Thm 5.5.1

let $k_\Delta\Gamma = \mathbb{Q}(z_j : j=1,\cdots,n)$ where z_j's are tetrahedral parameters associated with the ideal triangulation of $\mathbb{H}^3/\Gamma$

then $k_\Delta\Gamma = k\Gamma$

pf) We need to show that $k_\Delta\Gamma \subseteq k\Gamma$

note that $\Gamma^{(2)}$ is a finite index subgp of Γ

from $k_\Delta\Gamma \subseteq \mathbb{Q}(\mathrm{tr}\Gamma)$, we get $k_\Delta\Gamma^{(2)} \subseteq \mathbb{Q}(\mathrm{tr}\Gamma^{(2)}) = k\Gamma$

Since $k_\Delta\Gamma$ is a commensurability invariant, $k_\Delta\Gamma^{(2)} = k_\Delta\Gamma$.

Conjecture (Neumann 2010)

Every non-real concrete number field k and every quarternion algebra over it arise as the invariant trace field and the invariant quarternion algebra of some hyperbolic manifold

(A concrete number field is a number field K with a chosen embedding into $\mathbb{C}$)

Thm 5.6.1

let Γ be a finitely generated kleinian group expressed as $\Gamma_0 *_H \Gamma_1$ or $\Gamma_0 *_H$ where H is a nonelementary kleinian group

then $k\Gamma = k\Gamma_0 \cdot k\Gamma_1$ or $k\Gamma_0$ respectively

compositum of fields.

pf) 1) HNN extension case

lemma 3.5.5 says $k\Gamma = \mathbb{Q}(\mathrm{tr}\,\Gamma^{sq})$ where $\Gamma^{sq} = \langle \gamma_1^2, \gamma_2^2, \cdots, \gamma_n^2 \rangle$

for some generator set $\{\gamma_1, \cdots, \gamma_n\}$ of Γ with $\mathrm{tr}\gamma_i \neq 0 \ \forall i$

Since $\Gamma_0 *_H = \langle \Gamma_0, t \rangle / \{tH_0t^{-1} = H_1\}$ if we let $\Gamma_1 := \langle t^2, \Gamma_0^{(2)} \rangle$ then

$k\Gamma = \mathbb{Q}(\mathrm{tr}\Gamma_1)$

claim: $\mathbb{Q}(\mathrm{tr}\Gamma_1) = k\Gamma_0$

($\because$) Since H_0 is nonelementary, $\exists\ g_0, h_0 \in H_0^{(2)}$ s.t. $\langle g_0, h_0 \rangle$ is irreducible. if $g_1 = tg_0t^{-1}$, $h_1 = th_0t^{-1}$ then $\langle g_1, h_1 \rangle$ is also irreducible and

$A\Gamma_0 = k\Gamma_0[1, g_0, h_0, g_0h_0] = k\Gamma_0[1, g_1, h_1, g_1h_1]$

Conjugation by t induces automorphism θ of $A\Gamma_0$ and by Skolem Noether Thm (Coro 2.9.9.) $\exists\ y \in A\Gamma_0^*$ s.t.

$\theta(a) = tat^{-1} = yay^{-1} \quad \forall a \in A\Gamma_0$

Thus $at^{-1}y = t^{-1}ya \quad \forall a$ so that $t^{-1}y = uI$

in $A\Gamma_0 \otimes \mathbb{C} = M_2(\mathbb{C})$, $t^{-1}y$ commutes with every $a \in M_2(\mathbb{C})$

$$u^2 = \det(t^{-1}y) \underset{\substack{\uparrow \\ t \in SL(2,\mathbb{C})}}{=} \det(y)$$

noting that $y^2 - tr(y)y + n(y) = y^2 - tr(y)y + (\det y)I = 0$. and $y^2 - tr(y)y \in A\Gamma_0$, we get $(\det y)I \in A\Gamma_0$ so that $\underset{\substack{\| \\ u^2}}{\det y} \in k\Gamma_0$

from $t^{-1}y = uI$, $t^2 = y^2u^{-2}I \in A\Gamma_0^1$

Hence $\langle t^2, \Gamma_0^{(2)}\rangle \subset A\Gamma_0^1$ and thus $tr\Gamma_1 \subset k\Gamma_0$

2) Amalgamated free product case

$\Gamma = \langle \Gamma_0, \Gamma_1\rangle$ and $\Gamma_0 \cap \Gamma_1 = H$

claim: $\mathbb{Q}(tr\langle \Gamma_0^{(2)}, \Gamma_1^{(2)}\rangle) = k\Gamma_0 \cdot k\Gamma_1$

