Thm 5.3.1

If Γ is a finite covolume Kleinian group with

a) $k\Gamma$ contains no proper subfield other than $\mathbb{Q}$

b) $A\Gamma$ is ramified at at least one infinite place of $k\Gamma$

Then Γ contains no hyperbolic elements

Consequently, Γ contains no nonelementary Fuchsian subgps

Thm 5.3.8

Let Γ be a noncpt Kleinian gp with finite covolume and

a) $k = \mathbb{Q}(\mathrm{tr}\,\Gamma)$ is of odd degree over $\mathbb{Q}$ and contains no proper subfield other than $\mathbb{Q}$

b) Γ has integral traces

Then Γ contains no cocpt Fuchsian subgps.

Due to the condition b) in 5.3.1, $A\Gamma$ in 5.3.1 has to be a division algebra

Thus Γ is in fact cocpt by Thm 3.3.8

5.3.8 is about the noncpt Γ's Γ_d

Recall that the Bianchi groups $PSL(2, O_d)$ have $k\Gamma_d = \mathbb{Q}(\sqrt{-d})$

and $A\Gamma_d = M_2(\mathbb{Q}(\sqrt{-d}))$

In §9.6, It will be proved that all Bianchi groups contain cocompact Fuchsian subgps

proof of 5.3.8)

Suppose Γ contains a cocpt Fuchsian gp F

F also has integral traces by condition b) and by condition a)

$\mathrm{tr}\,F \subset \mathbb{Z}$ ($\because$ $\mathbb{Q}(\mathrm{tr}F) \subsetneq \mathbb{Q}(\mathrm{tr}\Gamma)$ so $\mathbb{Q}(\mathrm{tr}F) = \mathbb{Q}$)

Consider AF and $\mathcal{O}F := \{ \sum a_i f_i \mid a_i \in \mathbb{Z},\ f_i \in F \}$ which is an order of AF

claim : $AF \simeq M_2(\mathbb{Q})$

Assuming the claim, using Skolem Noether thm, we can conjugate in $GL(2, \mathbb{C})$ so that $AF = M_2(\mathbb{Q})$

Since all maximal orders in $M_2(\mathbb{Q})$ are conju. to $M_2(\mathbb{Z})$ by 2.2.10, We can further conjugate so that $\mathcal{O}F \subset M_2(\mathbb{Z})$

Then F has to be a subgp of $SL(2, \mathbb{Z})$.

Since $\mathbb{H}^2/PSL(2,\mathbb{Z})$ has a cusp, F cannot be cocpt $\Rightarrow\Leftarrow$

Let's prove $A\mathcal{F} \simeq M_2(\mathbb{Q})$

If not, $A\mathcal{F}$ has to be a division algebra over $\mathbb{Q}$

Note that there is only one infinite place for $\mathbb{Q}$ represented by the usual absolute value $v(x) = |x|$

Thm 2.7.3 says the number of places v of k such that A is ramified at v is an even number and

by Thm 2.7.5 $\mathrm{Ram}(A\mathcal{F}) \neq \mathrm{Ram}(M_2(\mathbb{Q})) = \phi$.

Thus $A\mathcal{F}$ is ramified at at least one finite place associated with a prime $p \in \mathbb{Z}$

Note that $A\mathcal{F} \otimes_{\mathbb{Q}} k = A\Gamma$ $\left(\because A\Gamma = \left(\frac{tr^2 g - 4,\ tr[g,h]-2}{k\Gamma}\right) \forall g,h \text{ s.t. } \langle g,h\rangle \text{ is irreducible}\right)$

and $A\Gamma \simeq M_2(k)$ by Thm 3.3.8

let $\mathcal{P}_1, \cdots \mathcal{P}_g$ be the k-prime divisors of p then

$(A\mathcal{F} \otimes_{\mathbb{Q}} k)_{\mathcal{P}_i} \simeq M(2, k_{\mathcal{P}_i})$ $\forall i = 1, \cdots, g$

but $A\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra over $\mathbb{Q}_p$

