Entropy Rigidity and Measures on the Boundary of Negatively Curved Metric Spaces

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- Mostow strong rigidity and quasi-isometry rigidity
- Volume entropy rigidity
- Katok's rigidity conjecture
- Applications of measures on the bdy - equidistribution of orbits

Gromov hyperbolic spaces

CAT(-1), CAT(0) spaces

measures on the boundary

Mostow strong rigidity

X_1 ,X_2 : hyperbolic manifolds ($X_i = \mathbb{H}^n/\pi_1(X_i)$)

$(n \geq 3)$

If $\pi_1(X_1), \pi_1(X_2)$ are isomorphic,

then X_1, X_2 are isometric.

Idea of proof. (for compact case)

$G = \pi_1(X_1) \cong \pi_1(X_2)$ acts on $\mathbb{H}^n$ discretely, cocompactly, isometrically.

-> conformal action on the boundary $\partial\mathbb{H}^n$

quasi-equivariant quasi-isometry $f : \mathbb{H}^n \longrightarrow \mathbb{H}^n$

quasi-conformal homeomorphism $\partial f : \partial\mathbb{H}^n \rightarrow \partial\mathbb{H}^n$

Quasi-isometry rigidity

Any quasi-isometry $f : X \to X$ is at bounded distance from a unique isometry f':

$$d(f, f') = \sup d(f(x), f'(x)) < \infty$$

Pansu X :quaternionic hyperbolic space
or Cayley hyperbolic plane

Bourdon-Pajot X: Fuchsian buildings

Both follow Mostow's idea, using boundary of Gromov hyperbolic space!

Definition (blackboard)

Volume entropy rigidity : what is volume entropy?

- exponential volume growth of balls!

$\mathbb{R}^n$: vol(B(r)) poly of r => entropy = 0

$\mathbb{H}^n$: vol(B(r)) $\sim e^{(n-1)r}$ => entropy = n-1

Manifold : take its universal cover

Def (volume entropy) M closed Riem. mnfd

$$\mathrm{h}_{\mathrm{vol}} = \lim_{r\to\infty} \frac{1}{r} \log \mathrm{vol}(B(x,r))$$

where $\mathrm{B(x,r)} \subset \tilde{M}$

Def (volume entropy) $\mathrm{B(x,r)} \subset \tilde{M}$

$$\mathrm{h_{vol}} = \lim_{r\to\infty} \frac{1}{r} \log \mathrm{vol}(B(x,r))$$

- $h_{\mathrm{vol}} > 0$ iff $\pi_1(M)$ is of exponential growth (Milnor)
- equal to critical exponent (for compact mnfds)

(Eskin-McMullen, Dalbo-Peigne-Picaud-Sambusetti)

$$\delta(\pi_1(M)) = \lim_{r\to\infty} \frac{1}{r} \log \#B_{\pi_1(M)}(x,r),$$

where $B_{\pi_1(M)} = \{\gamma \in \pi_1(M) : d(x, \gamma x) < r\}$

- It captures the part of the fastest growth.
- equal to Hausdorff dim of limit set on boundary

- equal to topological entropy of the geodesic flow

="exponential growth of # of ε-separated geodesic segments in $\widetilde{M}$" (Manning)

$$h_{top} = \lim_{\varepsilon \to 0} \lim_{r \to \infty} \frac{1}{r} \log \# S(\varepsilon, r)$$

where

$$S(\varepsilon, r) = \{g : g(0) \in F,\ \ell(g) < r, d(g(t), g'(t)) > \varepsilon,\ \forall 0 \le t \le r\}$$

- equal to the exponential growth rate of # of closed geodesics in M (Margulis)

related to

- Cheeger Isoperimetric constant

$$Ch(M) = \inf_N \frac{Area(N)}{\min\{Vol(A), Vol(B)\}},$$

where N separates M into disjoint A and B.

$$Ch(M) \le h_{vol} : \frac{Area(S(x,r))}{Vol(B(x,r))} = \frac{Vol(B(x,r))'}{Vol(B(x,r))} \ge Ch(M)$$

$$Vol(B(x,r)) \simeq e^{h_{vol} r} \gtrsim e^{Ch(M) r}$$

- smallest eigenvalue of the Laplacian (Ledrappier)

(Brooks)

$$\lambda_1(M) \ge \frac{Ch(M)^2}{4}$$ (Cheeger)

(Gromov)

- Gromov's simplicial volume $||M|| \le C(n) h_{vol}^n(M)$

Entropy rigidity

Gromov Conjecture Among all volume 1 Riemannian metrics on a closed manifold of non-positive curvature, of dimension ≥ 2, the locally symmetric metric minimizes the volume entropy.

[Katok] Surfaces

[Besson-Courtois-Gallot] Rank-1 symmetric spaces

[Connell-Farb] lattices in products of rank-1 symmetric spaces

higher rk : still open!

Besson-Courtois-Gallot $n \geq 3$

X, Y : compact connected orientable n-dim mnfds of negative curv. $Vol(X, g_0) = Vol(Y, g) = 1$

g_0 : locally symmetric metric

$f : Y \to X$: continuous map of deg $\neq 0$

Then $h^n_{vol}(Y, g) \geq |\deg f| h^n_{vol}(X, g_0)$

$=$ holds iff g locally symm , f local isometry

Coro (Mostow strong rigidity)

Idea of proof.

