Strong maximum principle for harmonic function

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In this paper, I want to talk about harmonic function, especially strong maximum principle for it. We can prove strong maximum principle with mean-value property, however we can also do with weak maximum principle without mean-value property. So, let me introduce about harmonic function first, then continue some theorems and lemmas for approaching strong maximum property. Before we start, denote $\Omega \subseteq \mathbb{R}^n$ be an open and connected set whose closure $\bar{\Omega}$ is compact.

Definition 1. A function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ is called

1. **subharmonic** if $\nabla^2 u \geq 0$ for all $u \in \Omega$ (If equality is never satisfied, it is called strictly subharmonic)
2. **superharmonic** if $\nabla^2 u \leq 0$ for all $u \in \Omega$ (If equality is never satisfied, it is called strictly superharmonic)
3. **harmonic** if $\nabla^2 u = 0$ for all $u \in \Omega$

Now, we prove the following theorems and lemma about a function u, mentioned above.

Theorem 1. *If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ is strictly subharmonic, there is no $\mathbf{x}_0 \in \Omega$ such that $u(\mathbf{x_0}) \geq u(\mathbf{x})$ for all $\mathbf{x} \in \Omega$. Similarly, if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ is strictly superharmonic, there is no $\mathbf{x}_0 \in \Omega$ such that $u(x_0) \leq u(x)$ for all $\mathbf{x} \in \Omega$.*

Proof. Suppose that $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ is subharmonic. Then, for any $\mathbf{x}_0 \in \Omega$, there exists a neighborhood $\mathcal{U} \subseteq \Omega$ of $\mathbf{x}_0$. Now, without loss of generality, we can assume $\frac{\partial^2 u}{\partial^2 x_1} > 0$ because $\nabla^2 u$ is positive at $\mathbf{x}_0$. Then in $\mathcal{U}$, there exists $\mathbf{x}' = \mathbf{x} + h\mathbf{e}_1$ such that $u(\mathbf{x}_0) < u(\mathbf{x})$ because concave upward function cannot have its maximum unless it is a boundary point of the domain, yet $\mathbf{x}_0$ is not at the boundary of $\Omega \cap \{\pi_1(\mathbf{x}_0)\} \times \mathbb{R}^{n-1}$. Therfore, $\mathbf{x}_0$ is not a maximum point for any $\mathbf{x}_0 \in \Omega$. Modifying the proof slightly would work for strictly superharmonic function. $\mathfrak{Q.E.D}$.

The purpose of this paper is proving strong maximum principle for harmonic function with weak maximum priniple. So, we prove the weak maximum principle for harmonic function.

Theorem 2. ***Weak maximum principle.*** *If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ is harmonic, then $\max_{\mathbf{x}\in\Omega} u \leq \max_{\mathbf{x}\in\partial\Omega} u$.*

Proof. Suppose not, where $\max_{\mathbf{x}\in\Omega} u = u(\mathbf{x}_0),\ \mathbf{x}_0 \in \Omega$. Now, denote $\varepsilon = \max_{\mathbf{x}\in\Omega} u - \max_{\mathbf{x}\in\partial\Omega} u$. Cause $\bar{\Omega}$ is compact, there exists L such that $\pi_1(\mathbf{x}) < L$ for all $\mathbf{x} \in \Omega$. So we can define $w(\mathbf{x}) = u(\mathbf{x}) + (\varepsilon/2)e^{\pi_1(\mathbf{x})-L}$. Then, $\nabla^2 w = \nabla^2 u + (\varepsilon/2)e^{\pi_1(\mathbf{x})-L} > 0$ and for any $\mathbf{x} \in \partial\Omega$, $w(\mathbf{x}_0) - w(\mathbf{x}) = u(\mathbf{x}_0) - u(\mathbf{x}) + (\varepsilon/2)(e^{\pi_1(\mathbf{x}_0)-L} - e^{\pi_1(\mathbf{x})-L}) > \varepsilon - (\varepsilon/2) = \varepsilon/2$. Now, we can know that a maximum of w on $\bar{\Omega}$ is not achieved in $\partial\Omega$ cause $w(\mathbf{x}_0) > w(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$. Thus, it is achieved in Ω. However, this violates **Theorem 1**. Therefore, $\max_{\mathbf{x}\in\Omega} u \leq \max_{\mathbf{x}\in\partial\Omega} u$. $\mathfrak{Q}.\mathfrak{E}.\mathfrak{D}.$

