SUPPLEMENTARY MATERIAL FOR "LOCAL LAW AND TRACY–WIDOM LIMIT FOR SPARSE SAMPLE COVARIANCE MATRICES"

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Abstract. This is a supplementary material for "Local law and Tracy–Widom limit for sparse sample covariance matrices".

A. Recursive Moment Estimate

In this section, we prove Lemma 4.1. Throughout this section, we fix $t \in [0, 6\log N]$ and omit t from the notation in the matrix X_t, H_t, $G_t(z)$, and their elements. Given $\epsilon > 0$, we introduce the z-dependent control parameter $\Phi_\epsilon \equiv \Phi_\epsilon(z)$,

$$\begin{aligned}\Phi_\epsilon(z) &:= N^\epsilon \mathbb{E}\Big[\Big(\frac{1}{q_t^4} + \frac{\operatorname{Im} m_t}{N\eta} + \frac{N-M}{N^2}\Big)|P(m_t)|^{2D-1}\Big] + N^{-\epsilon/4} q_t^{-1}\mathbb{E}\Big[|m_t - \widetilde{m}_t|^2 |P(m_t)|^{2D-1}\Big] \\ &+ N^\epsilon q_t^{-1}\sum_{s=2}^{2D}\sum_{u'=0}^{s-2}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m_t}{N\eta} + \frac{N-M}{N^2}\Big)^{2s-u'-2}|P'(m_t)|^{u'}|P(m_t)|^{2D-s}\Big] + N^\epsilon q_t^{-8D} \\ &+ N^\epsilon\sum_{s=2}^{2D}\mathbb{E}\Big[\Big(\frac{1}{N\eta} + \frac{1}{q_t}\Big(\frac{\operatorname{Im} m_t}{N\eta} + \frac{N-M}{N^2}\Big)^{1/2} + \frac{1}{q_t^2}\Big)\Big(\frac{\operatorname{Im} m_t}{N\eta} + \frac{N-M}{N^2}\Big)^{s-1}|P'(m_t)|^{s-1}|P(m_t)|^{2D-s}\Big].\end{aligned} \tag{A.1}$$

By expanding zm we get

$$\mathbb{E}[(1+zm)P^{D-1}\overline{P^D}] = \mathbb{E}\Big[\frac{1}{N}\Big(\sum_{i,\alpha} X_{\alpha l} G_{\alpha i}\Big) P^{D-1}\overline{P^D}\Big] \tag{A.2}$$

Using the generalized Stein lemma, we get

$$\mathbb{E}[(1+zm)P^{D-1}\overline{P^D}] = \frac{1}{N}\sum_{r=1}^{\ell}\frac{\kappa^{(r+1)}}{r!}\mathbb{E}\Big[\sum_{\substack{1\le i\le N\\ N+1\le\alpha\le M+N}} \partial_{\alpha i}^r\big(G_{\alpha i}P^{D-1}\overline{P^D}\big)\Big] \tag{A.3}$$

where $\partial_{\alpha i} = \partial/(\partial X_{\alpha i})$ and $\kappa^{(k)}$ are the cumulants of $X_{\alpha i}$. For the notational simplicity, we set

$$I \equiv I(z, m, D) := (1+zm)P(m)^{D-1}\overline{P(m)^D}. \tag{A.4}$$

Then, we can rewrite the cumulant expansion (A.3) as

$$\mathbb{E}I = \sum_{1\le r\le \ell}\sum_{0\le u\le r} w_{I_{r,u}}\mathbb{E}I_{r,u} + \mathbb{E}\Omega_\ell(I), \tag{A.5}$$

where we set

$$I_{r,u} := \kappa^{(r+1)}\frac{1}{N}\sum_{i,\alpha}(\partial_{\alpha i}^{r-u}G_{\alpha i})(\partial_{\alpha i}^u(P^{D-1}\overline{P^D})). \tag{A.6}$$

The weights $w_{r,u}$ are combinatorial coefficient given by

$$w_{I_{r,u}} := \frac{1}{r!}\binom{r}{u} = \frac{1}{(r-u)!u!}. \tag{A.7}$$

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By using these condensed form, we get the expansion

$$\mathbb{E}[|P|^{2D}] = \sum_{1\le r\le \ell}\sum_{0\le u\le r} w_{I_{r,u}}\mathbb{E}I_{r,u} + \mathbb{E}\Big[\Big((1-\frac{1}{d})m + zm^2 + \frac{s^{(4)}}{q^2}m^2\big(zm+1-\frac{1}{d}\big)^2\Big)P^{D-1}\overline{P^D}\Big] + \mathbb{E}\Omega_\ell(I). \tag{A.8}$$

We will use the following bound frequently in the estimates.