$(A\mathcal{F} \otimes_{\mathbb{Q}} k) \otimes_k k_{\mathcal{P}_i} \simeq (A\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} k_{\mathcal{P}_i}$ says

$A\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is split by $k_{\mathcal{P}_i}$'s

note that $[k : \mathbb{Q}] = \sum_{i=1}^{g} e_i [k_{\mathcal{P}_i} : \mathbb{Q}_p]$ says at least one of $[k_{\mathcal{P}_i} : \mathbb{Q}_p]$ is odd because $[k : \mathbb{Q}]$ is odd by condition a)

but then $A\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ cannot split by the $k_{\mathcal{P}_i}$ s.t. $[k_{\mathcal{P}_i} : \mathbb{Q}_p]$ is odd

($\because$) Exercise 2.3 3) says
If $[K:F]$ odd, then for $a,b \in F^*$, $\left(\frac{a,b}{F}\right)$ splits iff $\left(\frac{a,b}{K}\right)$ splits

example.

in §4.5, two bridge knot of the form $(p/_{p-2})$ $p = 4m+3$

has $\{e_1, e_2, \cdots e_{4m+2}\} = \{1, -1, 1, -1, \cdots -1, \underset{2m+1}{1}, \underset{2m+2}{1}, -1, 1, -1, \cdots, -1, 1\}$.

$S^3 \setminus (p/_{p-2}) = \mathbb{H}^3/\Gamma$, $k\Gamma = \mathbb{Q}(z)$.

the minimal polynomials for z for $m = 1, 2$ are

$1 + 2z - 3z^2 + z^3$, $1 + 3z - 13z^2 + 16z^3 - 7z^4 + z^5$

Since these are irreducible, there is no proper subfield between $\mathbb{Q}$ and $\mathbb{Q}(z)$ and $[\mathbb{Q}(z) : \mathbb{Q}] = 2m+1$

A : a quaternion algebra over F $\quad A_0$: pure quaternions

$B(x,y) := \frac{1}{2}\left[n(x+y)-n(x)-n(y)\right] = \frac{1}{2}\left[x\bar{y}+y\bar{x}\right]$

$O(A_0,n) = \{T : A_0 \to A_0 \mid T \text{ is linear, } n(Tx)=n(x)\ \forall x\in A_0\}$

$C : A^* \longrightarrow O(A_0,n)$, $\quad C(\alpha)(x) = \alpha x \alpha^{-1}$ $\quad \alpha\in A^*,\ x\in A_0$

induces an isomorphism $A^*/Z(A^*) \xrightarrow{\sim} SO(A_0,n)$

In particular when $A=\mathcal{H}^1$, we get $Z(A^*)=\{\pm1\}$ and an iso.

$SU(2)/\{\pm1\} \simeq SO(3,\mathbb{R})$ $\quad \left(\because\ \mathcal{H}^1 \simeq SU(2) \text{ by } \sigma : a+bi+cj+dk \longmapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}\right)$

A Hurwitz quaternion is defined as

$H := \{a+bi+cj+dk \in \mathcal{H} \mid a,b,c,d\in\mathbb{Z}, \text{ or } a,b,c,d\in\mathbb{Z}+\frac{1}{2}\}$

The group of units of H is known as BA_4, the binary tetrahedral group $\{\pm1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm1 \pm i \pm j \pm k)\}$ whose order is 24

BA_4 can be presented as

$\langle s,t \mid (st)^2 = s^3 = t^3 \rangle$, $\quad s=\frac{1}{2}(1+i+j+k)$, $t=\frac{1}{2}(1+i-j+k)$ $\quad (st=i)$

A_4, the orientation preserving index 2 subgp of symmetric gp S_4, can be regarded to act on 4 vertices in A_0 $\leftarrow$ 3-dim'l real vector space

$i+j+ij$, $\quad i-j-ij$, $\quad -i+j-ij$, $\quad -i-j+ij$

$\Big($ for example, $C(st)=C(i)$: $i\pm j\pm ij \longmapsto i(i\pm j\pm ij)(-i) = i\mp j\mp k$

$\quad C(s) = C(\frac{1}{2}(1+i+j+k))$ fixes $i+j+ij$, and rotates the other 3 vertices $\Big)$

Note $\underset{\subset \mathcal{H}^1}{BA_4}$ is the inverse image of $\underset{\subset SO(3,\mathbb{R})}{A_4}$ under the iso. $\mathcal{H}^1/\{\pm1\} \simeq SO(3,$

Some facts about finite Möbius gps

Thm (Klein)

There exist only finitely many orie. preserving isometries of $\mathbb{R}^3$ i.e.

