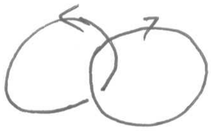
4:30PM

AROUND BENNEQUIN'S INEQUALITY

(ALEXANDER STOIMENOW)

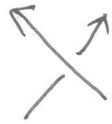
1. Bennequin's inequality:

Knot $S^1 \hookrightarrow S^3$ 

links $S^1 \cup .. \cup S^1 \hookrightarrow S^3$ 

$c(D)$ # crossing 


(+)


(−)

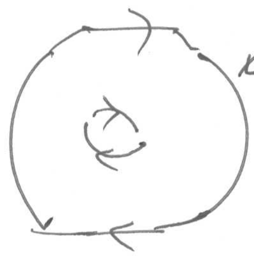
+ + − − = 0

$w(\ ) = 0$

$w(D)$ = sum of signs of crossing

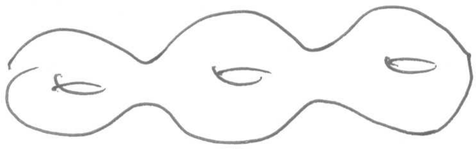
$s(D)$ = # of Seifert circles

E.g.



$s(\downarrow) = 3$ $\quad S($

<u>Genus</u>: (Seifert) $K \subset S^3 \quad \exists S$

 $\hookrightarrow S^3$

$\partial S = K$ Seifert Surface.

$g(S) = \text{genus}(S)$

$g(K) = \min\{ g(S) : S \text{ Seifert surface for } K \}$

genus of K

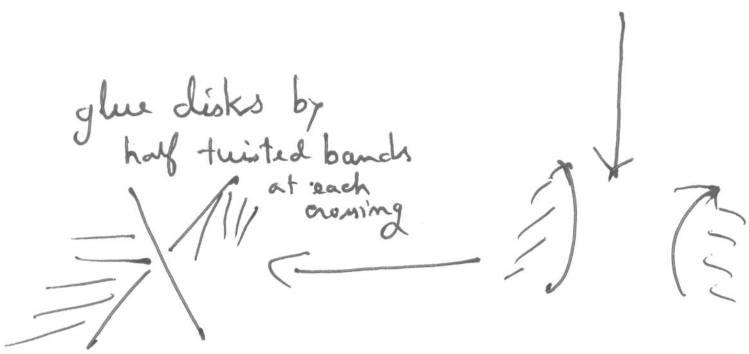
$g(K) = 0 \Leftrightarrow$ 

 K trivial

Seifert algorithm

diagram D of K  $\xrightarrow{\text{Smooth out}}$ 

glue disks by half twisted bands at each crossing



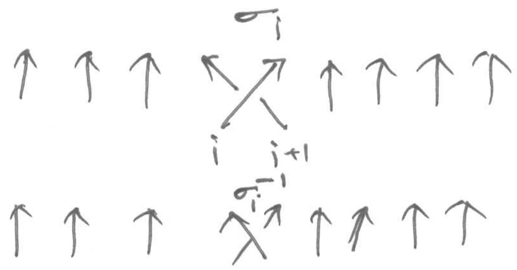
Canonical Seifert surface of D

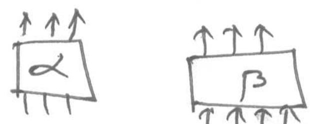
Braid representation:

$$B_n = \left\langle \sigma_1, \sigma_2 \ldots \sigma_{n-1} \;\middle|\; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \right\rangle$$
$$1 \le i \le n-1$$

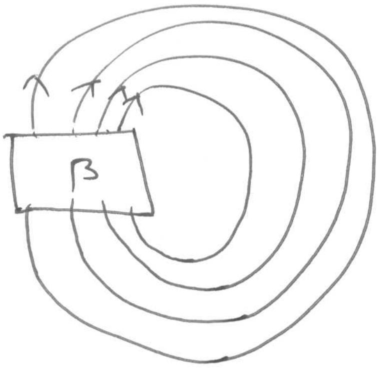
↑

Braid group.





closure

$\hat{\beta} =$  $=$ Link in S^3

L

β is a braid representation for $L = \hat{\beta}$.

$$B_n / [B_n, B_n] = \mathbb{Z} \quad \text{abelianization}$$

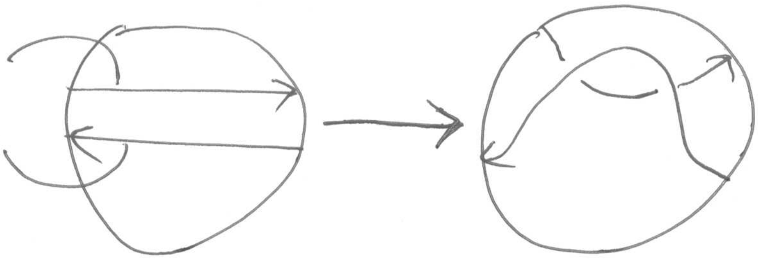
$$B_n \to \mathbb{Z}$$
$$\sigma_i \to 1 \qquad \text{exponent sum}$$

<u>Theorem</u>: (Bennequin, '83): $K = \hat{\beta}$ then $\beta \in B_n$
$n = n(\beta) = \#$ strands of β.

$$2\, g(K) \geq |T\beta| - n(\beta) + 1.$$

extends Braid representation → diagrams.