$f : Y \to X$: continuous map

$$\begin{tikzcd}
\mathcal{M}(\partial\widetilde{Y}) \arrow[r, "\partial f_*"] \arrow[dr, phantom, "\curvearrowright"] & \mathcal{M}(\partial\widetilde{X}) \arrow[d] \\
Y \arrow[u] \arrow[r, "f"] & X
\end{tikzcd}$$

Patterson-Sullivan measure; Barycenter map

Patterson-Sullivan measure : family of measures

$\pi_1(Y)$ -invariant conformal density of dimension δ :

$$\frac{d\mu_{y'}}{d\mu_y}(\xi) = e^{-\delta\beta_\xi(y',y)}, \qquad \gamma_*\mu_y = \mu_{\gamma y}$$

where $\beta_\xi(y',y) = \lim_{t\to\infty}\{d(y',\xi_t) - d(y,\xi_t)\}$

Bowen-Margulis measure : measure on the space of geodesics $\mathcal{G}(\widetilde{X})$

$$dm(u) = \frac{d\mu_x(\xi)d\mu_x(\eta)ds}{d_x(\xi,\eta)^{2\delta}}$$

$$= e^{\delta\beta_\xi(x,u)+\delta\beta_\eta(x,u)}d\mu_x(\xi)d\mu_x(\eta)ds$$

where $(\xi,\eta,s) = (g^{-\infty}u, g^{\infty}u, \beta_{g^{-\infty}u}(u,o))$

Rmk: PS, BM measures can be defined on CAT(-1)-sp!

(Roblin)

It is the unique measure of maximal measure-theoretic entropy.

Variational principle : $h_{\mathrm{vol}} = \sup_{\mu:g-\mathrm{inv}} \{h_\mu\}$

g : geodesic flow

Two natural g-invariant measures on $\mathcal{G}(\widetilde{X})$

- Bowen-Margulis measure: of maximal measure-theoretic entropy
- Liouville measure : "volume times angular measure"

Katok's Rigidity Conjecture For a closed Riemannian manifold of negative curvature,

Liouville measure $=$ Bowen-Margulis m iff the metric is locally symmetric.

- [Katok] surfaces
- Still open for higher dimension

Thm [Ledrappier-L, 09]

Δ : regular hyperbolic building

X : A compact quotient of Δ, with a hyp. metric

Liouville measure $\neq$ Bowen-Margulis measure

Rmk Contrast to Katok's conjecture.

Why buildings?

1. Non-archimedean analogue of symm. spaces.

$$G = SL_3(\mathbb{F}_q((t))) : \qquad G/I \text{ : affine flag variety}$$

$$K = SL_3(\mathbb{F}[[t]]) \xrightarrow{\Theta} SL_3(\mathbb{F}_q)$$

$$I = \Theta^{-1}(B(\mathbb{F}_q)) \xrightarrow{\Theta} B(\mathbb{F}_q)$$

2. We want to start with a singular space with a large group of isometries.

Examples

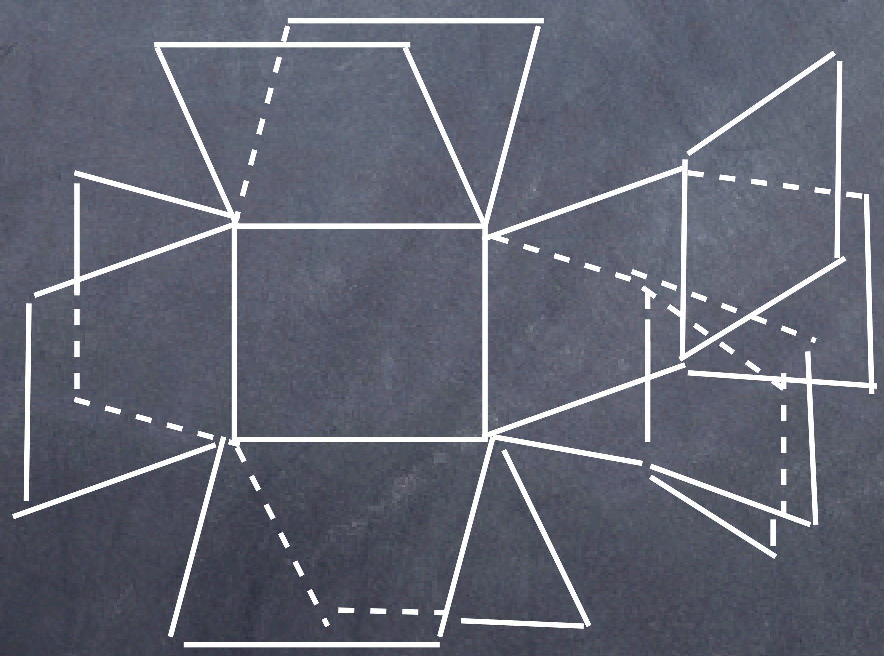
dimension 1: locally finite (uniform) trees

