Hyperbolic Structures from Ideal Triangulations

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Geometric structures on 3-manifolds

Thurston's idea: We would like to find geometric structures (or metrics) on 3-manifolds which are *locally homogeneous*: any two points have isometric neighbourhoods.

Our spaces should also be *complete* as metric spaces.

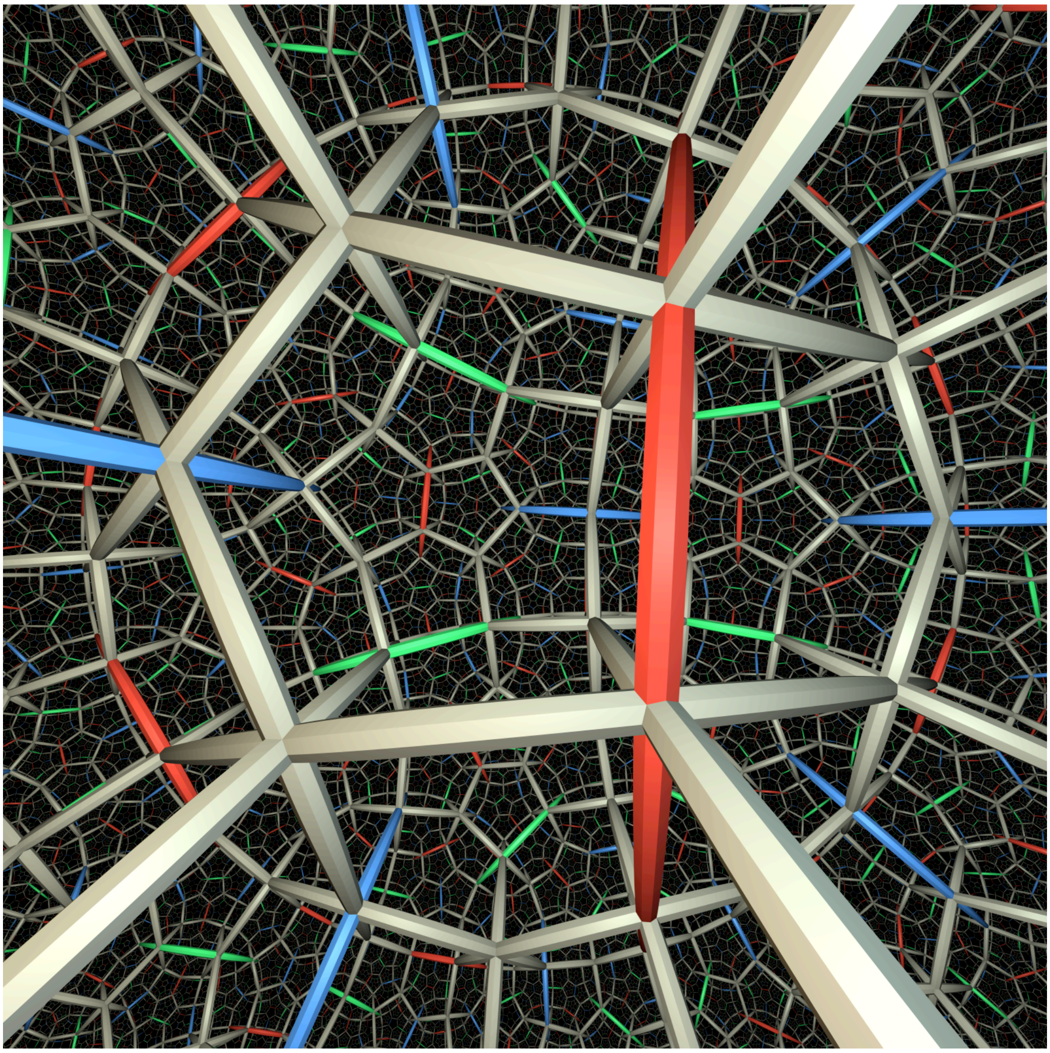
There are 8 kinds of geometry needed: three constant curvature geometries, five products or twisted products.

The most important and most commonly occurring is **hyperbolic geometry**.

- This has constant negative curvature -1.
- It is completely symmetric: looks the same near every point and in every direction.
- Angle sums in triangles are less than π.