Lemma A.1. *For any* $1 \le \mathfrak{i} \le M+N$,

$$\frac{1}{N}\sum_{\mathfrak{j}=1}^{M+N}|G_{\mathfrak{ij}}|^2 \prec \frac{\operatorname{Im} m(z)}{N\eta} + \frac{N-M}{N^2}. \quad (z\in\mathbb{C}^+) \tag{A.9}$$

Proof. Fix $\mathfrak{i}$. Let $\mathbf{x}_\mathfrak{n}$ be the normalized eigenvector of $(N+M)\times(N+M)$ matrix H, and $\lambda_\mathfrak{n}$ be the corresponding eigenvalue. Then we obtain

$$\begin{aligned}\frac{1}{N}\sum_{\mathfrak{j}=1}^{N+M}|G_{\mathfrak{ij}}|^2 &= \frac{1}{N}\sum_{\mathfrak{j}=1}^{M+N}\sum_{\mathfrak{n}=1}^{M+N}\frac{\mathbf{x}_\mathfrak{n}(\mathfrak{i})\overline{\mathbf{x}_\mathfrak{n}(\mathfrak{j})}}{\lambda_\mathfrak{n}}\sum_{\mathfrak{j}=1}^{M+N}\frac{\overline{\mathbf{x}_\mathfrak{m}(\mathfrak{i})}\mathbf{x}_\mathfrak{m}(\mathfrak{j})}{\lambda_\mathfrak{m}}\\ &= \frac{1}{N}\sum_{\mathfrak{n},\mathfrak{m}=1}^{M+N}\frac{\mathbf{x}_\mathfrak{n}(\mathfrak{i})\langle\mathbf{x}_\mathfrak{n},\mathbf{x}_\mathfrak{m}\rangle\overline{\mathbf{x}_\mathfrak{m}(\mathfrak{i})}}{\lambda_\mathfrak{n}\lambda_\mathfrak{m}}\\ &= \frac{1}{N}\sum_{\mathfrak{n}=1}^{M+N}\frac{|\mathbf{x}_\mathfrak{n}(j)|^2}{|\lambda_\mathfrak{n}|}.\end{aligned} \tag{A.10}$$

By the delocalization of eigenvectors,

$$\frac{1}{N}\sum_{\mathfrak{n}=1}^{M+N}\frac{|\mathbf{x}_\mathfrak{n}(\mathfrak{i})|^2}{|\lambda_\mathfrak{n}|} \prec \frac{1}{N^2}\sum_{\mathfrak{n}=1}^{M+N}\frac{1}{|\lambda_\mathfrak{n}|^2}.$$

Recall the definition of H. Suppose that $\mathbf{x}_\mathfrak{n} = (u_\mathfrak{n}(1), u_\mathfrak{n}(2),\cdots,u_\mathfrak{n}(n), v_\mathfrak{n}(1), v_\mathfrak{n}(2),\cdots,v_\mathfrak{n}(m))$ for some $\mathbf{u}_\mathfrak{n} = (u_\mathfrak{n}(1),\cdots,u_\mathfrak{n}(n))$ and $\mathbf{v}_\mathfrak{n} = (v_\mathfrak{n}(1),\cdots,v_\mathfrak{n}(m))$, which are vectors in $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. Then from $H\mathbf{x}_\mathfrak{n} = \lambda_\mathfrak{n}\mathbf{x}_\mathfrak{n}$, we obtain

$$\begin{aligned}-z\mathbf{u}_\mathfrak{n} + X^\dagger\mathbf{v}_\mathfrak{n} &= \lambda_\mathfrak{n}\mathbf{u}_\mathfrak{n},\\ X\mathbf{u}_\mathfrak{n} - \mathbf{v}_\mathfrak{n} &= \lambda_\mathfrak{n}\mathbf{v}_\mathfrak{n}.\end{aligned} \tag{A.11}$$

By combining two equations in (A.11), we get

$$X^\dagger X\mathbf{u}_\mathfrak{n} = (\lambda_\mathfrak{n}+1)(\lambda_\mathfrak{n}+z)\mathbf{u}_\mathfrak{n}. \tag{A.12}$$

It means that $\mathbf{u}_\mathfrak{n}$ is an eigenvector of $X^\dagger X$, with corresponding eigenvalue

$$\mu_\mathfrak{n} = (\lambda_\mathfrak{n}+1)(\lambda_\mathfrak{n}+z).$$

If we consider the equation above as the quadratic equation of $\lambda_\mathfrak{n}$, we get

$$\lambda_\mathfrak{n} = \frac{-(z+1)\pm\sqrt{(z+1)^2-4(z-\mu_\mathfrak{n})}}{2}. \tag{A.13}$$

Thus for each nonzero eigenvalue μ_n of $X^\dagger X$, there exist a pair of eigenvalues $\lambda_{n_a}, \lambda_{n_b}$ of H satisfying the above equation. So there are M pairs of such eigenvalues, and the other $(N-M)$ eigenvalues are equal to -1. Note that for each pair of eigenvalue $\lambda_{n_a}, \lambda_{n_b}$,

$$\frac{1}{|\lambda_{n_a}|^2} + \frac{1}{|\lambda_{n_b}|^2} = \frac{|\lambda_{n_a}|^2+|\lambda_{n_b}|^2}{|\lambda_{n_a}\lambda_{n_b}|^2} \le \frac{c}{|\mu_n - z|^2},$$

for some constant c, which obtained from the boundedness of eigenvalues of $X^\dagger X$. Hence from (A.10),

$$\begin{aligned}\frac{1}{N^2}\sum_{\mathfrak{n}}\frac{1}{|\lambda_{\mathfrak{n}}|^2} &\le \frac{1}{N^2}\sum_{i=1}^{N}\frac{c}{|\mu_i - z|^2} + \frac{N-M}{N^2}\\ &= \frac{\operatorname{Im} m(z)}{N\eta} + \frac{N-M}{N^2},\end{aligned} \tag{A.14}$$

which shows (A.9). □

A.1. **Truncation of the cumulant expansion.** In this subsection, we bound the error term $\mathbb{E}\Omega_\ell(I)$ in (A.8) for large ℓ. For $1 \le i \le N$ and $N+1 \le \alpha \le N+M$, let $E^{[\mathfrak{ij}]}$ denote the $(N+M)\times(N+M)$ matrix determined by

$$(E^{[\mathfrak{ij}]})_{ab} = \begin{cases} \delta_{\mathfrak{ia}}\delta_{\mathfrak{jb}} + \delta_{\mathfrak{ib}}\delta_{\mathfrak{ja}}, & \text{if } \mathfrak{i}\neq\mathfrak{j}, \\ \delta_{\mathfrak{ia}}\delta_{\mathfrak{jb}}, & \text{if } \mathfrak{i}=\mathfrak{j}, \end{cases} \qquad (1 \le \mathfrak{i},\mathfrak{j},\mathfrak{a},\mathfrak{b} \le N+M). \tag{A.15}$$