-> compact quotients are graphs

E.g. in dimension 1: locally finite regular trees

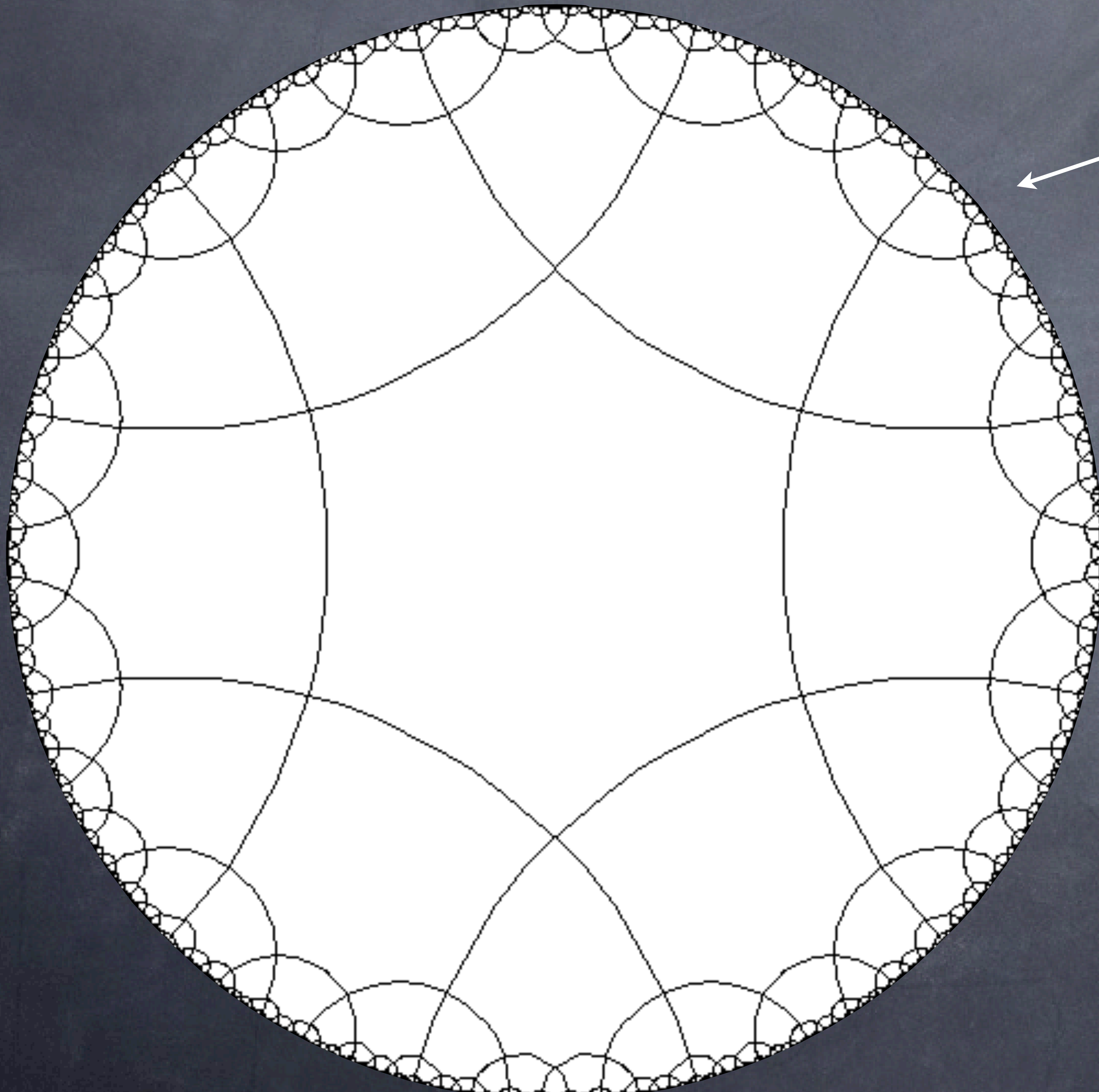
two axioms of buildings (blackboard)