In fact, $\quad angle\ sum\ = \pi - area$.

- Can have *ideal* polygons (and polyhedra) with vertices at infinity, but finite area (volume).
- Geodesics (lines) diverge exponentially fast.



(from Not Knot video, Geometry Center, Minneapolis)

Hyperbolic 3-manifolds

A *hyperbolic 3-manifold* M is a space locally modelled on $\mathbb{H}^3$, i.e. a Riemannian 3-manifold of constant curvature -1. If M is *complete* as a metric space, then the universal cover of M is isometric to $\mathbb{H}^3$ and $M = \mathbb{H}^3/\Gamma$ where $\Gamma \cong \pi_1(M)$ is a subgroup of $Isom(\mathbb{H}^3)$.
If M is orientable then we can regard Γ as a discrete, torsion free subgroup of $PSL_2(\mathbb{C})$.

A discrete subgroup Γ of $PSL_2(\mathbb{C})$ is called a *Kleinian group*. In general, the quotient $\mathbb{H}^3/\Gamma$ is a *hyperbolic* 3*-orbifold*.

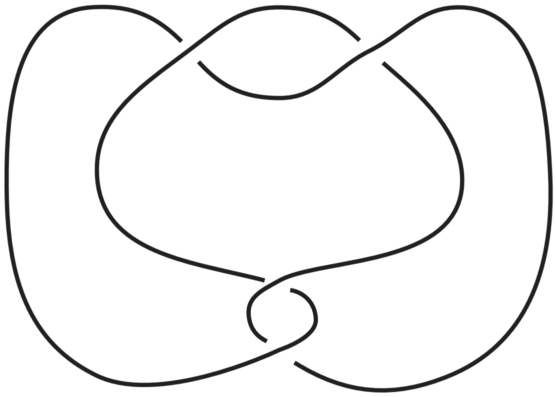
Example: Seifert-Weber dodecahedral space

This is a closed 3-manifold obtained from a dodecahedron by identifying each pentagonal face to the opposite face by a $3/10$ rotation.

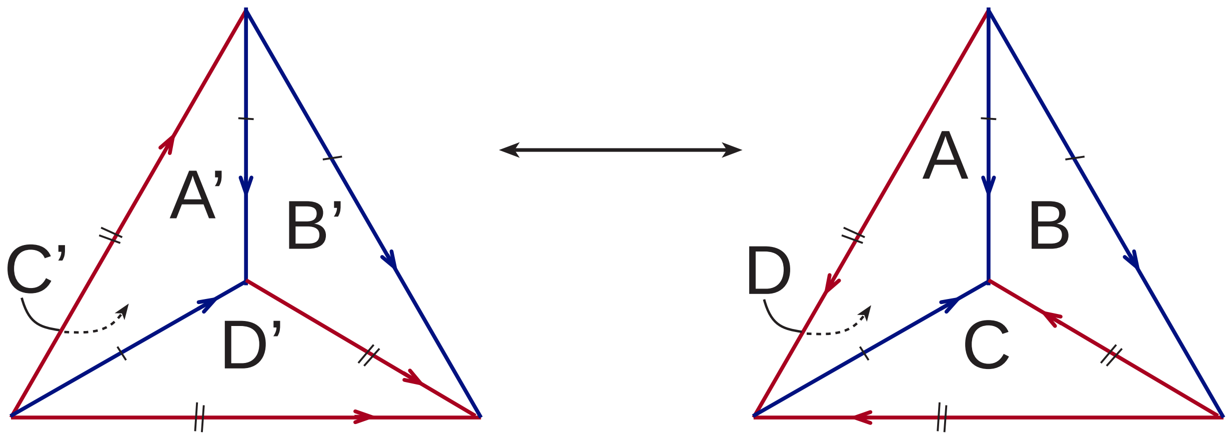
Here we find the edges are identified in 6 groups of 5. By a continuity argument there is a regular dodecahedron in $\mathbb{H}^3$ with all dihedral angles $360/5 = 72$ degrees. (Exercise!) Gluing together the faces by isometries gives a complete hyperbolic structure.

