Entropy Rigidity and Measures on the Boundary of Negatively Curved Metric Spaces

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Mostow strong rigidity and quasi-isometry rigidity

Volume entropy rigidity

Katok’s rigidity conjecture

Applications of measures on the bdy – equidistribution of orbits

Gromov hyperbolic spaces

CAT(-1), CAT(0) spaces

measures on the boundary
Mostow strong rigidity

\[ X_1, X_2 : \text{hyperbolic manifolds (} X_i = \mathbb{H}^n / \pi_1(X_i) \text{)} \]

If \( \pi_1(X_1), \pi_1(X_2) \) are isomorphic, then \( X_1, X_2 \) are isometric.

Idea of proof. (for compact case)

\[ G = \pi_1(X_1) \cong \pi_1(X_2) \text{ acts on } \mathbb{H}^n \text{ discretely, cocompactly, isometrically.} \]

\[ \rightarrow \text{conformal action on the boundary } \partial \mathbb{H}^n \]

quasi-equivariant quasi-isometry \( f : \mathbb{H}^n \rightarrow \mathbb{H}^n \)

quasi-conformal homeomorphism \( \partial f : \partial \mathbb{H}^n \rightarrow \partial \mathbb{H}^n \)
Quasi-isometry rigidity

Any quasi-isometry $f : X \rightarrow X$ is at bounded distance from a unique isometry $f'$:

$$d(f, f') = \sup d(f(x), f'(x)) < \infty$$

Pansu $X$ : quaternionic hyperbolic space
or Cayley hyperbolic plane

Bourdon-Pajot $X$ : Fuchsian buildings

Both follow Mostow’s idea, using boundary of Gromov hyperbolic space!
Volume entropy rigidity: what is volume entropy?

- exponential volume growth of balls!
  
  $\mathbb{R}^n$: $\text{vol}(B(r))$ poly of $r \Rightarrow \text{entropy} = 0$
  
  $\mathbb{H}^n$: $\text{vol}(B(r)) \sim e^{(n-1)r} \Rightarrow \text{entropy} = n-1$

Manifold: take its universal cover

**Def** (volume entropy) $M$ closed Riem. mnfd

$$h_{\text{vol}} = \lim_{r \to \infty} \frac{1}{r} \log \text{vol}(B(x, r))$$

where $B(x, r) \subset \tilde{M}$
Def (volume entropy) \( B(x,r) \subset \tilde{M} \)

\[
h_{vol} = \lim_{r \to \infty} \frac{1}{r} \log \operatorname{vol}(B(x, r))
\]

- \( h_{vol} > 0 \) iff \( \pi_1(M) \) is of exponential growth (Milnor)
- equal to critical exponent (for compact mnfds)

\[
\delta(\pi_1(M)) = \lim_{r \to \infty} \frac{1}{r} \log \# B_{\pi_1(M)}(x, r),
\]

where \( B_{\pi_1(M)} = \{ \gamma \in \pi_1(M) : d(x, \gamma x) < r \} \)

- It captures the part of the fastest growth.
- equal to Hausdorff dim of limit set on boundary
• equal to topological entropy of the geodesic flow
  = “exponential growth of # of $\varepsilon$-separated geodesic segments in $\tilde{M}$” (Manning)

$$h_{top} = \lim_{\varepsilon \to 0} \lim_{r \to \infty} \frac{1}{r} \log \# S(\varepsilon, r)$$

where

$$S(\varepsilon, r) = \{ g : g(0) \in F, \ell(g) < r, d(g(t), g'(t)) > \varepsilon, \forall 0 \leq t \leq r \}$$

• equal to the exponential growth rate of # of closed geodesics in $M$ (Margulis)
• Cheeger Isoperimetric constant

\[ Ch(M) = \inf_{N} \frac{Area(N)}{\min\{Vol(A), Vol(B)\}} , \]

where N separates M into disjoint A and B.

\[ Ch(M) \leq h_{vol} : \quad \frac{Area(S(x, r))}{Vol(B(x, r))} = \frac{Vol(B(x, r))'}{Vol(B(x, r))} \geq Ch(M) \]

\[ Vol(B(x, r)) \approx e^{h_{vol} r} \geq e^{Ch(M) r} \]

• smallest eigenvalue of the Laplacian

\[ \lambda_1(M) \geq \frac{Ch(M)^2}{4} \quad (\text{Cheeger}) \]

• Gromov’s simplicial volume

\[ \|M\| \leq C(n) h_{vol}^n (M) \quad (\text{Gromov}) \]

related to
Entropy rigidity

Gromov Conjecture Among all volume 1 Riemannian metrics on a closed manifold of non-positive curvature, of dimension $\geq 2$, the locally symmetric metric minimizes the volume entropy.