E.g. in dimension 2 : product of two trees



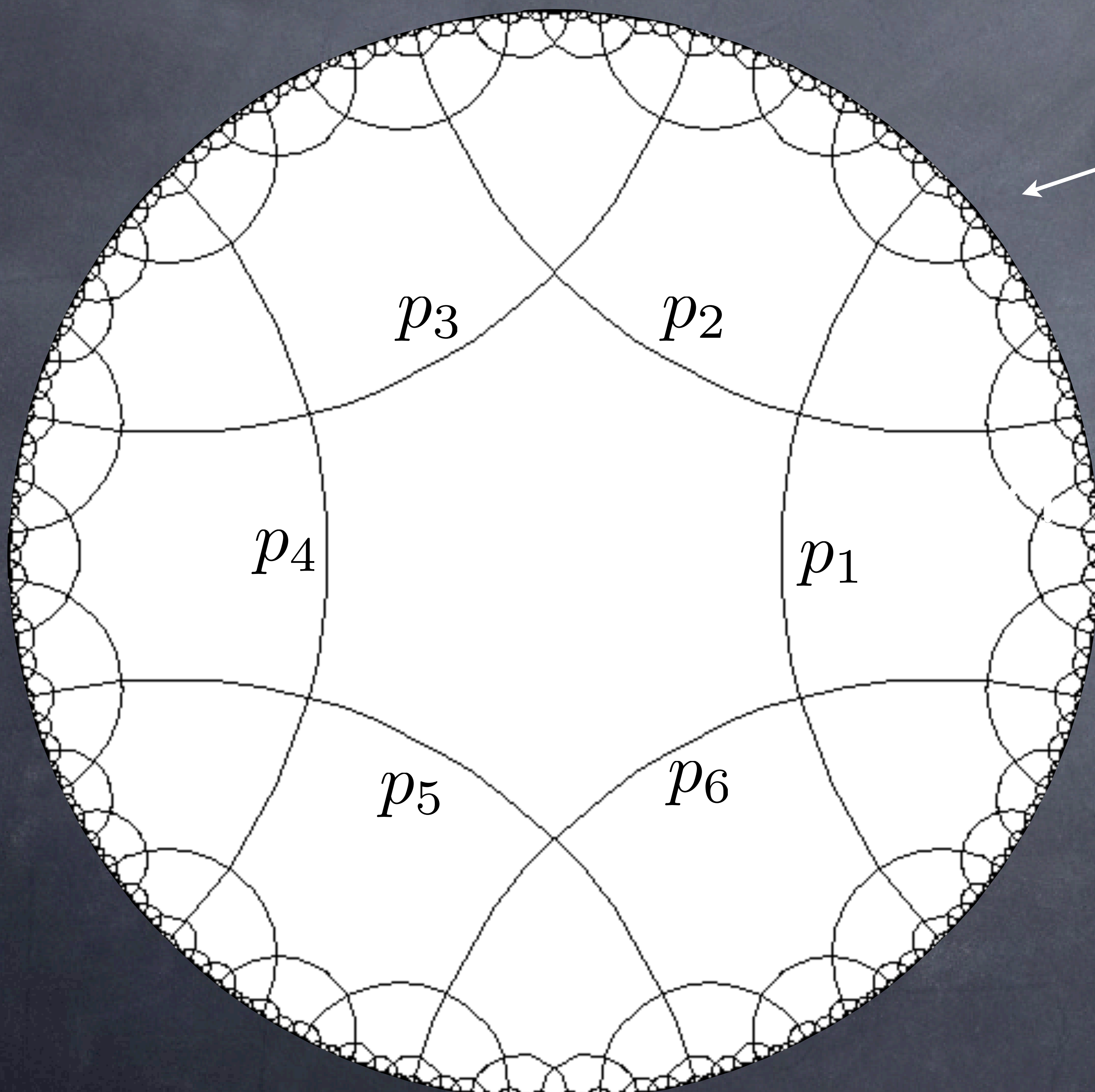
: Euclidean building

dim 2 : Right-angled Fuchsian building



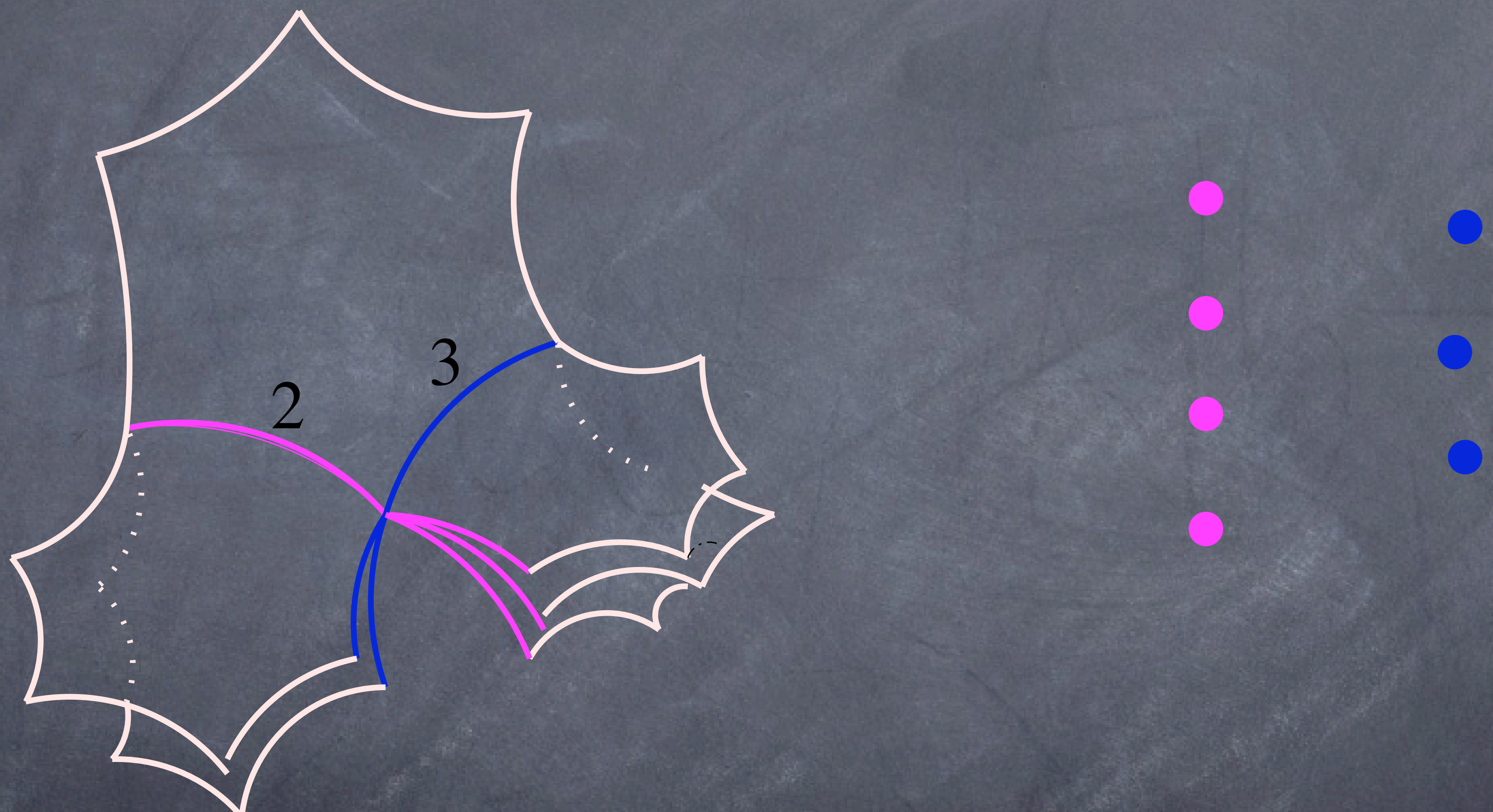
Each apartment
looks like this.

dim 2 : Right-angled Fuchsian building



Each apartment looks like this.