Cyclic gp C_d, Dihedral gp D_d, tetrahedral gp A_4 (or T)

Octahedral gp S_4, icosahedral group A_5 (or I)

Thm. (Cayley 1879)

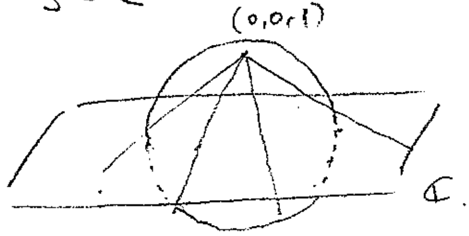
Let $h : S^2 \longrightarrow \hat{\mathbb{C}}$ be the stereographic projection defined by

$h(x_0,x_1,x_2) = \dfrac{x_0+ix_1}{1-x_2}$ and

let $R_{\theta,x}$ be the rotation with axis $\mathbb{R}\cdot x$ and angle θ

where $x=(x_0,x_1,x_2)\in S^2$

Then $R_{\theta,x}$ corresponds to $\zeta \longmapsto \dfrac{z\zeta - \bar{w}}{w\zeta + \bar{z}}$ $\quad \zeta \in \mathbb{C}$



where $z = \cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right) x_2$

$w = \sin\left(\frac{\theta}{2}\right)\left(x_1 + i x_0\right)$

Consider 4 points on S^2

$\frac{1}{\sqrt{3}}(1,1,1)$, $\frac{1}{\sqrt{3}}(1,-1,-1)$, $\frac{1}{\sqrt{3}}(-1,1,-1)$, $\frac{1}{\sqrt{3}}(-1,-1,1)$

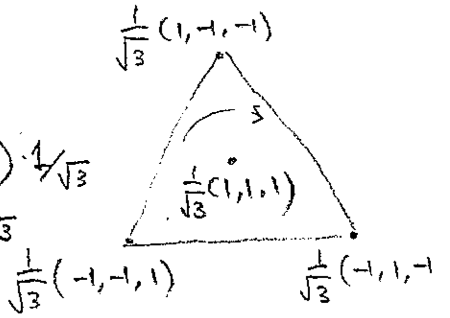
by the thm. of Cayley,

$R_{\pi,(1,0,0)} \Rightarrow \zeta \longmapsto 1/\zeta$

$R_{\pi,(0,1,0)} \Rightarrow \zeta \longmapsto -1/\zeta$

$R_{\pi,(0,0,1)} \Rightarrow \zeta \longmapsto -\zeta$

so we have D_2 of order 4 $\quad \zeta \longmapsto \pm\zeta, \pm\frac{1}{\zeta}$



$R_{\frac{2}{3}\pi, \frac{1}{\sqrt{3}}(1,1,1)} \Rightarrow \zeta \longmapsto \dfrac{z\zeta - \bar{w}}{w\zeta + \bar{z}}$ $\quad z = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\cdot 1/\sqrt{3}$, $\quad w = \sin\left(\frac{\pi}{3}\right)(1+i)/\sqrt{3}$

$$= \frac{\frac{1}{2}(1+i)\zeta - \frac{1}{2}(1-i)}{\frac{1}{2}(1+i)\zeta + \frac{1}{2}(1-i)} = \frac{\zeta + i}{\zeta - i}$$

similarly, for $\theta = \frac{2}{3}\pi, \frac{4}{3}\pi$, $x =$ vertices of tetrahedron, we get 8 Möbius transformations $\zeta \longmapsto \pm i\dfrac{\zeta+1}{\zeta-1}, \pm i\dfrac{\zeta-1}{\zeta+1}, \pm\dfrac{\zeta+i}{\zeta-i}, \pm\dfrac{\zeta-i}{\zeta+i}$

so we have got A_4 whose order is 12.