Example: The figure eight knot complement

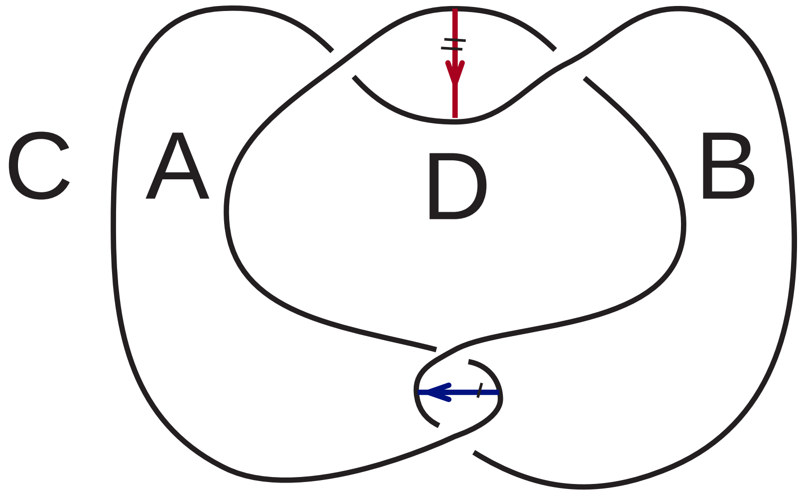
The figure eight knot K in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is shown below.



The complement $S^3 - K$ can be represented as the union of the two ideal tetrahedra (i.e. compact tetrahedra *with the vertices removed*).

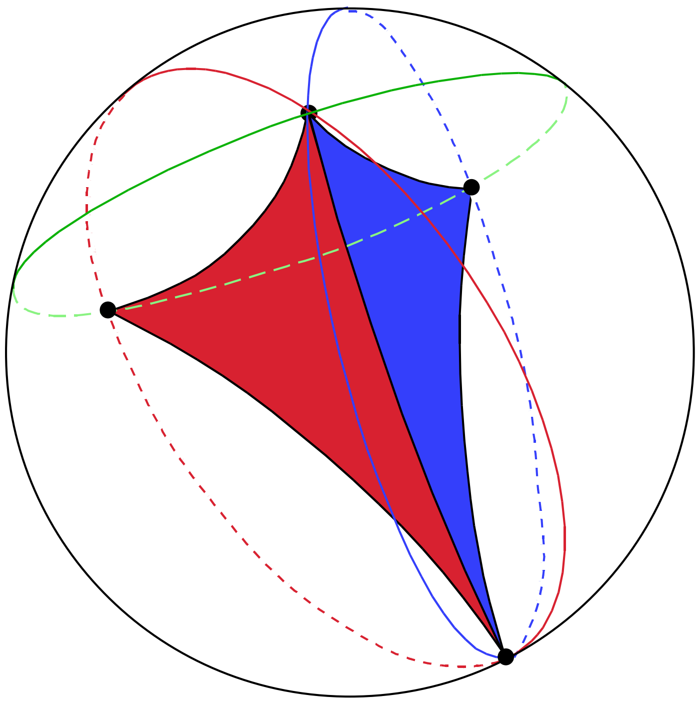


To see this, we can span the knot by a 2-complex consisting of K, two extra edges and four 2-cells A, B, C, D as shown below.



Cutting along this 2-complex divides S^3 into two open 3-balls. By looking carefully at the pattern of faces on the boundary of the balls, and *removing* the edges *along the knot* we obtain the topological description given above.

Now the edges of the two tetrahedra fit together in two groups of 6. Consider a regular *ideal* tetrahedron in $\mathbb{H}^3$, with all its vertices on the sphere at infinity. This has all dihedral angles 60 degrees.



Taking two regular ideal tetrahedra and gluing their faces together by isometries gives a hyperbolic structure on X (complete, finite volume).