Here, we use the Latin letters such as $\mathfrak{i},\mathfrak{j},\mathfrak{a},\mathfrak{b}$ to denote the indices that can be in $[1,N]$. For each pair of indices $(\mathfrak{i},\mathfrak{j})$, we define the matrix $H^{(\mathfrak{ij})}$ from H through the decomposition

$$H = H^{(\mathfrak{ij})} + H_{\mathfrak{ij}}E^{[\mathfrak{ij}]}. \tag{A.16}$$

With this notation we have the following estimate.

Lemma A.2. *Suppose that X satisfies Assumption 2.6 with $\phi > 0$. Let $1 \le \mathfrak{i},\mathfrak{j} \le N+M$, $D \in \mathbb{N}$ and $z \in \mathcal{E}$. Define the function $F_{\alpha i}$ by*

$$F_{\mathfrak{ij}(H)} := G_{\mathfrak{ji}}P^{D-1}\overline{P^D}, \tag{A.17}$$

where $G \equiv G^H(z)$ and $P \equiv P(m(z))$. Choose an arbitrary $\ell \in \mathbb{N}$. Then, for any (small) $\epsilon > 0$,

$$\mathbb{E}\Big[\sup_{x\in\mathbb{R},|x|\le q_t^{-1/2}} |\partial_{\mathfrak{ij}}^\ell F_{\mathfrak{ji}}(H^{(\mathfrak{ij})} + xE^{[\mathfrak{ij}]})|\Big] \le N^\epsilon, \tag{A.18}$$

uniformly $z \in \mathcal{E}$, for N sufficiently large. Here $\partial_{\mathfrak{ij}}^\ell$ denotes the partial derivative $\frac{\partial^\ell}{\partial H_{\mathfrak{ij}}^\ell}$.

Proof. Fix two pairs of indicies $(\mathfrak{a},\mathfrak{b})$ and $(\mathfrak{i},\mathfrak{j})$. From the definition of the Green function and (A.16) we get

$$G_{\mathfrak{ab}}^{H^{(\mathfrak{ij})}} = G_{\mathfrak{ab}}^H + H_{\mathfrak{ij}}(G^{H^{(\mathfrak{ij})}}E^{[\mathfrak{ij}]}G^H)_{\mathfrak{ab}} = G_{\mathfrak{ab}}^H + H_{\mathfrak{ij}}G_{\mathfrak{ai}}^{H^{(\mathfrak{ij})}}G_{\mathfrak{jb}}^H + H_{\mathfrak{ij}}G_{\mathfrak{aj}}^{(H^{(\mathfrak{ij})}}G_{\mathfrak{ib}}^H,$$

where we omit the z-dependence. Letting $\Lambda_o^{H^{(\mathfrak{ij})}} := \max_{\mathfrak{a},\mathfrak{b}}|G_{\mathfrak{ab}}^{H(\mathfrak{ij})}|$ and $\Lambda_o^H := \max_{\mathfrak{a},\mathfrak{b}}|G_{\mathfrak{ab}}^H|$, we get

$$\Lambda_o^{H^{(\mathfrak{ij})}} \prec \Lambda_o^H + \frac{1}{q_t}\Lambda_o^H\Lambda_o^{H^{(\mathfrak{ij})}}.$$

By (2.19) we have $|H_{\mathfrak{ij}}| \prec q_t^{-1}$ and by (2.53) we have $\Lambda_o^H \prec 1$, uniformly in $z \in \mathcal{E}$. It follows that $\Lambda_o^{H^{(\mathfrak{ij})}} \prec \Lambda_o^H \prec 1$, uniformly in $z \in \mathcal{E}$, where we used (2.53). Similarly, for $x \in \mathbb{R}$, we have

$$G_{\mathfrak{ab}}^{H^{(\mathfrak{ij})}+xE^{[\mathfrak{ij}]}} = G_{\mathfrak{ab}}^{H^{(\mathfrak{ij})}} - x\big(G_{\mathfrak{ab}}^{H^{(\mathfrak{ij})}}E^{[\mathfrak{ij}]}G_{\mathfrak{ab}}^{H^{(\mathfrak{ij})}xE^{[\mathfrak{ij}]}}\big)_{\mathfrak{ab}},$$

and we get

$$\sup_{|x|\le q_t^{-1/2}}\max_{\mathfrak{a},\mathfrak{b}}|G_{\mathfrak{ab}}^{H^{(\mathfrak{ij})}+xE^{[\mathfrak{ij}]}}| \prec \Lambda_o^{H^{(\mathfrak{ij})}} \prec 1, \tag{A.19}$$

uniformly in $z \in \mathcal{E}$, where we used once more (2.53).