If we identify S^3 with $\mathbb{H}^1$ by $z + jw \mapsto (z,w) \in \mathbb{C}^2$ with $|z|^2 + |w|^2 = 1$

and take the inverse image BA_4 of A_4

under the projection $p: SU(2) \longrightarrow SU(2)/\{\pm I\} \subset PSL(2,\mathbb{C}) = \text{Möb}(\hat{\mathbb{C}})$

BA_4 corresponds to $\pm(z+jw)$'s where z, w comes from the Cayley's Thm.

Thm. (Gábor, Finite Möbius groups ...)

Any finite subgp of $SU(2)$ is either cyclic or conjugate to one of D_d^*, T^*, O^*, I^*

where $*$ means the inverse image under p

lemma 5.4.1

let Γ be a kleinian group of finite covolume with $A\Gamma$ over a number field k and contains a subgp $\underset{iso.}{\cong} A_4$ then $A \cong \left(\frac{-1,-1}{k}\right)$

pf) Note that since A_4 is generated by s, t and $s^3 = t^3 = -1$, $A_4 = A_4^{(2)} \subset \Gamma^{(2)}$

by the above thm, we can conjugate Γ so that $\sigma(BA_4) \subset \rho^{-1}(\Gamma^{(2)}) \subset SL(2,\mathbb{C})$

Let $A_0 := \{ \sum a_i g_i : a_i \in \mathbb{Q},\ g_i \in \sigma(BA_4) \}$ then

Since $1, i, j, ij \in BA_4$, $A_0 = \left(\frac{-1,-1}{\mathbb{Q}}\right)$

$\{\sum a_i g_i : a_i \in k,\ g_i \in \sigma(BA_4)\}$ is also a quaternion algebra isomorphic to $A_0 \otimes_{\mathbb{Q}} k$ so $A \cong \left(\frac{-1,-1}{k}\right)$ ☒

by Thm 2.6.6.

A splits over all nondyadic p-adic fields i.e.

$A \otimes_k k_p \cong M_2(k_p)$

lemma 5.4.2

let Γ be a finite covolume kleinian group and contains a subgp $\cong A_5$. If $[k\Gamma : \mathbb{Q}] = 4$ then $A\Gamma$ has no finite ramification.

pf) by 5.4.1, $A\Gamma$ is ramified at all real places of which there are either 0 or 2 $\leftarrow$ $r_1 + 2r_2 = 4$ if $r_2 = 0$, $\Gamma \overset{\text{conj}}{\hookrightarrow} SL(2,\mathbb{R})$

Since Γ contains A_5, $\mathbb{Q}(\sqrt{5}) \subset k\Gamma$ $\leftarrow$ $\cos 72° = \frac{-1+\sqrt{5}}{4}$

by lemma 0.3.10 $2\mathcal{O}_5 = \mathfrak{p}$ for some prime $\mathfrak{p}$ in $\mathbb{Q}(\sqrt{5})$

i.e. 2 is inert w.r.t. $\mathbb{Q}(\sqrt{5}) | \mathbb{Q}$

Now w.r.t. $k\Gamma | \mathbb{Q}(\sqrt{5})$, if $\mathfrak{p}$ ramifies or inert, (i.e. $\mathfrak{p} R_{k\Gamma} = \mathfrak{P}$ or $\mathfrak{P}^2$) then $\exists$ only one dyadic prime $\mathfrak{P}$

but at $\mathfrak{P}$, $A\Gamma$ cannot be ramified due to Hilbert's reciprocity.

Thus $\mathfrak{p} = \mathfrak{P}_1 \mathfrak{P}_2$ so that $k_{\mathfrak{P}_1} \cong k_{\mathfrak{P}_2} \cong \mathbb{Q}(\sqrt{5})_{\mathfrak{p}}$ $\leftarrow$??