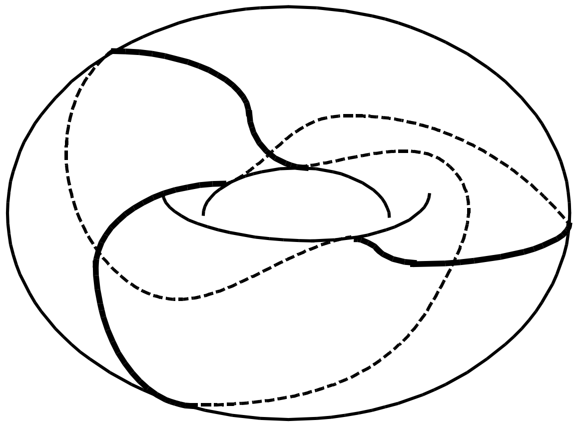
The Geometrization Theorem

Assume all manifolds are *compact* and *orientable*.

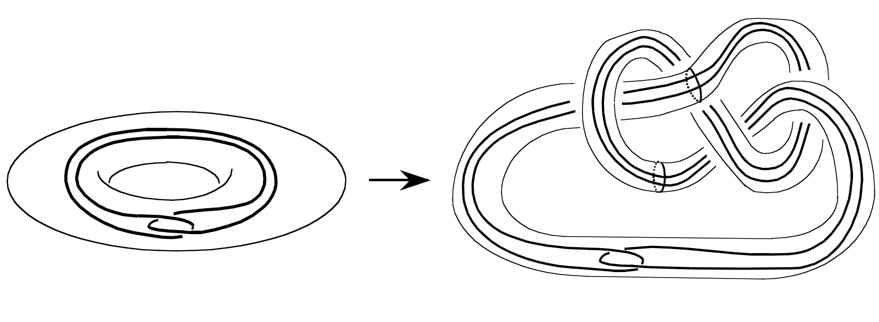
Geometrization Theorem (Thurston, Perelman) Let M be a compact, orientable, prime 3-manifold. Then there is a finite collection of disjoint, embedded incompressible tori in M (given by the Jaco-Shalen, Johannson torus decomposition), so that each component of the complement admits a geometric structure modelled on one of Thurston's eight geometries.

Application: knot complements

Let K be a knot in $S^3 = \mathbb{R}^3 \cup \infty$. Then K is called a *torus knot* if it can be placed on the surface of a standard torus.



A knot K' is called a *satellite knot* if it is obtained by taking a non-trivial embedding of a circle in a small solid torus neighbourhood of a knot K.

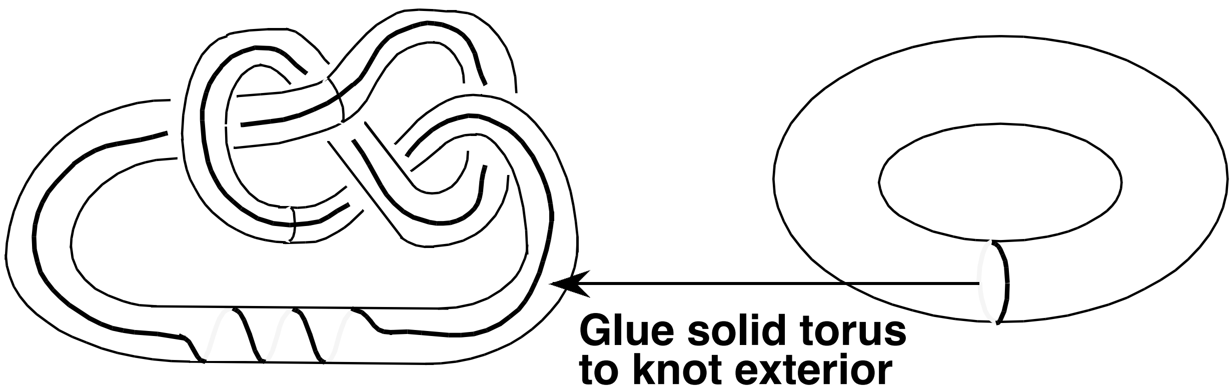


Corollary (Thurston) Let K be a knot in S^3. Then $S^3 - K$ has a geometric structure if and only if K is not a satellite knot. Further, $S^3 - K$ has a hyperbolic structure if and only if K is not a satellite knot or a torus knot.

Thus, “most” knot complements are hyperbolic. Similarly, “most” link complements are hyperbolic.

The next result shows that closed hyperbolic 3-manifolds are very abundant. First note that an orientable hyperbolic 3-manifold which has *finite volume* but is non-compact is homeomorphic to the interior of a compact 3-manifold $\overline{M}$ with boundary $\partial\bar{M}$ consisting of tori. We then call M a *cusped* hyperbolic manifold. (Example: the figure eight knot complement.)