Recall that P is a polynomial of degree 4 in m. Then $F_{\mathfrak{ji}}$ is a multivariate polynomial of degree $4(2D-1)+1$ in the Green function entries and the normalized trace m whose number of member terms in bounded by 4^{2D-1}. Hence $\partial_{\mathfrak{ij}}^\ell F_{\mathfrak{ji}}$ is a multivariate polynomial of degree $4(2D-1)+1+\ell$ whose number of member terms is roughly bounded by $4^{2D-1}\times(4(2D-1)+1+2\ell)^\ell$. Next, to control the individual monomials in $\partial_{\mathfrak{ij}}^\ell F_{\mathfrak{ji}}$, we apply (A.19) to each factor of Green function entries (at most $4(2D-1)+1+\ell$ times). Thus, altogether we obtain

$$\mathbb{E}\Big[\sup_{|x|\le q_t^{-1/2}}|(\partial_{\mathfrak{ij}}^\ell F_{\mathfrak{ji}})(H^{(\mathfrak{ij})} + xE^{[\mathfrak{ij}]})|\Big] \le 4^{2D}(8D+\ell)N^{(8D+\ell)\epsilon'}, \tag{A.20}$$

for any small $\epsilon' > 0$ and sufficiently large N. Choosing $\epsilon' = \epsilon/(2(8D+\ell))$ with get (A.18). □

The error term $\mathbb{E}\Omega_\ell(I)$ in (A.8) is controlled by the following result.

Corollary A.3. *Let $\mathbb{E}\Omega_\ell(I)$ be as in (A.8). With the assumptions and notation of Lemma A.2, we have, for any (small) $\epsilon > 0$,*

$$|\mathbb{E}\Omega_\ell(I)| \le N^\epsilon \Big(\frac{1}{q_t}\Big)^\ell, \tag{A.21}$$

uniformly in $z \in \mathcal{E}$, for N sufficiently large. In particular, the error $\mathbb{E}\Omega_\ell(I)$ is negligible for $\ell \ge 8D$.

Proof. First, fix a pair of indices $(\mathfrak{j}, \mathfrak{i})$, $\mathfrak{j} \neq \mathfrak{i}$. Denoting $\mathbb{E}_{\mathfrak{j}\mathfrak{i}}$ the partial expectation with respect to $H_{\mathfrak{i}\mathfrak{j}} = X_{\mathfrak{i}\mathfrak{j}}$, we have from Lemma 2.11, with $Q = q_t^{-1/2}$,

$$\begin{aligned}|\mathbb{E}_{\mathfrak{i}\mathfrak{j}}\Omega_\ell(H_{\mathfrak{i}\mathfrak{j}}F_{\mathfrak{j}\mathfrak{i}})| \le C_\ell \mathbb{E}_{\mathfrak{i}\mathfrak{j}}[|H_{\mathfrak{i}\mathfrak{j}}|^{\ell+2}] \sup_{|x|\le q_t^{-1/2}} |\partial_{\mathfrak{i}\mathfrak{j}}^{\ell+1} F_{\mathfrak{j}\mathfrak{i}}(H^{(\mathfrak{i}\mathfrak{j})} + xE^{[\mathfrak{i}\mathfrak{j}]})| \\ + C_\ell \mathbb{E}_{\mathfrak{i}\mathfrak{j}}[|H_{\mathfrak{i}\mathfrak{j}}|^{\ell+2}\mathbb{1}(|H_{\mathfrak{i}\mathfrak{j}} > q_t^{-1/2})] \sup_{x\in\mathbb{R}} |\partial_{\mathfrak{i}\mathfrak{j}}^{\ell+1} F_{\mathfrak{j}\mathfrak{i}}(H^{(\mathfrak{i}\mathfrak{j})} + xE^{[\mathfrak{i}\mathfrak{j}]})|,\end{aligned} \tag{A.22}$$

with $C_\ell \le (C\ell)^\ell/\ell!$, for some numeral constant C. To control the full expectation of the first term on the right side, we use the moment assumption (2.19) and Lemma A.2 to conclude that, for any $\epsilon > 0$,

$$C_\ell \mathbb{E}\Big[\mathbb{E}_{\mathfrak{i}\mathfrak{j}}[|H_{\mathfrak{j}\mathfrak{i}}|^{\ell+2}] \sup_{|x|\le q_t^{-1/2}} |\partial_{\mathfrak{i}\mathfrak{j}}^{\ell+1} F_{\mathfrak{j}\mathfrak{i}}(H^{(\mathfrak{i}\mathfrak{j})} + xE^{[\mathfrak{i}\mathfrak{j}]})|\Big] \le C_\ell \frac{(C(\ell+2))^{c(\ell+2)}}{Nq_t^\ell} N^\epsilon \le \frac{N^{2\epsilon}}{Nq_t^\ell},$$

for N sufficiently large. To control the second term on the right side of (A.22), we use the deterministic bound $\|G(z)\| \le \eta^{-1}$ to conclude that

$$\sup_{x\in\mathbb{R}} |\partial_{\mathfrak{i}\mathfrak{j}}^{\ell+1} F_{\mathfrak{j}\mathfrak{i}}(H^{(\mathfrak{i}\mathfrak{j})} + xE^{[\mathfrak{i}\mathfrak{j}]})| \le 4^{2D}(8D+\ell)\Big(\frac{C}{\eta}\Big)^{8D+\ell}, \quad (z \in \mathbb{C}^+); \tag{A.23}$$