Now, we are very close to the final goal. Before we begin, I wonder if the readers have grasped the motivation of the previous proof. Why should we take such w? The key of the roof was that adding a "very thin" strictly subharmonic function on the given function so that there is not that much difference but the function becomes stirctly subharmonic.

However, the previous proof requires $\max_{\mathbf{x}\in\Omega} u > u(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$. However, this is very strong assumption; what if we want to target only a small partion of $\partial\Omega$? Then we should modify the "very thin" function much more delicately. This is the difficult one. I would proceed the proof with some lemmas.

Lemma 1. *Suppose $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ is harmonic and $\mathbf{x}_0 \in \Omega$. Now, if there exists an $\mathbf{w}_0 \in \mathbb{R}^n \setminus \bar{\Omega}$ such that $u(\mathbf{x}) < u(\mathbf{x}_0)$ for all $\mathbf{x} \in \{\mathbf{x} \in \partial\Omega \mid \|\mathbf{x} - \mathbf{w}_0\| \leq \|\mathbf{x}_0 - \mathbf{w}_0\|\}$, then $u(\mathbf{x}_0) \neq \max_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x})$.*

Proof. Suppose that $u(\mathbf{x}_0) = \max_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x}) = M$. Cause $S = \{\mathbf{x} \in \partial\Omega \mid \|\mathbf{x} - \mathbf{w}_0\| \leq \|\mathbf{x}_0 - \mathbf{w}_0\|\}$ is compact by Heine - Borel theorem, u, a continuous function, has an maximum $\max_{\mathbf{x}\in S} u(\mathbf{x}) < M$. Denote $k = M - \max_{\mathbf{x}\in S} u(\mathbf{x})$. Moreover, cause $\frac{1}{\|\mathbf{x}-\mathbf{w}_0\|^2}$ is also a continuous function on S, we can denote $\max_{\mathbf{x}\in S} \frac{1}{\|\mathbf{x}-\mathbf{w}_0\|^2} = k'$. then, define a function $w(\mathbf{x}) = \frac{\varepsilon}{\|\mathbf{x}-\mathbf{w}_0\|^2}$ on $\bar{\Omega}$, where

$$\varepsilon = \frac{k}{k' - \frac{1}{\|\mathbf{x}-\mathbf{w}_0\|^2}}.$$

Moreover, $u(\mathbf{x}_0) + w(\mathbf{x}_0) > u(\mathbf{x}) + w(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$, and $u(\mathbf{x}) + w(\mathbf{x})$ is a harmonicfunction on Ω cause $w(\mathbf{x})$ is a harmonic function on $\mathbb{R}^n \setminus \{\mathbf{w}_0\}$. It makes contradiction with the **Weak maximum principle** of harmonic functions. Therefore,

$u(\mathbf{x}_0) \neq \max\limits_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x})$. $\mathfrak{Q}.\mathfrak{E}.\mathfrak{D}.$

By previous lemma tells us we can know whether a harmonic function $u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}(\bar{\Omega})$ can't have the maximum at the given point in the region just by comparing u at the given point and specific partition of boundary of region decided by the specific point outside of the region. It is seemed to be difficult to apply. However, we can prove very critical lemma for final goal with the previous lemma.