(pf) $A\Gamma_0 \otimes_{k\Gamma_0} k\Gamma_0 \cdot k\Gamma_1 \simeq A\Gamma_1 \otimes_{k\Gamma_1} k\Gamma_0 \cdot k\Gamma_1 := A$

because $AH \otimes_{kH} k\Gamma_0 \cdot k\Gamma_1 \simeq (AH \otimes_{kH} k\Gamma_i) \otimes_{k\Gamma_i} k\Gamma_0 \cdot k\Gamma_1$

$\simeq A\Gamma_i \otimes_{k\Gamma_i} k\Gamma_0 \cdot k\Gamma_1 \quad i = 0, 1$

so $\Gamma_0^{(2)}, \Gamma_1^{(2)}$ are subgroups of A^1 and thus

$\mathbb{Q}(tr\langle \Gamma_0^{(2)}, \Gamma_1^{(2)}\rangle) \subset k\Gamma_0 \cdot k\Gamma_1$ □

Coro. 5.6.2 Mutation preserves the invariant trace field.

Thm 5.6.4 $K = \mathbb{Q}(\sqrt{-d_1}, \cdots, \sqrt{-d_r})$ $d_1, \cdots, d_r$ are square free positive integers then K is the invariant trace field of a finite volume hyperbolic 3-manifold

We need the following

Thm (C. Adams 1985)

let S be an incompressible thrice punctured sphere properly embedded in a finite volume hyper. 3-mfd M. then S can be isotoped to be totally geodesic in M.

lemma 5.6.5

$M = \mathbb{H}^3/\Gamma$ $f: D \hookrightarrow M$ an incompressible twice punctured disk then $f_*(\pi_1 D) \subset \Gamma$ is conjugate in $PSL(2,\mathbb{C})$ to the level 2 congruence subgroup of $PSL(2,\mathbb{Z})$

$:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \,\middle|\, c \equiv 0 \mod 2 \right\}$

pf) $\pi_1 D = \langle a, b\rangle$ a, b are parabolic in M, ab is also parabolic.

We can conjugate so that $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$.

$ab = \begin{pmatrix} 1+2r & 2 \\ r & 1 \end{pmatrix}$ is parabolic so

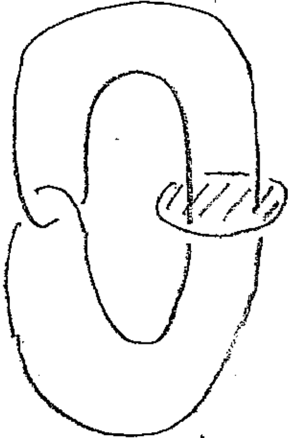
$2+2r = \pm 2$ if it is $+2$, $r=0$ so $ab = a$ $\times$

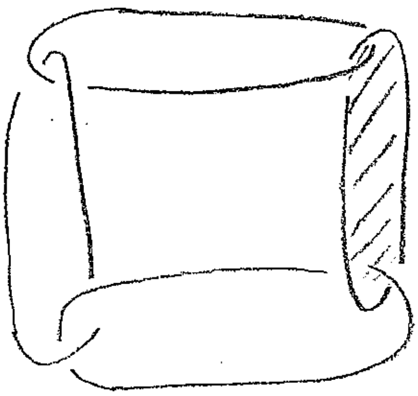
Thus $r=-2$ $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ generates the level 2 Congruence gp

lemma 5.6.6.

$\forall d$, $PSL(2, O_d)$ contains a torsion free subgp G_d s.t.
$\mathbb{H}^3/G_d$ contains an embedded totally geodesic twice punctured disc.

pf) for $d=1, 3$.





Whitehead link L (see §4.5)

$S^3 \setminus L$ has $k\Gamma = \mathbb{Q}(i)$

$A\Gamma = M_2(\mathbb{Q}(i))$

4-chain link L'

$S^3 \setminus L'$ is hyperbolic again and

$k\Gamma = \mathbb{Q}(\sqrt{3}\,i)$

$A\Gamma = M_2(\mathbb{Q}(\sqrt{3}\,i))$

for $d \neq 1, 3$
We need the result of Fine, Frohman (1986)

Now we can glue various $\mathbb{H}^3/G_d$'s along totally geodesic 3-punctured sphere so that
we get $\mathbb{Q}(\sqrt{-d_1}) \cdot \mathbb{Q}(\sqrt{-d_2}) \cdots \mathbb{Q}(\sqrt{-d_r}) = \mathbb{Q}(\sqrt{-d_1}, \cdots \sqrt{-d_r})$
as the invariant trace field of the glued hyper. manifold.
by Thm 5.6.1 $\square$

Thm 8.2.2 says any number field with exactly one complex place can arise as the invariant trace field of a Kleinian group