see the paragraph above (A.20). On the other hand, we have from Hölder's inequality and the moment assumptions in (2.19) that, for any $D' \in \mathbb{N}$,

$$\mathbb{E}_{\mathfrak{i}\mathfrak{j}}[|H_{\mathfrak{i}\mathfrak{j}}|^{\ell+2}\mathbb{1}(|H_{\mathfrak{i}\mathfrak{j}} > q_t^{-1/2})] \le \Big(\frac{C}{q}\Big)^{D'},$$

for N sufficiently large. Using that $q \ge N^\phi$ by (2.20), we obtain, for any $D' \in \mathbb{N}$,

$$C_\ell \mathbb{E}_{\mathfrak{i}\mathfrak{j}}[|H_{\mathfrak{i}\mathfrak{j}}|^{\ell+2}\mathbb{1}(|H_{\mathfrak{i}\mathfrak{j}} > q_t^{-1/2})] \sup_{x\in\mathbb{R}} |\partial_{\mathfrak{i}\mathfrak{j}}^{\ell+1} F_{\mathfrak{j}\mathfrak{i}}(H^{(\mathfrak{i}\mathfrak{j})} + xE^{[\mathfrak{i}\mathfrak{j}]})| \le \Big(\frac{C}{q}\Big)^{D'}, \tag{A.24}$$

uniformly on $\mathbb{C}^+$, for N sufficiently large.

Next, summing over α, i and choosing $D' \ge \ell$ sufficiently large in (A.24) we obtain, for any $\epsilon > 0$,

$$\Big|\mathbb{E}\Big[\Omega_\ell\big((1+zm)P^{D-1}\overline{P^D}\big)\Big]\Big| = \Big|\mathbb{E}\Big[\Omega_\ell\big(\frac{1}{N}\sum H_{\mathfrak{i}\mathfrak{j}}F_{\mathfrak{j}\mathfrak{i}}\big)\Big]\Big| \le \frac{N^\epsilon}{q_t^\ell}, \tag{A.25}$$

uniformly on $\mathcal{E}$, for N sufficiently large. This proves (A.21). □

Remark A.4. We will also consider slight generalizations of the cumulant expansion in (A.3). Let $1 \le \mathfrak{i}, \mathfrak{j} \le N + M$. Let $n \in \mathbb{N}_0$ and choose indices $1 \le \mathfrak{a}_1, \cdots, \mathfrak{a}_n, \mathfrak{b}_1, \cdots, \mathfrak{b}_n \le N$. Let $D \in \mathbb{N}$ and choose $0 \le u_1, u_2, u_3, u_4 \le N$. Fix $z \in \mathcal{E}$. Define the function $F_{\mathfrak{j}\mathfrak{i}}$ by setting

$$F_{\mathfrak{j}\mathfrak{i}} := G_{\mathfrak{j}\mathfrak{i}} \prod_{l=1}^n G_{\mathfrak{a}_l\mathfrak{b}_l} P^{D-u_1}\overline{P^{D-u_2}}(P')^{u_3}(\overline{P'})^{u_4}. \tag{A.26}$$

It is then straightforward to check that we have the cumulant expansion

$$\mathbb{E}\Big[\frac{1}{N}\sum_{\mathfrak{i}\neq\mathfrak{j}} H_{\mathfrak{i}\mathfrak{j}}F_{\mathfrak{i}\mathfrak{j}}\Big] = \sum_{r=1}^\ell \frac{\kappa_t^{(r+1)}}{r!}\mathbb{E}\Big[\frac{1}{N}\sum_{\mathfrak{i}\neq\mathfrak{j}} \partial_{\mathfrak{i}\mathfrak{j}}^r F_{\mathfrak{i}\mathfrak{j}}\Big] + \mathbb{E}\Omega_\ell\Big(\frac{1}{N}\sum_{\mathfrak{i}\neq\mathfrak{j}} H_{\mathfrak{i}\mathfrak{j}}F_{\mathfrak{i}\mathfrak{j}}\Big), \tag{A.27}$$

where the error $\mathbb{E}\Omega(\cdot)$ satisfies the same bound as in (A.21). This follows by extending Lemma A.2 and Corollary A.3.

A.2. **Truncated cumulant expansion.** In the remainder of this section we derive the following result from which Lemma 4.1 follows directly.

Lemma A.5. *Fix $D \geq 2$ and $\ell \geq 8D$. Let $I_{r,u}$ be given by (A.6). Then for any (small) $\epsilon > 0$, we have*

$$w_{I_{1,0}}\mathbb{E}[I_{1,0}] = -\mathbb{E}\Big[\big((1-\frac{1}{d})m + zm^2\big)P(m)^{D_1}\overline{P(m)^D}\Big] + O(\Phi_\epsilon), \quad w_{I_{2,0}}\mathbb{E}[I_{2,0}] = O(\Phi_\epsilon)$$

$$w_{I_{3,0}}\mathbb{E}[I_{3,0}] = -\frac{s^{(4)}}{q_t^2}\mathbb{E}\Big[m^2\big(zm+1-\frac{1}{d}\big)^2P(m)^{D_1}\overline{P(m)^D}\Big] + O(\Phi_\epsilon), \tag{A.28}$$

and

$$w_{I_{r,0}}\mathbb{E}[I_{r,0}] = O(\Phi_\epsilon), \quad (4 \leq r \leq \ell) \tag{A.29}$$

uniformly in $z \in \mathcal{E}$, for any sufficiently large N. Moreover, for any (small) $\epsilon > 0$,

$$w_{I_{r,u}}\mathbb{E}[I_{r,u}] = O(\Phi_\epsilon), \quad (1 \leq u \leq r \leq \ell), \tag{A.30}$$

uniformly in $z \in \mathcal{E}$, for any sufficiently large N.