Lemma 2. *Suppose $u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}(\bar{\Omega})$ is harmonic and $B_{3r}(\mathbf{x}_0) \subseteq \Omega$. If u does not attain its $\max\limits_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x})$ on $\bar{B}_r(\mathbf{x}_0)$, then neither does on $\bar{B}_{2r}(\mathbf{x}_0)$.*

Proof. Cause Ω is open, we can suppose u is a harmonic function on Ω and $B_{3r}(\mathbf{x}_0) \subseteq \Omega$. Also, suppose u does not attain its $\max\limits_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x})$ on $\bar{B}_r(\mathbf{x}_0)$. Denote $\mathbf{x} \in \partial B_{2r}(\mathbf{x}_0)$, then also denote $\mathbf{z} = (5/8)\mathbf{x}_0 + (3/8)\mathbf{x}$, $\mathbf{w} = (9/16)\mathbf{x}_0 + (7/16)\mathbf{x}$, and $\mathbf{y} = (-1/4)\mathbf{x}_0 + (5/4)\mathbf{x}$. Furthermore, it is trivial that there exists a unique $(n-2)-$ *sphere* $S^{n-2} = \partial B_r(\mathbf{x}_0) \cap \partial B_{\|\mathbf{x}-\mathbf{z}\|}(\mathbf{z})$.

Note that every open ball in Eucledian space is convex. It means that there exists a line segment in $\bar{B}_r(\mathbf{x}_0) \cap \bar{B}_{\|\mathbf{x}-\mathbf{z}\|}(\mathbf{z})$ whose endpoints are $\mathbf{w}$ and each point of $S^{n-2} = \partial B_r(\mathbf{x}_0) \cap \partial B_{\|\mathbf{x}-\mathbf{z}\|}(\mathbf{z})$. So, we can define $S_1 = \{\mathbf{p} \mid \mathbf{p} = t\mathbf{q} + (1-t)\mathbf{w}, t \in [0,1], q \in S^{n-2} = \partial B_r(\mathbf{x}_0) \cap \partial B_{\|\mathbf{x}-\mathbf{z}\|}(\mathbf{z})\}$. Also, there is a unique $(n-1)-$ *sphere* $S^{n-1} \subset B_{3r}(\mathbf{x}_0)$ such that $S^{n-2} \cup \{\mathbf{y}\} \subset S^{n-1}$ where $S^{n-2} = \partial B_r(\mathbf{x}_0) \cap \partial B_{\|\mathbf{x}-\mathbf{z}\|}(\mathbf{z})$. Then it can divided by two parts by $S^{n-2} = \partial B_r(\mathbf{x}_0) \cap \partial B_{\|\mathbf{x}-\mathbf{z}\|}(\mathbf{z})$. Denote S_2 be an one of those two divided parts such that $\mathbf{y} \in S_2$. Now, we can construct an open and connected set $S \subseteq B_{3r}(\mathbf{x}_0)$ uniquely as $\partial S = S_1 \cup S_2$. Then, S contains $\mathbf{x}$, but its closure doesn't contain $\mathbf{z}$. Moreover, $S_1 = \bar{B}_r(\mathbf{x}_0) \cap \partial S = \bar{B}_{\|\mathbf{x}-\mathbf{z}\|}(\mathbf{z}) \cap \partial S$.

Now, if $u(\mathbf{x}) > \max\limits_{\mathbf{k}\in S_1} u(\mathbf{k})$, then $u(\mathbf{x}) \neq \max\limits_{\mathbf{k}\in \bar{S}} u(\mathbf{k})$ by **Lemma 1.** Therefore, $u(\mathbf{x}) \neq \max\limits_{\mathbf{k}\in\bar{\Omega}} u(\mathbf{k})$. If not, $u(\mathbf{x}) \neq \max\limits_{\mathbf{k}\in\bar{\Omega}} u(\mathbf{k})$ cause $\max\limits_{\mathbf{k}\in S_1} u(\mathbf{k}) \neq \max\limits_{\mathbf{k}\in\bar{\Omega}} u(\mathbf{k})$. Cause it is satisfying for all $\mathbf{x} \in \partial B_{2r}(\mathbf{x}_0)$, there is no $\mathbf{x} \in \bar{B}_{2r}(\mathbf{x}_0)$ such that $u(\mathbf{x}) = \max\limits_{\mathbf{k}\in\bar{\Omega}} u(\mathbf{k})$ by **Weak maximum principle.** $\mathfrak{Q}.\mathfrak{E}.\mathfrak{D}.$

