

Quasiconformal nonstability for isometry groups
in hyperbolic 4-space
(Youngju Kim) ①

- parabolic isometry
- Deformation spaces.

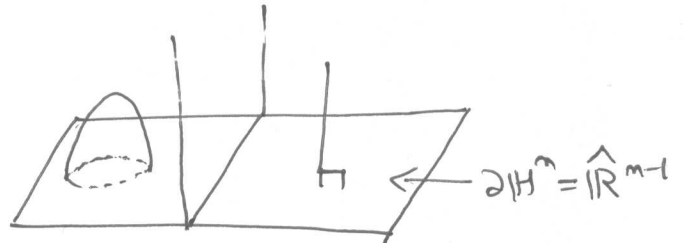
Introduction:

\mathbb{H}^m : hyperbolic m -space

$$= \{x \in \mathbb{R}^m \mid x_m > 0\}$$

the upper half space model

$$ds = \frac{dx}{x_m}$$



$$\partial\mathbb{H}^m = \hat{\mathbb{R}}^{m-1}$$

$$\text{Isom}(\mathbb{H}^m) = \text{Mob}(\hat{\mathbb{R}}^{m-1})$$

[all mappings are
or-preserving]

"
composition of reflections in hyperplanes
or S^{m-2}

< Classification of isometries >

elliptic if \exists a fixed $p \in \mathbb{H}^m$.

parabolic if no fixed $p \in \mathbb{H}^m$ and

loxodrom (hyperbolic) otherwise.

Parabolic on $\hat{\mathbb{R}}^m$, \int fixed ∞

$$x \rightarrow Ax + e_1, A \in O(m), Ae_1 = e_1$$

$n = 3$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

$A = I$: strictly parabolic (translation)

$A \neq I$: Screw parabolic $\left\{ \begin{array}{l} \text{rational if } f^m = \text{translations} \\ \text{irrational otherwise.} \end{array} \right.$ for some m .

Remark: ① no screw parabolic in $n < 4$.

② $\widehat{\mathbb{R}^3}(\mathbb{H}^4) \left\{ f_t = R_{2\pi t} \circ T \mid T x = x + e_1 \right.$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi t & -\sin 2\pi t \\ 0 & \sin 2\pi t & \cos 2\pi t \end{pmatrix}$$

$f_t \stackrel{\text{Möb}}{\simeq} f_s \iff t = s.$

~~\mathcal{G}_m parabolic~~

\mathcal{G}_m particular, $f_t \stackrel{\text{Möb}}{\not\sim} I.$

3. $G \leq \text{Isom}(\mathbb{H}^n)$
(discrete
torsion free)

$p \in \widehat{\mathbb{R}^{n-1}}$ parabolic fixed pt of G .

the maximal parabolic subgroup of G at p

$$G_p = \{ g \in G \mid p \in \text{fix}(g) \}.$$

If $\mathbb{H}^2, \mathbb{H}^3$, \exists a precisely invariant horoball B_p for the action G_p in G .

However, $m \geq 4$, no precisely invariant horoball
 if $G_p \ni s.p.$

But \exists "Margulis region".

4. In \mathbb{H}^4 , $\exists J, H \leq \text{Isom}(\mathbb{H}^4)$; geo finite groups.

$M^m = \mathbb{H}^m / G$: geometric finite if

\exists compact convex submanifold $\subseteq M$
 which carries all closed geodesics.

In \mathbb{H}^2 , geo finiteness $\Leftrightarrow \pi_1$: finitely generated.

But $J \cap H$ infinitely generated.

5. No irrational screw parabolic in $M = \mathbb{H}^m / G$,

$\text{Vol}(M) < \infty$.
 (\because the maximal parabolic group \ni irr. s.p.)
 can be only rank 1:

Fact: $f_+ \cong \Gamma$

consider $\mathbb{H}^4 / \hat{\mathbb{R}}^3$, i.e. $\exists \phi$ homeo: $\hat{\mathbb{R}}^3 \rightarrow \hat{\mathbb{R}}^3$ such that

$$\phi f_+ \phi^{-1} = \Gamma$$

Question: $f_+ \stackrel{q\text{-comb}}{\cong} \Gamma$.

$\exists ? \phi: \hat{\mathbb{R}}^3 \rightarrow \hat{\mathbb{R}}^3$ q-comb such that

$$\phi f_+ \phi^{-1} = \Gamma$$

Answer: "No"

Lemma: $f_t \neq \Gamma$ on $\widehat{\mathbb{R}}^3$

Lemma: $f_t \neq f_s$ if t : irrational

Lemma: $t, s \in \mathbb{Q}$

If $f_t \approx f_s$, then $o(f_t) = o(f_s)$

ex) $f_{\frac{\pi}{3}} \neq f_{\frac{2\pi}{3}} \approx_{p.c.} f_{\frac{4\pi}{3}}$
??

Deformation space:

$\Gamma \leq_{\text{discrete}} \text{Isom}(\mathbb{H}^m)$ generated by $\gamma_1, \gamma_2, \dots, \gamma_m$

$D(\Gamma) = \{ \rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^m) \}$

- discrete
- faithful
- type-preserving
- $\rho(g)$ -para $\Leftrightarrow g$ -parabolic.

Γ is an ϵ -def. if $|\rho(\gamma_i) - \gamma_i| < \epsilon$ ($i=1, 2, \dots, m$)

Γ is qconformally stable if $\exists \epsilon_0 > 0$ such that any ϵ -def with $\epsilon < \epsilon_0$ is qconformally conjugate to Id.

Remark: ① Mostow Rigidity theorem in $\mathbb{H}^m, m \geq 3$.
trivial def. space for a cocompact volume Γ .