Proof of Lemma 4.1. By the definition of Φ_ϵ in (A.1), it suffices to show that $\mathbb{E}[|P^{2D}(z)|] \leq \Phi_\epsilon(z)$, for all $z \in \mathcal{E}$, for N sufficiently large. Choosing $\ell \geq 8D$, Corollary A.3 asserts that $\mathbb{E}\Omega_\ell(I)$ in (A.8) is negligible. By Lemma A.5 the only non-negligible terms in the expansion of the first term on the right side of (A.8) are $w_{I_{1,0}}\mathbb{E}I_{1,0}$ and $w_{I_{3,0}}\mathbb{E}I_{3,0}$, yet these two terms cancel with the middle term on the right side of (A.8), up to negligible terms. Thus the whole right-hand side of (A.8) is negligible. This proves Lemma 4.1. ☐

Now we choose an initial small $\epsilon > 0$. In the remaining sections we bound $\mathbb{E}I_{r,u}$ to prove Lemma A.5.

A.3. **Estimate on $I_{1,u}$.** By the definition of $I_{1,0}$ in (A.5),

$$\begin{aligned}\mathbb{E}I_{1,0} &= \frac{\kappa^{(2)}}{N}\mathbb{E}\Big[\sum_{i,\alpha}(\partial_{\alpha i}G_{\alpha i})(P^{D-1}\overline{P^D})\Big]\\ &= -\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}(G_{\alpha\alpha}G_{ii} + G_{\alpha i}G_{\alpha i})(P^{D-1}\overline{P^D})\Big]\\ &= -\mathbb{E}\Big[\big((1-\frac{1}{d})m + zm^2\big)P^{D-1}\overline{P}^D\Big] - \Big[\frac{1}{N^2}\sum_{i,\alpha}(G_{\alpha i})^2P^{D-1}\overline{P}^D\Big]\end{aligned} \tag{A.31}$$

The last term on the last line is negligible since

$$\Big|\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}(G_{\alpha i})^2P^{D-1}\overline{P}^D\Big]\Big| \leq N^\epsilon\mathbb{E}\Big[\big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\big)|P|^{2D-1}\Big],$$

where we used Lemma A.1. We thus obtain

$$\Big|I_{1,0} + \mathbb{E}\big[zm^2P^{D-1}\overline{P^D}\big]\Big| \leq \Phi_\epsilon. \tag{A.32}$$

Consider $I_{1,1}$ next. We have

$$\begin{aligned}\mathbb{E}I_{1,1} &= \frac{1}{N^2}\mathbb{E}\Big[\sum_{i,\alpha}(G_{\alpha i})(\partial_{\alpha i}P^{D-1}\overline{P^D})\Big]\\ &= -\frac{1}{N^2}\mathbb{E}\Big[\frac{2(D-1)}{N}\sum_{i,\alpha}G_{\alpha i}P'(m)\sum_j G_{ji}G_{\alpha j}P^{D-2}\overline{P}^D\Big]\\ &\quad - \frac{1}{N^2}\mathbb{E}\Big[\frac{2D}{N}\sum_{i,\alpha}G_{\alpha i}\overline{P'(m)}\sum_j\overline{G_{ji}G_{\alpha j}}P^{D-1}\overline{P}^{D-1}\Big]\\ &= -2(D-1)\mathbb{E}\Big[\frac{1}{N^3}\sum_{i,j,\alpha}G_{\alpha i}G_{ji}G_{\alpha j}P'(m)P^{D-2}\overline{P^D}\Big]\\ &\quad - 2D\mathbb{E}\Big[\frac{1}{N^3}\sum_{i,j,\alpha}G_{\alpha i}\overline{G_{ji}G_{\alpha j}P'(m)}P^{D-1}\overline{P^{D-1}}\Big].\end{aligned} \tag{A.33}$$

Here the fresh summation index j originated from $\partial_{\alpha i}P(m) = P'(m)\frac{1}{N}\sum_{j=1}^{N}\partial_{\alpha i}G_{jj}$. Using Lemma A.1, we get

$$|\mathbb{E}I_{1,1}| \le (4D-2)\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)^{3/2}|P'||P|^{2D-2}\Big] + O(\Phi_\epsilon), \tag{A.34}$$

for N sufficiently large. This proves (A.30) for $r = u = 1$.

A.4. **Estimate on $I_{2,0}$.** By the definition of $I_{r,s}$ in (A.5), we have

$$I_{2,0} := \kappa_t^{(3)}\frac{1}{N}\sum_{i,\alpha}(\partial_{\alpha i}^2 G_{\alpha i})P^{D-1}\overline{P^D}.$$

We then notice that $I_{2,0}$ contains terms with one or three off-diagonal Green functions entries $G_{\alpha i}$. We split accordingly

$$w_{I_{2,0}}I_{2,0} = w_{I_{2,0}^{(1)}}I_{2,0}^{(1)} + w_{I_{2,0}^{(3)}}I_{2,0}^{(3)},$$

where $I_{2,0}^{(p)}$ contains terms with p off-diagonal Green function entries and $(3-p)$ diagonal entries and $w_{I_{2,0}^{(p)}}$ denote the respective weights. Explicitly,

$$\begin{aligned}\mathbb{E}I_{2,0}^{(1)} &= \kappa_t^{(3)}\frac{1}{N}\mathbb{E}\Big[\sum_{i,\alpha}G_{\alpha i}G_{ii}G_{\alpha\alpha}P^{D-1}\overline{P^D}\Big],\\ \mathbb{E}I_{2,0}^{(3)} &= \kappa_t^{(3)}\frac{1}{N}\mathbb{E}\Big[\sum_{i,\alpha}(G_{\alpha i})^3P^{D-1}\overline{P^D}\Big],\end{aligned} \tag{A.35}$$

where $w_{I_{2,0}} = 1, w_{I_{2,0}^{(1)}} = 3, w_{I_{2,0}^{(3)}} = 1$.

