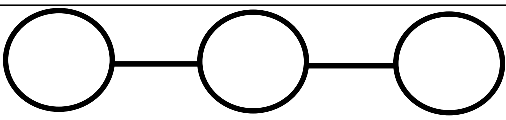
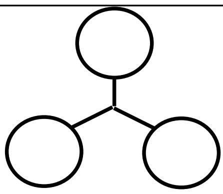
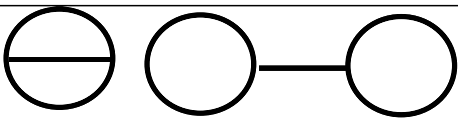


Next, we remove repeated projections by finding a canonical description for each one using the ideas behind plantri.

Finally we distinguish the graphs using hyperbolic invariants computed using Orb, e.g. volumes of associated orbifolds and Kojima's canonical decomposition.

Other recent work with D. Heard, B. Martelli, C. Petronio looks at enumeration and classification of knotted trivalent graphs in general closed 3-manifolds.

The following table summarizes the
prime trivalent graphs in S^3:
up to 4 vertices and 7 crossings.

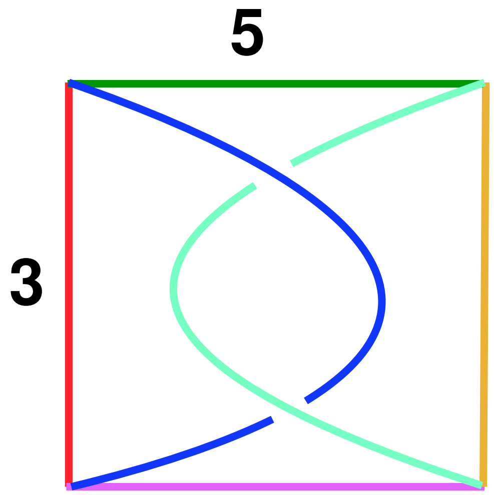
basic graph	no. of circle components 0	1	2
	90	50	4
	48	9	0
	810	143	3
	554	121	3
	529	29	0
	60	3	0
	57	0	0
	8	0	0
	8	0	0

Application 2: enumeration of low volume hyperbolic 3-orbifolds

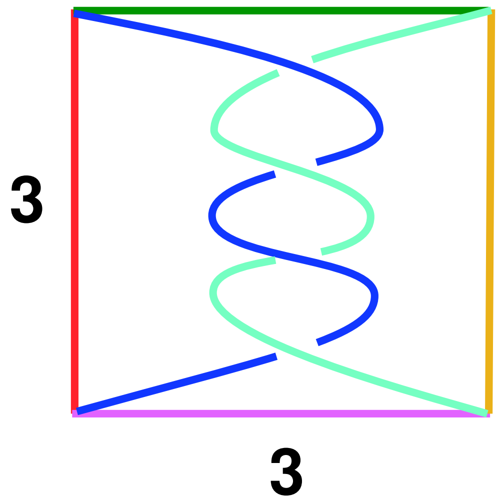
By varying the labels on the knotted graphs obtained above we can start generating hyperbolic orbifolds with underlying space S^3. This work is just beginning; currently we are looking at orbifolds with connected graphs as singular locus.

The following table shows a few of the lowest volume orbifolds. The first orbifolds on our list are all arithmetic!

(All edges are labelled 2 except where otherwise indicated.)