Attach p_i more hexagons to each edge.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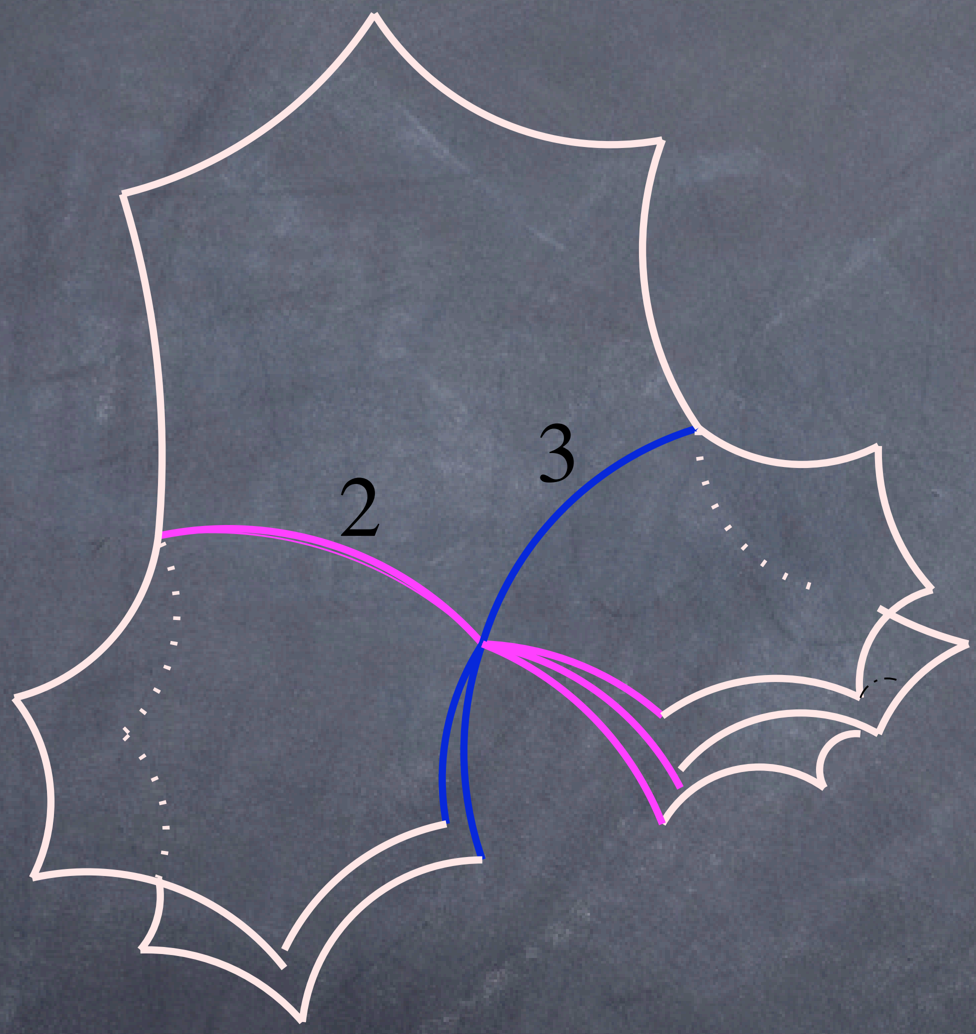
2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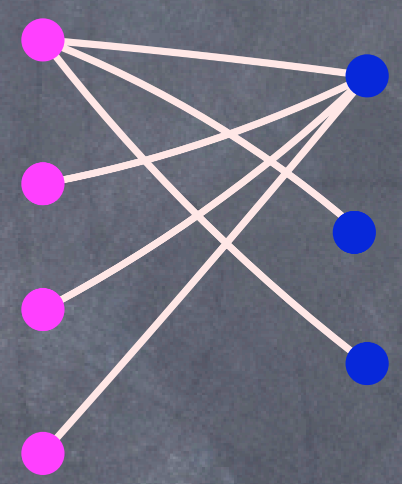
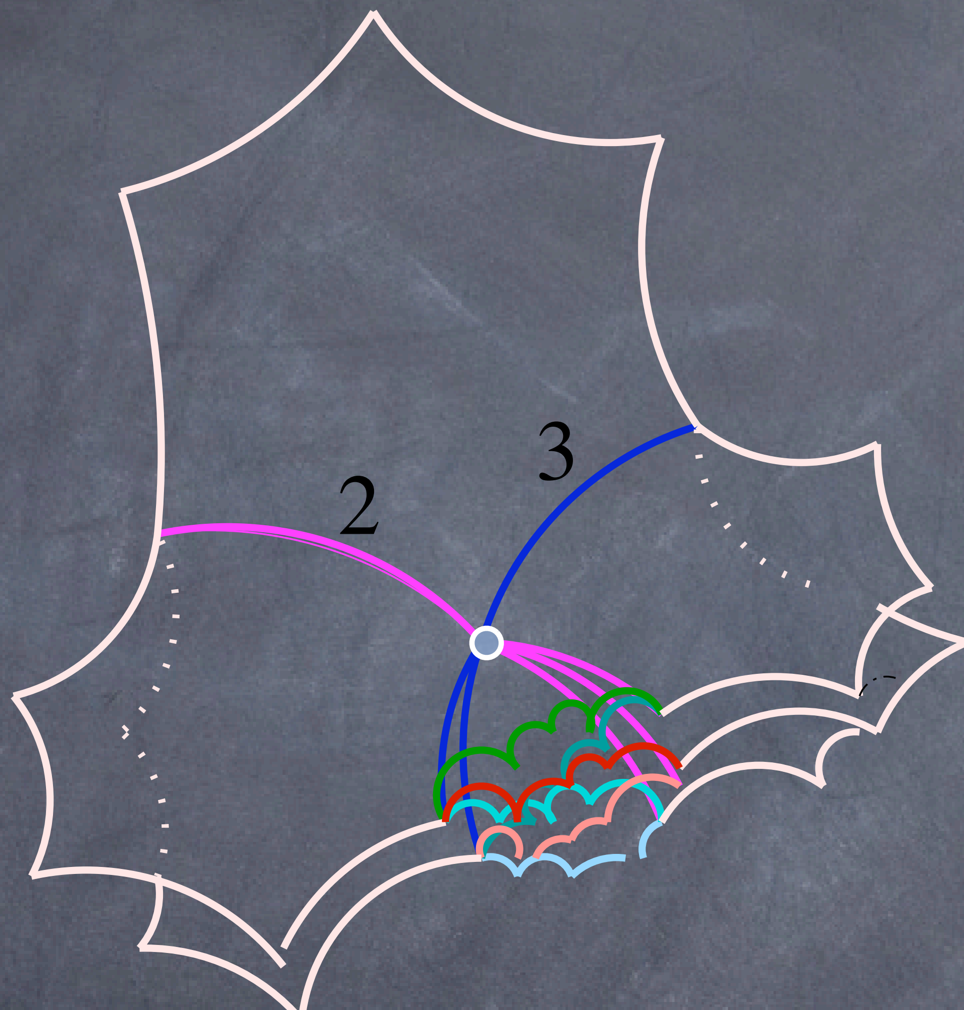
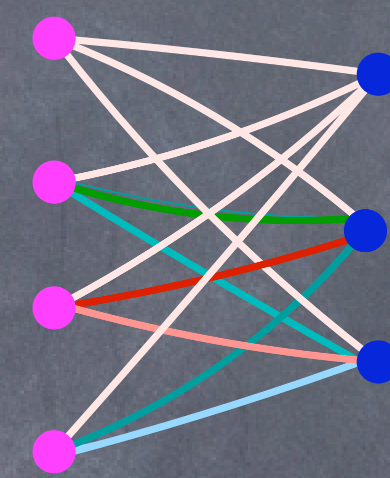