We can then form many closed manifolds by *Dehn filling*: attaching solid tori to $\partial\bar{M}$, e.g.



Hyperbolic Dehn Surgery Theorem [Thurston]

If M is a cusped hyperbolic 3-manifold, then "almost all" manifolds obtained from M by Dehn filling are hyperbolic. (A finite number of surgeries must be excluded for each cusp.)

Since every closed 3-manifold can be obtained by Dehn filling from a hyperbolic link complement, this shows that in some sense "most" closed 3-manifolds are hyperbolic!

Problem: How to compute and understand these hyperbolic structures.

Method for computing hyperbolic structures

(W. Thurston, J. Weeks)

Let M be the interior of a compact manifold with boundary consisting of tori. Decompose the manifold topologically into ideal tetrahedra with faces glued together in pairs.

Find shapes of ideal tetrahedra in $\mathbb{H}^3$ satisfying:

(1) **Edge conditions:** at each edge

(a) sum of dihedral angles $= 2\pi$

(b) there's no translation along edge

($\Leftrightarrow$ cross section orthogonal to edge isometric to disc in H^2)

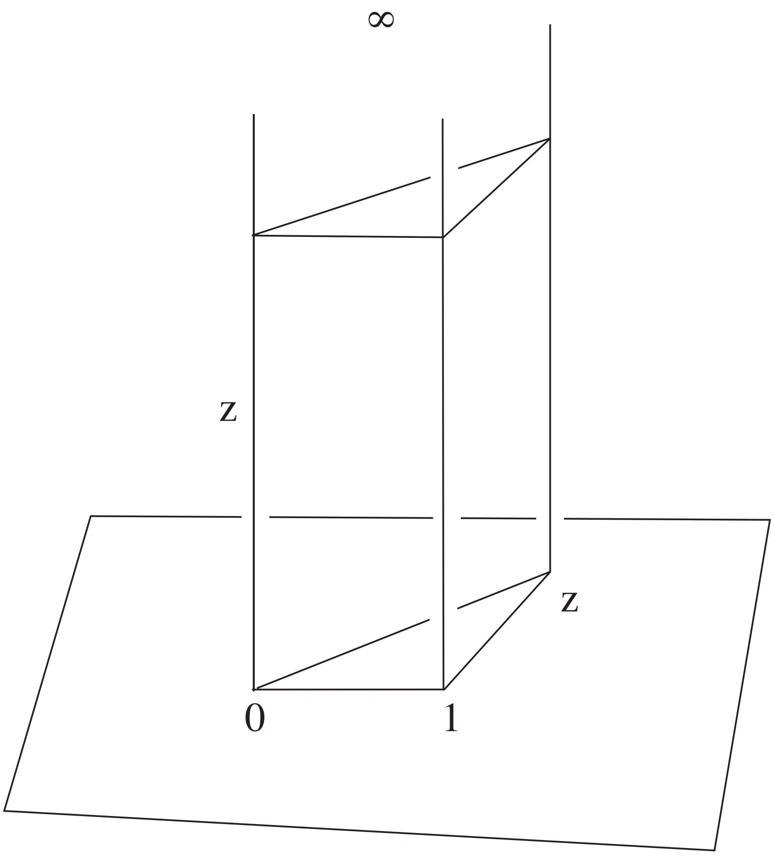
We can always glue faces together by isometries (any two ideal triangles are congruent), giving a hyperbolic structure on $M \backslash \{1 - skeleton\}$.

(1) $\Rightarrow$ get (possibly incomplete) hyperbolic structure on M.

(2) **Completeness conditions:**
horospherical triangles (in link of each ideal vertex) must fit together to give a closed Euclidean surface.