[Katok] Surfaces
[Connell-Farb] lattices in products of rank-1 symmetric spaces
[Besson-Courtois-Gallot] Rank-1 symmetric spaces

higher rk : still open!
Besson-Courtois-Gallot \( n \geq 3 \)

\( X, Y \) : compact connected orientable n-dim mnfds of negative curv. \( Vol(X, g_0) = Vol(Y, g) = 1 \)

\( g_0 \) : locally symmetric metric

\( f : Y \to X \) : continuous map of \( \deg \neq 0 \)

Then \( h^\text{vol}_n(Y, g) \geq |\deg f|h^\text{vol}_n(X, g_0) \)

= holds iff \( g \) locally symm, \( f \) local isometry

Coro (Mostow strong rigidity)
Idea of proof. \( f : Y \to X \) : continuous map

\[
\mathcal{M}(\partial \tilde{Y}) \xrightarrow{\partial f^*} \mathcal{M}(\partial \tilde{X})
\]

Patterson-Sullivan measure \( \uparrow \bigcirc \downarrow \) Barycenter map

\[
Y \xrightarrow{f} X
\]

Patterson-Sullivan measure : family of measures

\( \pi_1(Y) \)-invariant conformal density of dimension \( \delta \):

\[
\frac{d\mu_{y'}}{d\mu_y}(\xi) = e^{-\delta \beta_\xi(y',y)}, \quad \gamma^*\mu_y = \mu_{\gamma y}
\]

where

\[
\beta_\xi(y', y) = \lim_{t \to \infty} \{d(y', \xi_t) - d(y, \xi_t)\}
\]
Bowen-Margulis measure: measure on the space of geodesics $\mathcal{G}(\tilde{X})$

$$dm(u) = \frac{d\mu_x(\xi)d\mu_x(\eta)ds}{d_x(\xi, \eta)^{2\delta}}$$

$$= e^{\delta \beta_x(x,u) + \delta \beta_\eta(x,u)}d\mu_x(\xi)d\mu_x(\eta)ds$$

where $(\xi, \eta, s) = (g^{-\infty}u, g^{\infty}u, \beta_{g^{-\infty}}u(u, o))$

Rmk: PS, BM measures can be defined on $\text{CAT}(-1)$-sp! (Roblin)

It is the unique measure of maximal measure-theoretic entropy.

Variational principle: $h_{vol} = \sup_{\mu: g^{-\text{inv}} \mu} \{h_\mu\}$

$\mathcal{G}$ : geodesic flow
Two natural $g$-invariant measures on $G(\tilde{X})$

- Bowen-Margulis measure: of maximal measure-theoretic entropy
- Liouville measure: “volume times angular measure”

Katok’s Rigidity Conjecture
For a closed Riemannian manifold of negative curvature,

Liouville measure $\parallel$ the metric is locally symmetric
Bowen-Margulis measure $\iff$ [Katok] surfaces
Still open for higher dimension
**Thm** [Ledrappier-L, 09]

\( \Delta : \) regular hyperbolic building

\( X : \) A compact quotient of \( \Delta \), with a hyp. metric

\( \text{Liouville measure} \neq \text{Bowen-Margulis measure} \)

**Rmk** Contrast to Katok’s conjecture.
Why buildings?

1. Non-archimedean analogue of symm. spaces.

\[ G = SL_3(\mathbb{F}_q((t))) : \quad G/I : \text{affine flag variety} \]

\[ K = SL_3(\mathbb{F}[[t]]) \quad \Theta \rightarrow SL_3(\mathbb{F}_q) \]

\[ I = \Theta^{-1}(B(\mathbb{F}_q)) \quad \Theta \rightarrow B(\mathbb{F}_q) \]

2. We want to start with a singular space with a large group of isometries.

Examples

dimension 1: locally finite (uniform) trees

\( \rightarrow \) compact quotients are graphs
E.g. in dimension 1: locally finite regular trees

two axioms of buildings (blackboard)

E.g. in dimension 2: product of two trees

: Euclidean building
dim 2: Right-angled Fuchsian building

Each apartment looks like this.
dim 2: Right-angled Fuchsian building

Each apartment looks like this.