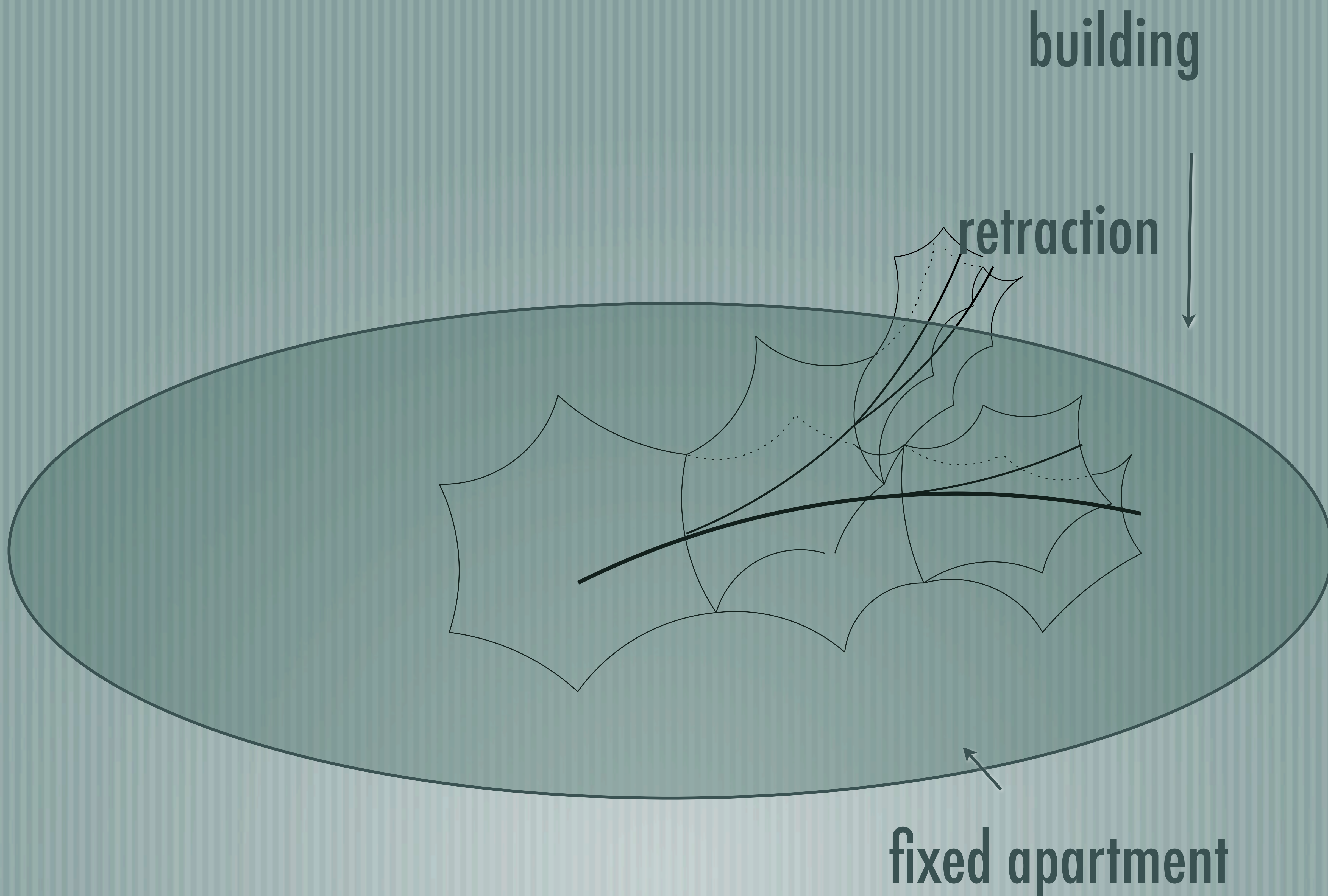
Lk(∘): complete bipartite graph



gen. 2-gon : diameter 2,
length of shortest cycle 4

Apartments : tessellations of H^n

Chambers : polygons


building
retraction
fixed apartment

Description of entropy of the building

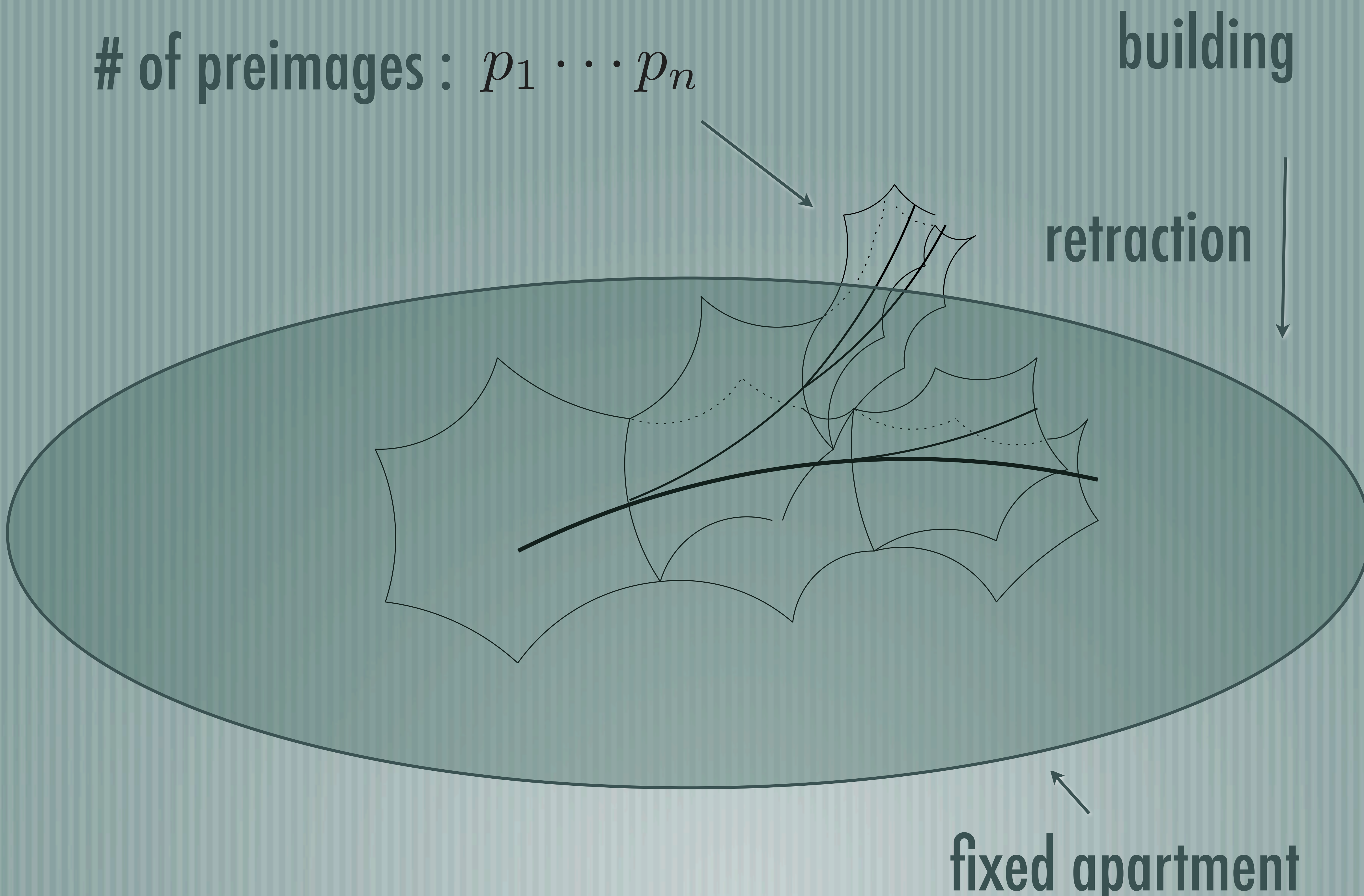
Thm [Ledrappier-L] The entropy of any regular building can be separated into the growth of an apartment and the growth coming from tree-like branching.

Idea of proof.

Assume that the quotient is one polygon.

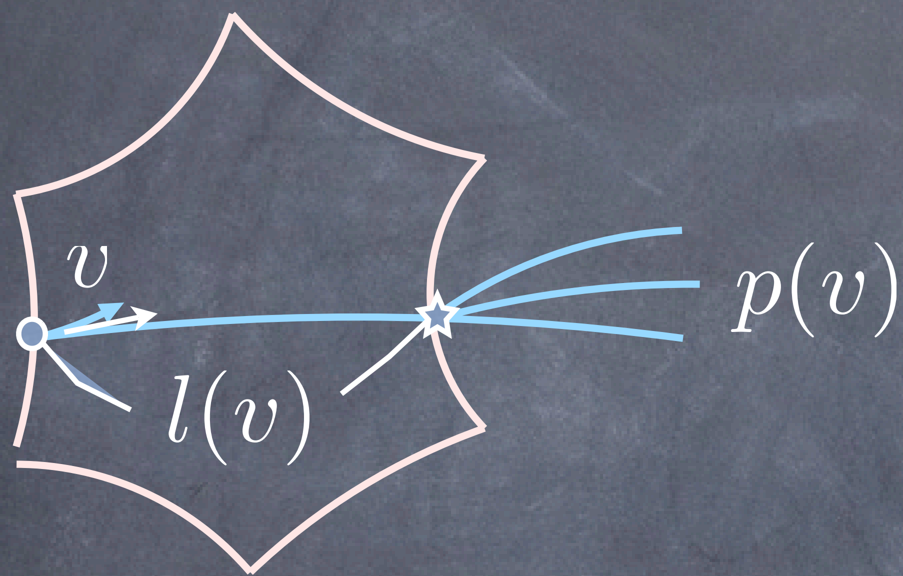
h_{vol} = exp. growth of geodesic segments

= exp. growth of geodesics in $\mathbb{H}^2$ multiplied by # of preimages under the retraction:



# of preimages : $p_1 \cdots p_n$
building
retraction
fixed apartment

$$1 = \int_0^l \frac{1}{l}\,dt \qquad \log p = \int_0^l \frac{\log p}{l}\,dt$$



$$= \int_{\bullet}^{\star} \frac{\log p(g_t(v))}{l(g_t(v))}\,dt$$