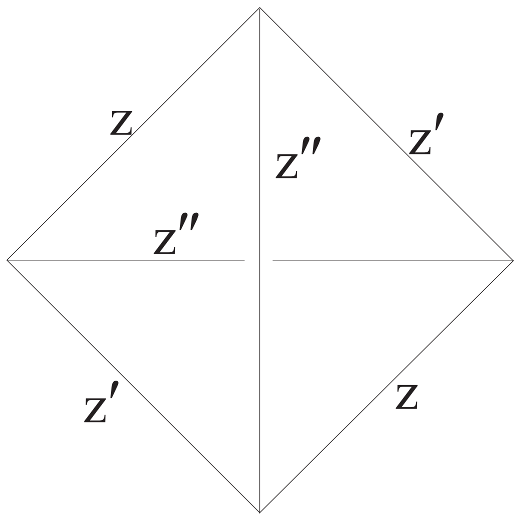
Paremetrizing ideal tetrahedra in $\mathbb{H}^3$

In the upper half space model, we can move the four ideal vertices to $0, 1, z, \infty \in \mathbb{C} \cup \{\infty\}$ by a hyperbolic isometry. Then the complex parameter $z \in \mathbb{C} \setminus \{0, 1\}$ describes the shape of the tetrahedron.



The parameter z is associated with the edge from 0 to ∞; the other edges have complex parameters

$$z, \quad z' = \frac{z-1}{z}, \quad z'' = \frac{1}{1-z}.$$

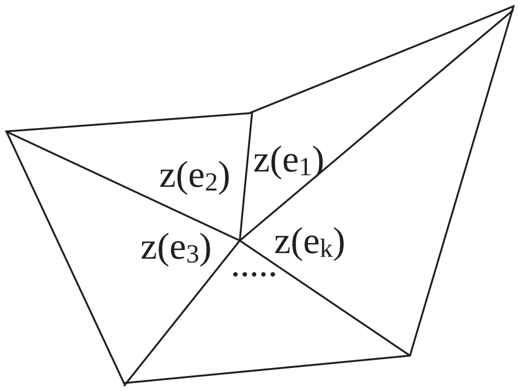


Note that these satisfy

$$zz'z'' = -1.$$

Edge conditions

Given n ideal tetrahedra with complex edge parameters $z(e)$, we want Euclidean triangles to fit together around each edge:



This gives

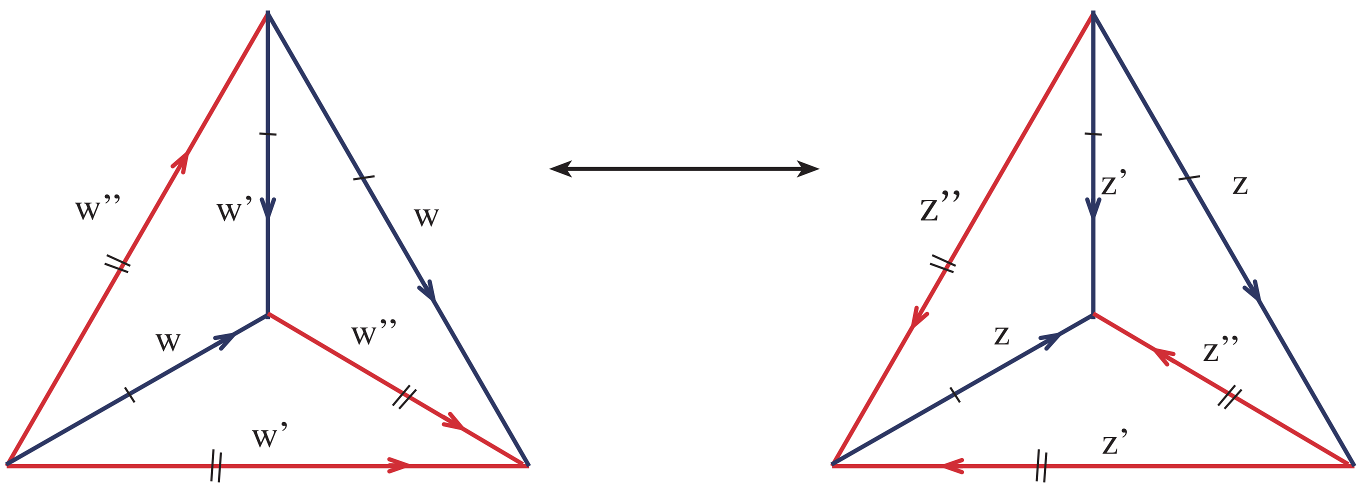
$$z(e_1)z(e_2)\cdots z(e_k) = 1$$

and

$$\arg z(e_1) + \arg z(e_2) + \ldots + \arg z(e_k) = 2\pi$$

(so the angle sum is 2π, not $4\pi, 6\pi, \ldots$...).

