Quasiconformal nonstability for isometry groups in hyperbolic 4-space

(Youngju Kim).

- parabolic isometry
- Deformation spaces.

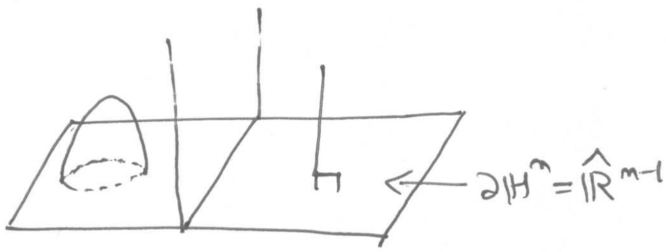
Introduction:

$\mathbb{H}^n$: hyperbolic n-space

$= \{ x \in \mathbb{R}^n \,/\, x_n > 0 \}$

the upper half space model

$ds = \dfrac{dx}{x_n}$.



$\partial\mathbb{H}^n = \hat{R}^{n-1}$

$\mathrm{Isom}(\mathbb{H}^n) = \mathrm{Mob}(\hat{\mathbb{R}}^{n-1})$ [all mapping are or-preserving]

$\parallel$

composition of reflections in hyperplanes or S^{n-2}.

⟨Classification of isometries⟩

elliptic if $\exists$ a fixed pt $\in \mathbb{H}^n$.

parabolic if no fixed pt $\in \mathbb{H}^n$ and

loxodim (hyperbolic) otherwise.

Parabolic on $\hat{\mathbb{R}}^n$, f fixed ∞

$x \to Ax + e_1$, $A \in O(n)$, $Ae_1 = e_1$

$n = 3$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$A = I$: strictly parabolic (translation)

$A \neq I$: Screw parabolic $\begin{cases} \text{rational if } f^m = \text{translations for some } m. \\ \text{irrational otherwise.} \end{cases}$

<u>Remark</u>: ① no screw parabolic in $n < 4$.

② $\hat{\mathbb{R}}^3$ $(\mathbb{H}^4)$ $\{ f_t = R_{2\pi t} \circ T \mid Tx = x + e_1$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi t & -\sin 2\pi t \\ 0 & \sin 2\pi t & \cos 2\pi t \end{pmatrix}$$

$f_t \overset{\text{Möb}}{\simeq} f_s \iff t = s$.

~~In parabolic p~~

In particular, $f_t \overset{\text{Möb}}{\not\simeq} T$.

3. $G \leq \mathrm{Isom}(\mathbb{H}^n)$
(discrete
torsion free)

$p \in \hat{\mathbb{R}}^{n-1}$ parabolic fixed pt of G.

the maximal parabolic subgroup of G at p

$$G_p = \{ g \in G \mid p \in \mathrm{fix}(g) \}.$$

If $\mathbb{H}^2, \mathbb{H}^3$, $\exists$ a precisely invariant horoball B_p for the action G_p in G.

However, $n \geq 4$, no precisely invariant horoball

if $G_p \ni$ s.p.

But $\exists$ "Margulis region".

4. In $\mathbb{H}^4$, $\exists\ J, H \leq \mathrm{Isom}(\mathbb{H}^4)$; geo finite groups.

$M^n = \mathbb{H}^n / G$: geometric finite if

$\exists$ compact convex submanifold $\subseteq M$

which carries all closed geodesics.

In $\mathbb{H}^2$, geo finiteness $\Leftrightarrow \pi_1$: finitely generated.

But $J \cap H$ infinitely generated.

5. No irrational screw parabolic in $M = \mathbb{H}^n / G$,

$\mathrm{Vol}(M) < \infty$.

($\because$ the maximal parabolic group $\ni$ irr. s.p

can be only rank 1 :

<u>Fact</u>: $f_t \overset{top}{\simeq} T$.

consider $\mathbb{H}^4 / \hat{\mathbb{R}}^3$, i.e. $\exists\ \phi$ homeo : $\hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ such that

$\phi f_t \phi^{-1} = T$.

<u>Question</u>: $f_t \overset{q\text{-conf}}{\simeq} T$.

$\exists$? $\phi : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ q. conf such that

$\phi f_t \phi^{-1} = T$.

<u>Answer</u>: "No"

Lemma: $f_t \not\simeq I$ on $\widehat{\mathbb{R}}^3$

Lemma: $f_t \not\simeq f_s$ if t : irrational

Lemma: $t, s \in \mathbb{Q}$

If $f_t \simeq f_s$, then $o(f_t) = o(f_s)$

??

ex) $f_{\frac{2\pi}{3}} \not\simeq$

$f_{\frac{2\pi}{3}} \overset{??}{\underset{q.c}{\simeq}} f_{\frac{4\pi}{3}}$.

Deformation space:

$\underset{\text{discrete}}{\Gamma \leq} \mathrm{Isom}(\mathbb{H}^n)$ generated by $\gamma_1, \gamma_2 \ldots \gamma_m$.

$D(\Gamma) = \{ \rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$

discrete

faithful

type-preserving

$\rho(g)$ - para $\Leftrightarrow$ g - parabolic.

Γ - is an ϵ-def. if $|\rho(\gamma_i) - \gamma_i| < \epsilon \quad (i = 1, 2 \ldots m)$

Γ is qconformally stable if $\exists\, \epsilon_0 > 0$ such that any ϵ-def with $\epsilon < \epsilon_0$ is qconf conjugate to Id.

Remark: ① Mostow Rigidity theorem in H^n, $n \geq 3$.

trivial defi. space for a cofinite volume Γ.

② Marden Stability (in $\mathbb{H}^2, \mathbb{H}^3$)

geo. finite Kleinian group is qconf stable.

Q: qconf. stability in $\mathbb{H}^n$, $n \geq 4$?

Theorem (08 -)

$\exists\ \Gamma \leq \mathrm{Isom}(\mathbb{H}^4)$, geo. finite but not qconf stable.

Def^n: 3-punctured sphere group is a non-elementary discrete Möb group generated by two parabolics f, g with $f \cdot g$: parabolic.

Remark: ① In $\mathbb{H}^2, \mathbb{H}^3$, $D(\Gamma) = \{pt\}$.

② In $\mathbb{H}^4$, $\exists$ infinitely many distinct "3-punct sp gps)

($\because$ Screw parabolics).

$\Gamma_0 \leq \mathrm{Isom}(\mathbb{H}^2)$ - 3-punct sp. gp.

Theorem (08 -)

$\exists$ 2-dim parameter space $P(\Gamma_0) \subseteq D(\Gamma_0)$ in $\mathrm{Isom}(\mathbb{H}^4)$

s.t. ① Each def $\underset{g.e.}{\not\cong}$ Fd

② all qconf distinct to each other except measure '0' set.

③ geo. finite.

④ all have the same marked length spectrum.

(isospectral $\not\Rightarrow$ isometry in $\mathbb{H}^4$).

⑤ The quotient manifolds have no simple closed geodesics.

Theorem (08-)

Γ_0 - qconf rigid.

($\Leftrightarrow$ Every non-trivial def of Γ_0 $\exists$ is qconf distinct from Id).

Question:

1. $2 \leq \dim D(\Gamma_0)$?

Remark: 2. Can we find the def. space ?

e.g. angle-preserving ?

3. In all dimensions convex compact (= geo finite without para)

$\Rightarrow$ q.conf stability.