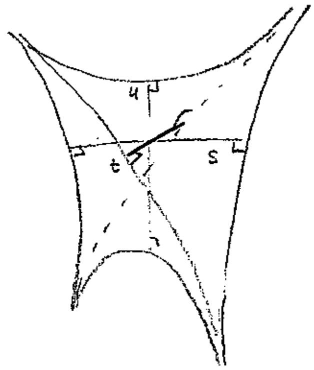
$\left(\frac{-1,-1}{\mathbb{Q}(\sqrt{5})}\right)$ splits in the field $\mathbb{Q}(\sqrt{5})_{\mathfrak{p}}$ again by Hilbert's reciprocity

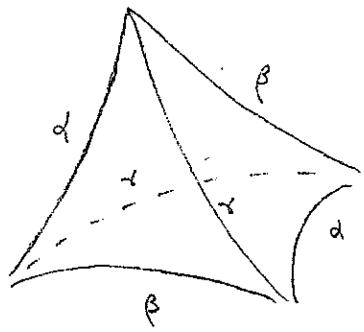
Hence, $\left(\frac{-1,-1}{k\Gamma}\right)$ splits in $k_{\mathfrak{P}_1}, k_{\mathfrak{P}_2}$ and $A\Gamma$ has no finite ramification.

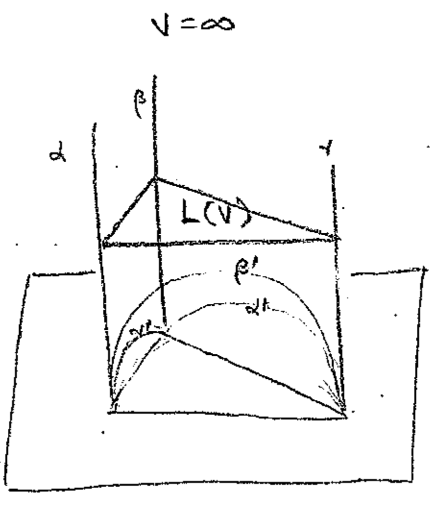
- tetrahedral parameter

Every cusped hyperbolic 3-manifold with finite volume can be triangulated by ideal tetrahedra (Thurston)

For closed hyperbolic manifolds?

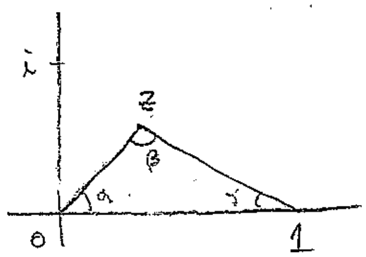






$L(v)$ = the link of an ideal vertex v

$$\begin{aligned}\pi &= \alpha'+\beta'+\gamma \\ &= \alpha+\beta'+\gamma' \\ &= \alpha+\beta+\gamma \\ &= \alpha'+\beta+\gamma'\end{aligned} \Rightarrow \begin{aligned}\alpha'&=\alpha \\ \beta'&=\beta \\ \gamma'&=\gamma\end{aligned}$$



We can assume that 3 ideal vertices are $0, 1, \infty$

$\alpha \leftrightarrow z \qquad \beta \leftrightarrow \dfrac{z-1}{z} \qquad \gamma \leftrightarrow \dfrac{1}{1-z}$

let $M = \mathbb{H}^3/\Gamma$ be a cusped finite volume hyperbolic 3-mfd with ideal triangulation $M = S_1 \cup S_2 \cup \cdots \cup S_n$ and let z_i (or $\frac{z_i-1}{z_i}$, $\frac{1}{1-z_i}$) be the tetrahedral parameter of S_i

Denote $\mathbb{Q}(z_i : i=1,\cdots n)$ by $k_\Delta M$

Thm 5.5.1 $k_\Delta M = k\Gamma$. so $k_\Delta M$ is indep. of the choice of triangulatio[n]

pf) Let V be the vertices of all lifted S_i's in $\partial_\infty \mathbb{H}^3$ and let k_1 be the field generated by all cross ratios of 4-tuples of V