Yamada & Vogel: Thm: One can turn every link diagram into a closed braid diagram by



preserves # Seifert circles
preserves the writhe.

Cor: Assume D is a diagram of K

$$2g(K) \geqslant |w(D)| - g(D) + 1$$

2. Link polynomials $\{\text{links in } S^3\} \to \mathbb{Z}[t^{\frac12}, t^{-\frac12}]$

$\mathbb{Z}[t^{\pm1}, t^{\pm1}]$

$$\Delta(L_+) - \Delta(L_-) = (t^{\frac12} - t^{-\frac12})\,\Delta(L_0)$$

Alexander polynomials.

$$t^{-1}V(L_+) - tV(L_-) = (t^{\frac12} - t^{-\frac12})\,V(L_0)$$ Jones polynomial

$$\Delta(0) = V(0) = 1.$$

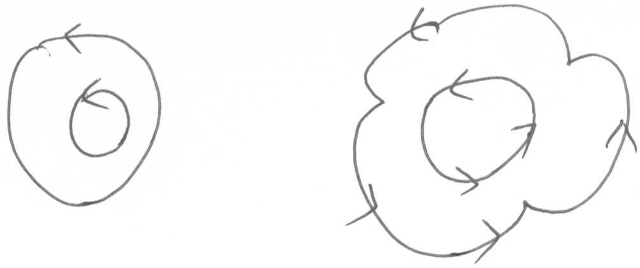
$$l^{-1}P(L_+) - lP(L_-) = -mP(L_0)$$

$\Gamma(K) = \mathbb{Z}[l^{\frac12}, l^{-\frac12}]$ HOMFLY(-PT)) Shem polynomial.

Theorem: $g(K) \geqslant \frac12 \operatorname{span} \Delta(K)$.

Ex:  trefoil link K. $\Delta(K) = t - 1 + t^{-1}$.

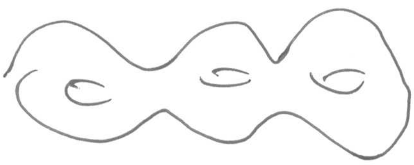
$\operatorname{span}(\Delta) = 2 \Rightarrow g(K) = 1$.



$e(D) = 3$

$s(D) = 2$

$$\text{genus}(\text{canonical Surface}) = \frac{c(D) - s(D) + 1}{2}$$



$1 - 2g = s - c.$

g (canonical Surface of

) $= \frac{3-2+1}{2} = 1$

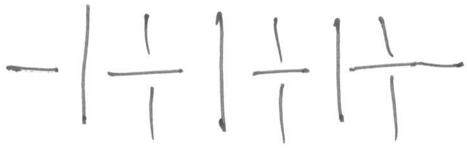
$\Rightarrow g$ (

) $\leq 1.$

<u>Theorem</u>: (Crowell)

If D is alternating diagram then $2g(S) = 2g(K) = \text{Span}\,\Delta(D)$.

Canonical Surface



3. Signature

$$\sigma : \{\text{Knots}\} \to 2\mathbb{Z}.$$

$$\sigma(O) = 0$$

$\downarrow$

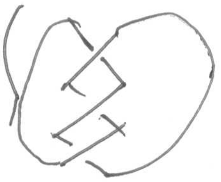
circle in plane

$\sigma($  $) - \sigma($  $) \in \{0, 2\}$

$$\sigma(K) \equiv 0(4) \iff \det(K) = |\underbrace{\Delta(K)}_{\mathbb{Z}[t,t^{-1}]}(-1)| \equiv 1(4)$$

$\downarrow$

$0 \pmod 4$. $\equiv 3 \pmod 4$

<u>Ex:</u> $\sigma($  $)^{\in\{0,2\}} - \sigma($  $)^{=0} \in \{0,2\}$

$\det(\text{trifoil}) = 3$, $\det(O) = |\Delta(O)(-1)| = 1$, $\Delta(O) = 1$

$$\Delta(\text{trifoil}) = t - 1 + t^{-1}$$

$t = -1$, $\Delta(-t) = 3$.

$$\sigma \equiv 2 \pmod 4.$$

$\Rightarrow \sigma \geq 2$. $g \geq 1$.

<u>Theorem:</u> $g_4(K) \geq \frac{1}{2}|\sigma(K)|$.