We first note that $I_{2,0}^{(3)}$ satisfies, for N sufficiently large,

$$|\mathbb{E}I_{2,0}^{(3)}| \le \frac{N^\epsilon s^{(3)}}{q_t}\mathbb{E}\Big[\frac{1}{N^3}\sum_{i,\alpha}|G_{\alpha i}|^2|P|^{2D-1}\Big] \le \frac{N^\epsilon}{q_t}\mathbb{E}\Big[\big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\big)|P|^{2D-1}\Big] \le \Phi_\epsilon. \tag{A.36}$$

Remark A.6 (Power counting 1)**.** Consider the terms $I_{r,0}$, $r \ge 1$. For $n \ge 1$, we split

$$w_{I_{2n,0}}I_{2n,0} = \sum_{l=0}^{n} w_{I_{2n,0}^{(2l+1)}}I_{2n,0}^{(2l+1)}, \quad w_{I_{2n-1,0}}I_{2n-1,0} = \sum_{l=0}^{n} w_{I_{2n-1,0}^{(2l+1)}}I_{2n-1,0}^{(2l+1)}, \tag{A.37}$$

according to the parity of r. Now we bound the summands in (A.37) as follows. First, we note that each term in $I_{r,0}$ contains a factor of $q_t^{(r-2)_+}$. Second, for $\mathbb{E}I_{2n,0}^{(2l+1)}$ and $\mathbb{E}I_{2n-1,0}^{(2l)}$, with $n \ge 1, l \ge 1$, we can extract one factor of $(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2})$ by Lemma A.1. Other Green function entries are bounded using $|G_{\alpha i}| \prec 1$. Thus, for $n \ge 1$, $l \ge 1$,

$$\begin{aligned}|\mathbb{E}I_{2n,0}^{(2l+1)}| &\le \frac{N^\epsilon}{q_t^{2n-2}}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)|P|^{2D-1}\Big],\\ |\mathbb{E}I_{2n-1,0}^{(2l)}| &\le \frac{N^\epsilon}{q_t^{(2n-3)_+}}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)|P|^{2D-1}\Big],\end{aligned} \tag{A.38}$$

for N sufficiently large, and we conclude that all these terms are negligible.

Next consider $\mathbb{E}I_{2,0}^{(1)}$. Using $|G_{ii}| \prec 1$ and $|G_{\alpha\alpha}| \prec 1$ and Lemma A.1 we get

$$|\mathbb{E}I_{2,0}^{(1)}| \le \frac{N^\epsilon s^{(3)}}{q_t}\mathbb{E}\Big[\sum_{i,\alpha}\frac{1}{N^2}|G_{\alpha i}||P|^{2D-1}\Big] \le \frac{N^\epsilon}{q_t}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)^{1/2}|P|^{2D-1}\Big], \tag{A.39}$$

for N sufficiently large. This bound is, however, not negligible. We need to gain an additional factor of q_t^{-1} with which it will become negligible. We have the following lemma.

Lemma A.7. *For any (small) $\epsilon > 0$, we have, for all $z \in \mathcal{E}$,*

$$|\mathbb{E}I_{2,0}^{(1)}| \le \frac{N^\epsilon}{q_t^2}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)^{1/2}|P|^{2D-1}\Big] + \Phi_\epsilon$$
$$\le N^\epsilon \mathbb{E}\Big[\Big(q_t^{-4} + \frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)|P|^{2D-1}\Big] + \Phi_\epsilon, \tag{A.40}$$

for N sufficiently large. In particular, the term $\mathbb{E}I_{2,0}$ is negligible.

Proof. First we introduce some resolvent expansion formulas. Let $i \in [\![1, N]\!]$ and $\alpha \in [\![N+1, N+M]\!]$. Since $G = H^{-1}$,

$$1 = I_{ii} = (HG)_{ii} = -zG_{ii} + \sum_\alpha X^\dagger_{i\alpha} G_{\alpha i} = -zG_{ii} + \sum_\alpha X_{\alpha i} G_{\alpha i}, \tag{A.41}$$

respectively,

$$0 = I_{\alpha i} = (HG)_{\alpha i} = -G_{\alpha i}\sum_j X^\dagger_{j\alpha} G_{\alpha i} = -G_{\alpha i} + \sum_j X_{\alpha j} G_{\alpha i}. \tag{A.42}$$

This gives us resolvent expansion formula

$$1 + zG_{ii} = \sum_\alpha X_{\alpha i} G_{\alpha i}, \quad G_{\alpha i} = \sum_j X_{\alpha j} G_{\alpha i}. \tag{A.43}$$