We can choose an isometry so that 3 vertices of $\tilde{S}_1$ are $0, 1, \infty$

let k_2 be the field generated by remaining points of V

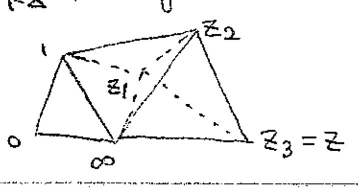
claim : $k_1 = k_2 = k_\Delta M$

$(\because)$ $k_2 \subseteq k_1$: cross ratio of $0, 1, \infty, z$ is just z

$k_1 \subseteq k_2$: clear

$k_\Delta M \subseteq k_1$: $z_i, \frac{z_i-1}{z_i}, \frac{1}{1-z_i}$ are cross ratios of $0, 1, \infty, z_i$

$k_2 \subseteq k_\Delta M$: given $z \in V \setminus \{0,1,\infty\}$ consider a sequence of $\tilde{S}_i$'s



$z_1 \in k_\Delta M \Rightarrow z_2 \in k_\Delta M \Rightarrow \cdots \Rightarrow z \in k_\Delta M$

$(\infty), 1, z_1 \in k_\Delta M$

tetrahedral parameter = cross ratio of $\infty, 1, z_1, z_2$

Given $\gamma \in \Gamma$, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma(0) = w_1$, $\gamma(1) = w_2$, $\gamma(\infty) = w_3$

i.e. $\left.\begin{array}{l} b - dw_1 = 0 \\ a + b - cw_2 - dw_2 = 0 \\ a - cw_3 = 0 \end{array}\right\}$ We can solve for a, b, c, d then
$a = \varepsilon A(w_1, w_2, w_3), \cdots \quad d = \varepsilon D(w_1, w_2, w_3)$
A, B, C, D are linear equations

To discard ε, $\gamma^2 = \dfrac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \in PSL(2, k_\Delta M)$

Thus, $k\Gamma \subseteq k_\Delta M$.

from 4.2.3, Any non cocpt Γ can be conjugated to lie in $PSL(2, \mathbb{Q}(tr\Gamma))$

Noting that V is the set of fixed pts of parabolic elements of Γ,

We get $V \subset \mathbb{Q}(tr\Gamma)$ $\left(\because \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = z, \; z = \dfrac{(a-d) \pm \sqrt{(a-d)^2 - 4bc}}{2c} \underset{\uparrow}{=} \dfrac{a-d}{2c} \right)$

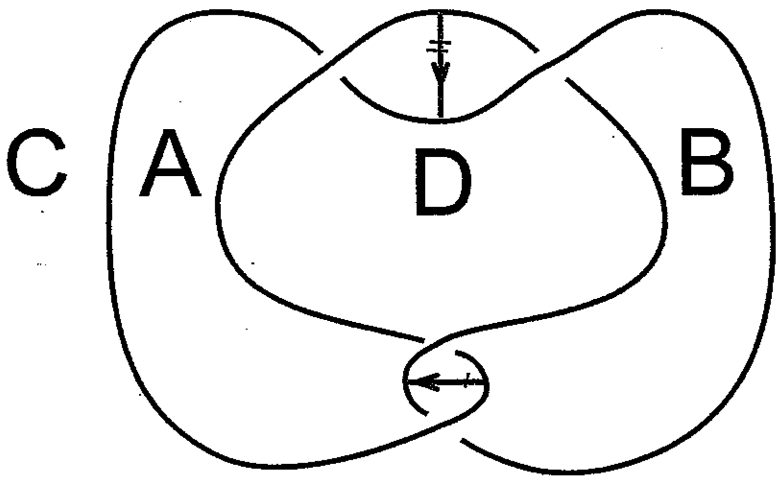
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is parabolic