Example: The figure eight knot complement



The edge equations are:

$$z^2 z' w^2 w' = 1, \quad z'(z'')^2 w'(w'')^2 = 1$$

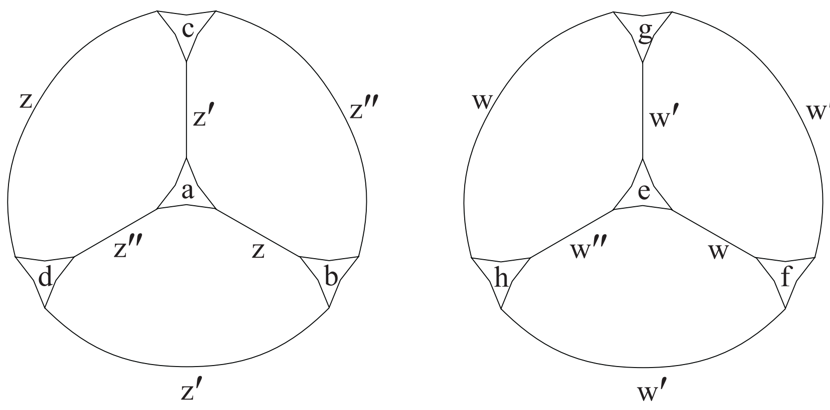
which simplify to

$$\begin{aligned} zw(1-z)(1-w) &= 1 \\ z^{-1}w^{-1}(1-z)^{-1}(1-w)^{-1} &= 1. \end{aligned}$$

These equations are equivalent (since $zz'z'' = ww'w'' = 1$), so there is a 1-complex dimensional solution space, giving *possibly incomplete* hyperbolic structures.

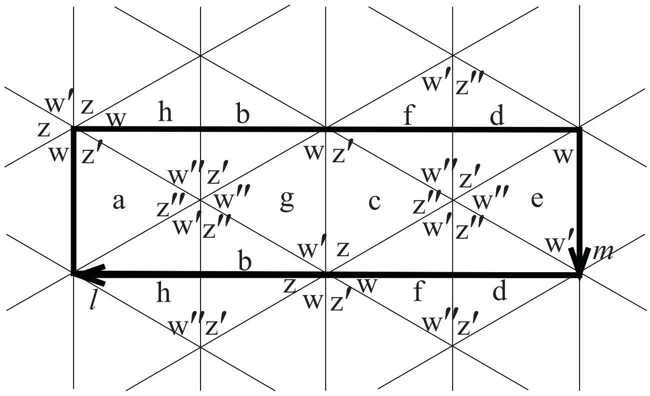
Completeness

We look at the developing map for a cusp cross section. First cut off corners of tetrahedra:



These give a triangulation of the cusp torus.

Now map these corners to horospherical triangles:



Can read off the "holonomies" of the standard longitude and the meridian: these are expansions by complex numbers

$$h(l) = z^2(1-z)^2$$

$$h(m) = w(1-z).$$

For a complete hyperbolic structure,

$$h(l) = h(m) = 1.$$

This gives a unique solution with $\mathrm{Im}(z)$ and $\mathrm{Im}(w) > 0$:

$$z = w = e^{\pi i/3}$$

i.e. both regular ideal tetrahedra.

(Note that we expect uniqueness by Mostow-Prasad Rigidity!)

Hyperbolic Dehn filling

Thurston showed that many of the *incomplete solutions* also have topological significance: their completions give hyperbolic structures on the closed manifolds obtained by Dehn filling.

For (p, q)-Dehn filling (so that $pm + ql$ bounds a disc in the added solid torus) need to solve:

edge conditions: as before

and

holonomy condition:

the holonomy of (p, q) curve is rotation by 2π,

$$\text{or} \;\; h(m)^p h(l)^q = 1$$

This is another polynomial equation in the complex edge parameters.