Attach $p_i$ more hexagons to each edge.
Lk(\(\cdot\)): complete bipartite graph

gen. 2-gon: diameter 2, length of shortest cycle 4

Apartments: tessellations of \(\mathbb{H}^n\)
Chambers: polygons
Description of entropy of the building

Thm [Ledrappier-L] The entropy of any regular building can be separated into the growth of an apartment and the growth coming from tree-like branching.

Idea of proof.

Assume that the quotient is one polygon.

\[ h_{\text{vol}} = \text{exp. growth of geodesic segments} \]
\[ = \text{exp. growth of geodesics in } \mathbb{H}^2 \]
multiplied by \# of preimages
under the retraction:
\[
1 = \int_0^l \frac{1}{l} \, dt \\
\log p = \int_0^l \frac{\log p}{l} \, dt \\
1 = \int_0^l \frac{1}{l} \, dt \\
\log p = \int_0^l \frac{\log p}{l} \, dt \\
\]

\[
p_1 \cdots p_n = e^{\log p_1 + \cdots + \log p_n} \\
= \exp \left( \int_0^T \frac{\log p}{l}(g_t(v)) \, dt \right)
\]
Idea of proof (continued).

\[
    h_{\text{vol}} = \text{exp. growth of geodesics in } \mathbb{H}^2 \\
    \text{multiplied by \# of preimages under the retraction map}
\]

\[
    = \text{exp. growth of geodesics in } \mathbb{H}^2 \\
    \text{multiplied by } \exp \int_0^T \frac{\log p}{l} (g_t(v)) dt
\]

Thm 2 [Ledrappier-L] \( \triangle \): Euclidean or hyperbolic regular building

\[
    h_{\text{vol}} (\triangle) = \mathcal{P}_A \left( \frac{\log p}{l} \right)
\]

topological pressure of geodesic flow (of \( \frac{\log p}{l} \))
Thm 2 [Ledrappier-L] \( \Delta : \) Bourdon’s building

\[
h_{\text{vol}}(\Delta) = P_{\mathbb{H}^2} \left( \frac{\log p}{l} \right)
\]

\[
= \sup_{\mu} \left\{ h_{\mu} + \int \frac{\log q}{l} d\mu \right\}
\]

Now let’s take some special measure \( \mu \), namely the Liouville measure on \( \mathbb{H}^2 \).
Now take $\mu_0 = \text{Liouville measure}:$

$$h_{\text{vol}} \geq h_{\mu_0} + \int \frac{\log p}{l} d\mu_0$$

$$= 1 + \frac{1}{\pi^2} \sum \log p_i l(e_i)$$

(Santalo’s formula)

Coro $X$: $p_i$ branching at edge $e_i$, with a hyperbolic metric. Then

$$h(X) > 1 + \frac{1}{\pi^2} \sum \log p_i l(e_i)$$
Thm (Ruelle)
If $P(f)$ and $P(g)$ have the same equilibrium measure, then $f$ and $g$ are cohomologous up to a constant:

$$f = g + h - h \circ \sigma + c$$

We know that $h_{\text{top}} = P(0)$ is attained by the Liouville measure, thus if $P\left(\frac{\log p}{l}\right)$ is also attained by the Liouville measure, then

$$\int \frac{\log p}{l} dm$$

should not depend on $m$.

Construct a family of measures!
Applications to equidistribution of orbits:

\[ x, y \in X, b \in X(\infty), S_x(\Omega_1) : \text{sector}, \quad \Omega_2 \subset X(\infty) \]

Q. \[ |\{ \gamma \in \Gamma : y \gamma \in S_x(\Omega_1) \cap B_T(x), b \gamma^{-1} \in \Omega_2 \}| = ? \]

\[ \sim m_x(\Omega_1)m_x(\Omega_2) \frac{\text{vol}(B_T(x))}{\text{vol}(X/\Gamma)} \]

[Margulis, Gorodnik–Oh, Oh–Shah, Roblin]

\[ \Gamma : \text{geometrically finite} \]

\[ \rightarrow \text{asymptotic formula involves BM-measure \& PS-measure}. \]