It's time to achieve the final goal, proving the strong maximum principle for harmonic function without mean-value property. Cause u is a continuous function on $\bar{\Omega}$, it suffices to prove that a set of point whose value of u is maximum on $\bar{\Omega}$ is dense in $\bar{\Omega}$ if at least one of them is in Ω. With previous theorems and lemmas, we can prove it. After proving the strong maximum principle for harmonic function, you can realize that strong maximum principle is not only for harmonic function. However, maybe you can't realize that if you prove the strong maximum principle for harmonic function with mean-value property.

Theorem 3. ***Strong maximum principle.*** *If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ is harmonic with $u(\mathbf{x}_0) = \max_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x})$ where $\mathbf{x_0} \in \Omega$, then u is constant on $\bar{\Omega}$.*

Proof. Let $\Omega_0 = \{\mathbf{x} \in \bar{\Omega} \mid u(\mathbf{x}) = \max_{\bar{\Omega}} u\}$ and $\mathbf{x}_0 \in \Omega \cap \Omega_0$. For arbitrary open set $\mathbf{x}_0 \notin D \subset \Omega$, choose one point $\mathbf{z}_0 \in D$. Cause Ω is path-connected, there exists a one-to-one path $\Gamma : [0, 1] \to \Omega$ such that $\Gamma(0) = \mathbf{z}_0$ and $\Gamma(1) = \mathbf{x}_0$. Cause $\Gamma([0, 1])$ is compact, we can denote $\varepsilon = d(\Gamma([0, 1]), \partial\Omega)$, where $d(A, B)$ is distance between two sets A and B.

Now, suppose $u(\mathbf{z}_0) \neq \max_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x})$. Cause u is continuous, there exists $\varepsilon > \delta > 0$ such that $\max_{\mathbf{x}\in\bar{B}_\delta(\mathbf{z}_0)} u(\mathbf{x}) \neq \max_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x})$. Denote $\mathbf{z}_n = \partial B_\delta(\mathbf{z}_{n-1}) \cap \Gamma([0, 1])$ for $n \in \mathbb{N}$, where $\Gamma^{-1}(\mathbf{z}_n) > \Gamma^{-1}(\mathbf{z}_{n-1})$ with the closest $\Gamma^{-1}(\mathbf{z}_n)$ to $\Gamma^{-1}(\mathbf{z}_{n-1})$. Then, there exists $M \in \mathbb{N}$ such that $\Gamma([0, 1]) \subset \bigcup_{i=0}^{M} B_\delta(\mathbf{z}_i) \subset \Omega$, cause $\Gamma([0, 1])$ is compact.

By **Lemma 2**, $\max_{\bigcup_{i=0}^{M} \bar{B}_\delta(\mathbf{z}_i)} u \neq \max_{\bar{\Omega}} u$. It contradicts to $\Gamma(1) = \mathbf{x}_0 \in \Omega_0$. So, $u(\mathbf{z}_0) = \max_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x})$, thus $\mathbf{z}_0 \in \Omega_0$. Therefore, Ω_0 is dense in Ω, it means $\Omega_0 = \bar{\Omega}$. Namely, u is constant on $\bar{\Omega}$, if $\max_{\Omega} u = \max_{\bar{\Omega}} u$. $\mathfrak{Q}.\mathfrak{E}.\mathfrak{D}$.

Now, we achieved the final goal. This proof without mean-value property give us possiblity of existence of strong maximum principle not only for harmonic function, but also for other functions who have weak maximum principle although it don't have mean-value property.

Actually, this proof of strong maximum principle obviously means that a function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ satisfying $\bigcup_{\imath=1}^{n} a_i(\mathbf{x}) \frac{\partial^2 u}{\partial x_i^2} = 0$ for $a_i(\mathbf{x}) \geq 0$ must have the strong maximum principle.