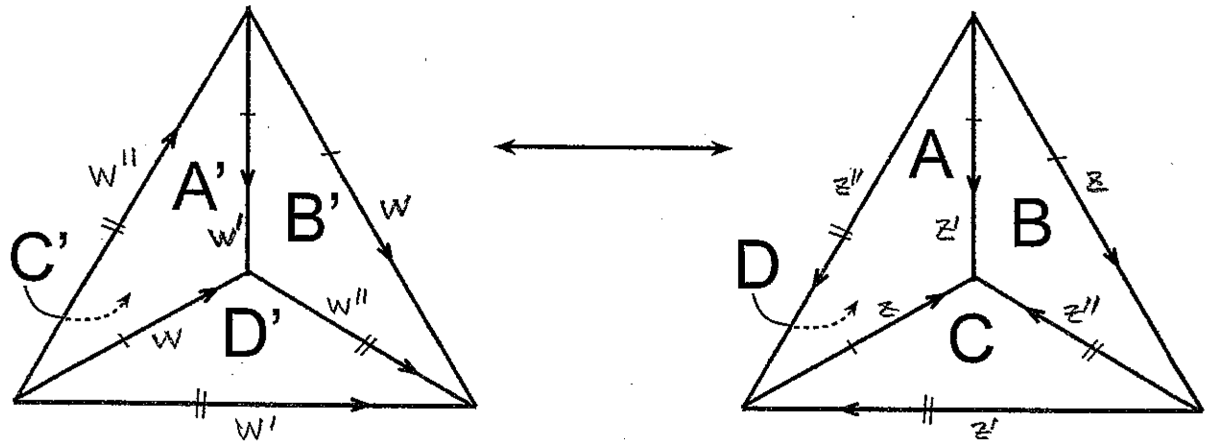
so $k_\Delta M \subset \mathbb{Q}(tr\Gamma)$

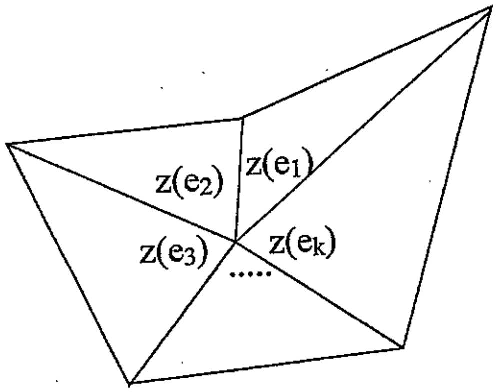
Since $k_\Delta M$ is commensurability invariant of Γ,

We get $k_\Delta M \subset k\Gamma$ $\square$

example)







$z(e_1)z(e_2)\cdots z(e_k)=1$

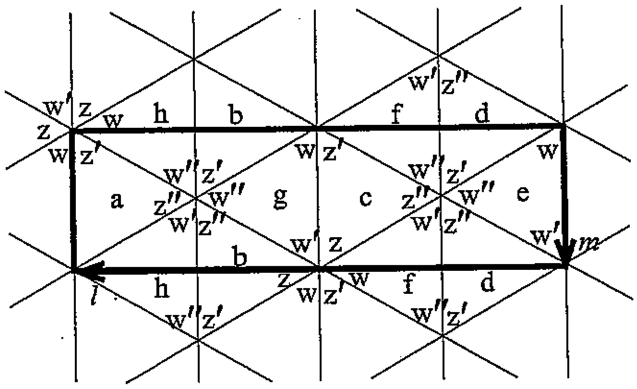
$\Leftrightarrow \ \arg z(e_1)+\cdots+\arg z(e_k)=2\pi$

$\Rightarrow \ z^2 z' w^2 w' = 1$ and

$z'(z'')^2 w'(w'')^2 = 1$

$\Leftrightarrow \ zw(1-z)(1-w) = 1$

$z^{-1}w^{-1}(1-z)^{-1}(1-w)^{-1} = 1$



for completeness

$h(l) = z^2(1-z)^2 = 1$

$h(m) = w(1-z) = 1$

These give a unique solution with $\operatorname{Im} z > 0$ $\operatorname{Im} w > 0$

i.e. $z = w = e^{\pi i/3}$

Thus $k\Gamma = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(z)$