$$p_1 \cdots p_n = e^{\log p_1 + \cdots + \log p_n}$$

$$= \exp\left(\int_0^T \frac{\log p}{l}(g_t(v))\,dt \right)$$

Idea of proof (continued).

h_{vol} = exp. growth of geodesics in $\mathbb{H}^2$
multiplied by # of preimages under
the retraction map

= exp. growth of geodesics in $\mathbb{H}^2$
multiplied by $\exp \int_0^T \frac{\log p}{l}(g_t(v))dt$

Thm 2 [Ledrappier-L] Δ : Euclidean or hyperbolic regular building

$$h_{\mathrm{vol}}(\Delta) = \mathcal{P}_{\mathcal{A}}\left(\frac{\log p}{l}\right)$$

topological pressure of geodesic flow

(of $\frac{\log p}{l}$)

Thm 2 [Ledrappier-L] Δ : Bourdon's building

$$h_{\mathrm{vol}}(\Delta) = \mathcal{P}_{\mathbb{H}^2}\left(\frac{\log p}{l}\right)$$

Variational Principle

$$= \sup_{\mu}\left\{h_\mu + \int \frac{\log q}{l}\,d\mu\right\}$$

flow invariant

Now let's take some special measure μ, namely the Liouville measure on $\mathbb{H}^2$.