Some computer programs

SnapPea by Jeff Weeks

Uses **ideal triangulations** to find hyperbolic structures on **cusped** hyperbolic 3-manifolds (finite volume, non-compact) and closed manifolds obtained from these by Dehn filling.

Can start by drawing a projection of a knot or link, and find hyperbolic structures on the link complement and on manifolds obtained by Dehn surgery.

Given a cusped 3-manifold M, SnapPea will:

• Find an ideal triangulation of M

• Simplify the triangulation (e.g. by 3 to 2, and 2 to 3 moves)

• Solve Thuston's gluing equations numerically using Newton's method (all regular tetrahedra as initial guess)

• Can also do hyperbolic Dehn surgery.

Given the hyperbolic structure SnapPea can:

• calculate many geometric invariants (e.g. volume, lengths of closed geodesics

• test for isometry between manifolds using the "canonical cell decomposition" of Epstein-Penner.

Open Problem:

Prove that there are efficient algorithms for computing hyperbolic structures on 3-manifolds and 3-orbifolds.

Algorithms exist [Casson, J. Manning], but not efficient.

In practice SnapPea seems to work extremely well — explain why!

Snap by Oliver Goodman
(working with W. Neumann, C.Hodgson)
An **exact** version of SnapPea: describes hyperbolic structures via algebraic numbers, and computes associated arithmetic invariants.

Snap is based on SnapPea and the number theory package Pari.

Note: Mostow-Prasad Rigidity implies that the complex edge parameters corresponding to the complete hyperbolic structure are *algebraic numbers*. (They lie in a 0-dimensional algebraic variety of solutions to Thurston's gluing equations: a system of polynomial equations with integer coefficients.)

Step 1. Compute hyperbolic structure *numerically to high precision* (e.g. 100-200 digits), using SnaPea's solution as a starting point.

From this we try to find **exact** solutions described as algebraic numbers!

Idea:
Step 2. Given an accurate numerical approximation to an algebraic number z *try to guess the minimal polynomial* for z.
Step 3. After doing this for all simplex parameters z_i, *verify the guess* by checking that Thurston's gluing equations are satisfied by exact computation in a suitable algebraic number field.

To guess minimal polynomials, use the LLL algorithm [Lenstra-Lenstra-Lovasz, 1982] for finding short vectors in an integer lattice with a given inner product:

If z approximates an algebraic number τ, we look for an integer polynomial of degree $\leq m$ with coefficients not too big, which is very small at z.

More precisely, we look at the quadratic form

$$(a_0, a_1, \ldots, a_m) \mapsto a_0^2 + a_1^2 + \ldots + a_m^2 + N|a_0 + a_1 z + \ldots + a_m^n|^2$$

where N is **large**, say $N \approx 10^{1.5d}$ if z is given to d decimal places.

If LLL finds a short integer vector $(a_0, \ldots a_m)$ then it is likely that
$a_0 + a_1\tau + \ldots + a_m\tau^m = 0$.
By factoring this, we guess the minimal polynomial for τ!

Similarly, we use LLL to find an algebraic number field containing all simplex parameters.

If this process succeeds, we obtain a rigorous proof that M is hyperbolic, and an exact description of its hyperbolic structure.

From this we can then compute many powerful arithmetic invariants, e.g. invariant trace field, invariant quaternion algebra, and decide if M is arithmetic or not.

Some References

W. Thurston, Geometry and topology of 3-manifolds, Lecture Notes, Princeton University (1979), http://www.msri.org/publications/books/gt3m/

W. Thurston, Three-dimensional Geometry & Topology, Princeton Univ. Press, 1997.

W. Neumann and D. Zagier, Volumes of hyperbolic three-manifolds, Topology 24 (1985), 307–332.

D. Coulson, O. Goodman, C. Hodgson and W. Neumann, Computing arithmetic invariants of 3-manifolds, Experimental Math. 9 (2000),127–152.

J. Weeks, SnapPea, a program for studying hyperbolic 3-manifolds, http://www.geometrygames.org/SnapPea/.

O. Goodman, Snap, http://www.ms.unimelb.edu.au/ snap/ and http://sourceforge.net/projects/snap-pari/

M. Culler and N. Dunfield, SnapPy, http://www.math.uic.edu/∼t3m/