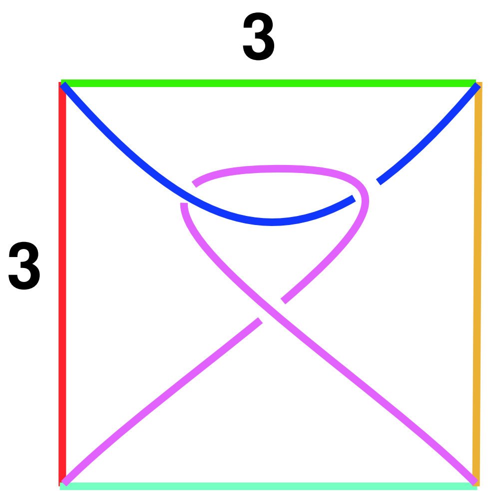


Vol: 0.03905

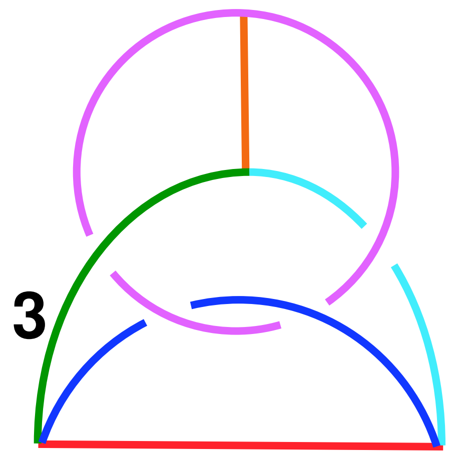


Vol: 0.04089

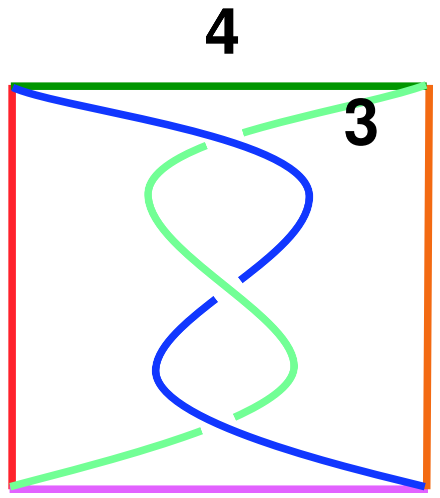
(the two smallest orbifolds: [Martin-Marshall])



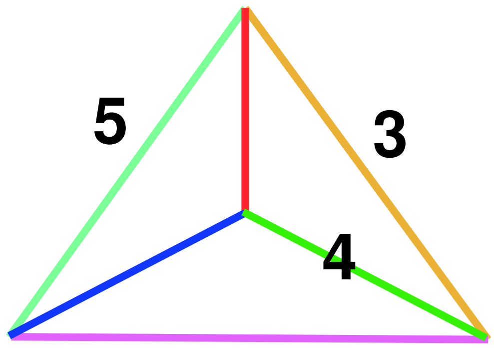
Vol: 0.05265



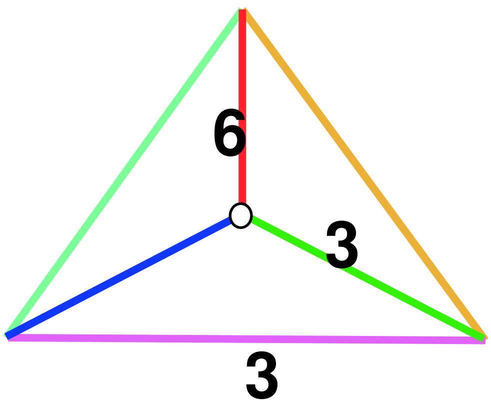
Vol: 0.065965



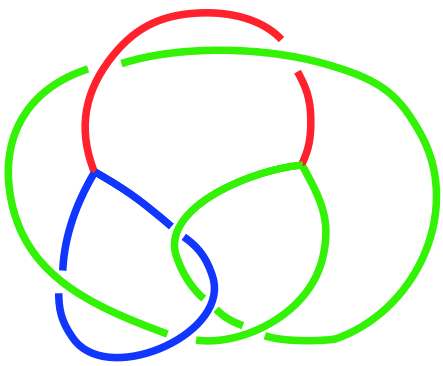
Vol: 0.06619



Vol: 0.071770



Vol: 0.0845785
smallest cusped
orbifold [Meyerhoff]



Vol: 0.117838
smallest 2 vertex
orbifold found

Some Open Problems

1. Extend the current results to give lists of the lowest volume hyperbolic 3-manifolds and 3-orbifolds of various types, e.g.

- closed orientable manifolds
- closed orientable orbifolds
- manifolds with totallly geodesic boundary
- closed non-rorientable manifolds

The ideas of Gabai-Meyerhoff-Milley (Mom technology and rigorous computation techniques) could be very useful here.

2. Extend the capabilities of Orb to produce exact hyperbolic structures and arithmetic invariants for hyperbolic 3-orbifolds.

(Currently can take an approximate representation produced by Orb, then use Snap to produce a high precision representation, then guess exact values of traces using the LLL algorithm. But the process not automated.)