Now take μ_0 = Liouville measure:

$$h_{\rm vol} \geq h_{\mu_0} + \int \frac{\log p}{l} d\mu_0$$
$$= 1 + \frac{1}{\pi^2} \sum \log p_i l(e_i)$$

(Santalo's formula)

<u>Coro</u> X : p_i branching at edge e_i, with a hyperbolic metric. Then

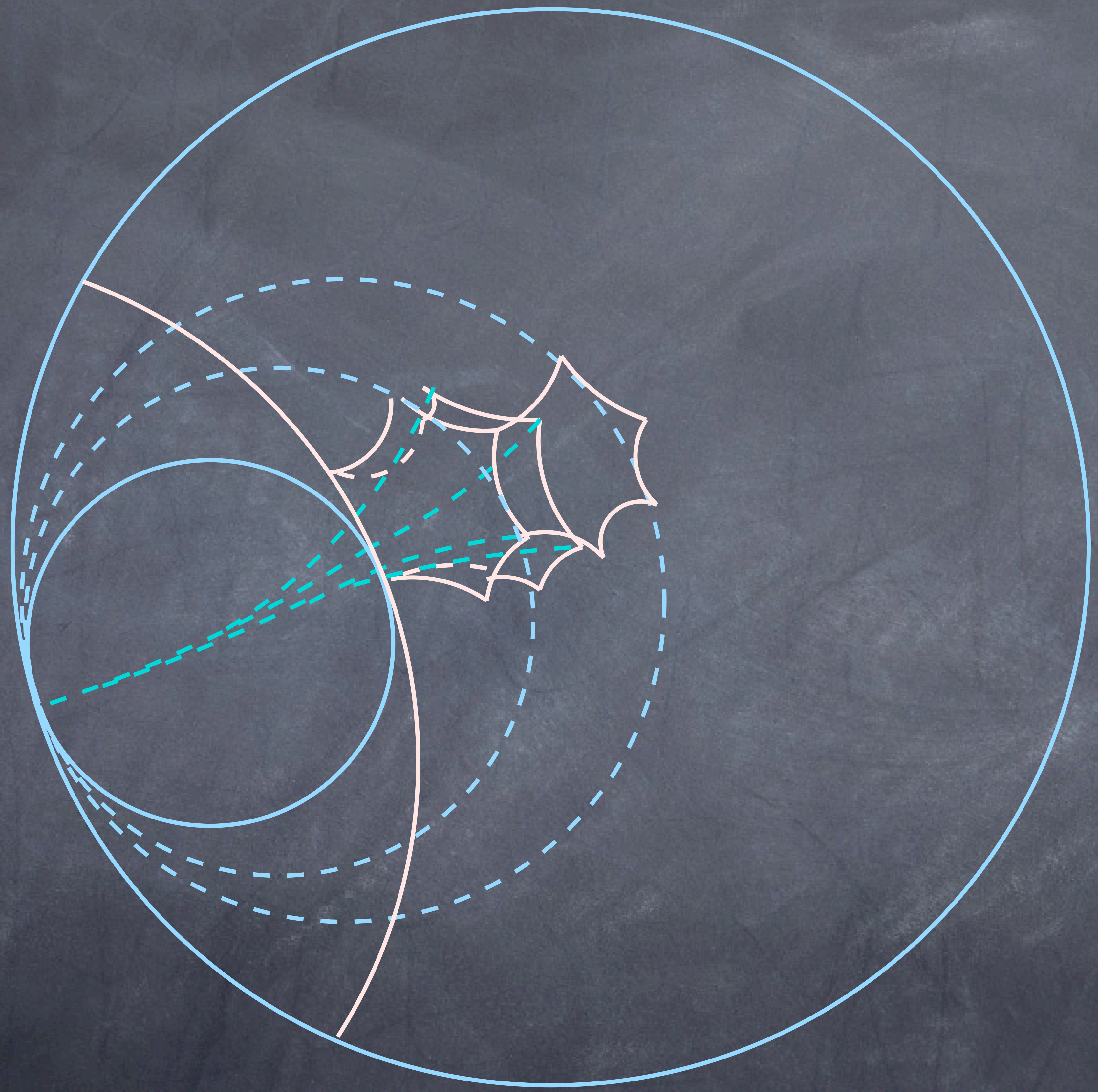
$$h(X) > 1 + \frac{1}{\pi^2} \sum \log p_i l(e_i)$$

Thm (Ruelle)
If $\mathcal{P}(f)$ and $\mathcal{P}(g)$ have the same equilibrium measure, then f and g are cohomologous up to a constant:

$$f = g + h - h \circ \sigma + c$$

We know that $h_{\text{top}} = \mathcal{P}(0)$ is attained by the Liouville measure, thus if $\mathcal{P}(\frac{\log p}{l})$ is also attained by the Liouville measure, then $\int \frac{\log p}{l} dm$ should not depend on m.

Construct a family of measures !



Applications to equidistribution of orbits :

$x, y \in X, b \in X(\infty),\ S_x(\Omega_1)$: sector, $\quad \Omega_2 \subset X(\infty)$

Q. $|\{\gamma \in \Gamma : y\gamma \in S_x(\Omega_1) \cap B_T(x), b\gamma^{-1} \in \Omega_2\}| =?$

$$\sim m_x(\Omega_1) m_x(\Omega_2) \frac{\mathrm{vol}(B_T(x))}{\mathrm{vol}(X/\Gamma)}$$

[Margulis, Gorodnik-Oh, Oh-Shah, Roblin]

Γ :geometrically finite

-> asymptotic formula involves BM-measure & PS-measure.