Now recalling (A.35), we have

$$\mathbb{E}I_{2,0}^{(1)} = \kappa_t^{(3)}\frac{1}{N}\mathbb{E}\Big[\sum_{i,\alpha} G_{\alpha i}G_{ii}G_{\alpha\alpha}P^{D-1}\overline{P^D}\Big]. \tag{A.44}$$

Using the resolvent formula (A.43) we expand in the index i to get

$$\mathbb{E}I_{2,0}^{(1)} = \kappa_t^{(3)}\frac{1}{N}\mathbb{E}\Big[\sum_{i,\alpha} X_{j\alpha}G_{ji}G_{ii}G_{\alpha\alpha}P^{D-1}\overline{P^D}\Big]. \tag{A.45}$$

For simplicity we abbreviate $\hat{I} = I_{2,0}^{(1)}$. Then for arbitrary $\ell' \in \mathbb{N}$, the cumulant expansion

$$\mathbb{E}I_{2,0}^{(1)} = \mathbb{E}\hat{I} = \sum_{r'=1}^{\ell'}\sum_{u'=0}^{r'} w_{\hat{I}_{r',u'}}\mathbb{E}\hat{I}_{r',u'} + O\Big(\frac{N^\epsilon}{q_t^{\ell'}}\Big), \tag{A.46}$$

with

$$\hat{I}_{r',u'} = N\kappa^{(r'+1)}N\kappa^{(3)}\frac{1}{N^3}\sum_{i,j,\alpha}\big(\partial_{ij}^{r'-u'}(G_{j\alpha}G_{ii}G_{\alpha\alpha})\big)(\partial_{\alpha i}^{u'}(P^{D-1}\overline{P^D})). \tag{A.47}$$

and $w_{\hat{I}_{r',u'}} := \frac{1}{r'!}\binom{r'}{u'}$. We first focus on $\hat{I}_{r',0}$. For $r' = 1$,

$$\mathbb{E}\hat{I}_{1,0} = -\frac{\kappa_t^{(3)}}{q_t}\mathbb{E}\Big[\frac{1}{N^3}\sum_{i,j,\alpha} G_{i\alpha}G_{jj}G_{ii}G_{\alpha\alpha}P^{D-1}\overline{P}^D\Big]$$
$$- 3\frac{\kappa_t^{(3)}}{q_t}\mathbb{E}\Big[\frac{1}{N^3}\sum_{i,j,\alpha} G_{ij}G_{j\alpha}G_{ii}G_{\alpha\alpha}P^{D-1}\overline{P}^D\Big]$$
$$- 2\frac{\kappa_t^{(3)}}{q_t}\mathbb{E}\Big[\frac{1}{N^3}\sum_{i,j,\alpha} G_{\alpha i}G_{ij}G_{j\alpha}G_{ii}P^{D-1}\overline{P}^D\Big]$$
$$=: \mathbb{E}\hat{I}_{1,0}^{(1)} + 3\mathbb{E}\hat{I}_{1,0}^{(2)} + 2\mathbb{E}\hat{I}_{1,0}^{(3)}, \tag{A.48}$$

where we organize the terms according to the off-diagonal Green functions entries. By Lemma A.1,

$$|\mathbb{E}\hat{I}_{1,0}^{(2)}| \le \frac{N^\epsilon}{q_t}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)|P|^{2D-1}\Big] \le \Phi_\epsilon, \tag{A.49}$$

and

$$|\mathbb{E}\hat{I}_{1,0}^{(3)}| \le \frac{N^\epsilon}{q_t}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)^{3/2}|P|^{2D-1}\Big] \le \Phi_\epsilon. \tag{A.50}$$

We rewrite $\hat{I}_{1,0}^{(1)}$ with $\widetilde{m}$ as

$$\begin{aligned}\mathbb{E}\hat{I}_{1,0}^{(1)} = &- \mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}\widetilde{m}G_{i\alpha}G_{ii}G_{\alpha\alpha}P^{D-1}\overline{P^D}\Big] \\ &- \mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}(m-\widetilde{m})G_{i\alpha}G_{ii}G_{\alpha\alpha}P^{D-1}\overline{P^D}\Big] + O(\Phi_\epsilon).\end{aligned} \tag{A.51}$$

By Schwarz inequality and the high probability bounds $|G_{ii}|, |G_{\alpha\alpha}| \le N^{\epsilon/8}$, for N sufficiently large, the second term in (A.51) is bounded as

$$\begin{aligned}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}(m-\widetilde{m})G_{i\alpha}G_{ii}G_{\alpha\alpha}P^{D-1}\overline{P^D}\Big] &\le N^{\epsilon/4}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}|m-\widetilde{m}||G_{i\alpha}||P|^{2D-1}\Big] \\ &\le N^{-\epsilon/4}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}|m-\widetilde{m}|^2|P|^{2D-1}\Big] + N^{3\epsilon/4}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}|G_{i\alpha}|^2|P|^{2D-1}\Big] \\ &= \frac{1}{d}N^{-\epsilon/4}\mathbb{E}\Big[|m-\widetilde{m}||P|^{2D-1}\Big] + N^{3\epsilon/4}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)|P|^{2D-1}\Big]\end{aligned} \tag{A.52}$$

Thus we obtain from (A.48), (A.49), (A.50), (A.51), and (A.52) that

$$\mathbb{E}\hat{I}_{1,0} = -\widetilde{m}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}G_{i\alpha}G_{ii}G_{\alpha\alpha}P^{D-1}\overline{P^D}\Big] + O(\Phi_\epsilon) = -\mathbb{E}\widetilde{m}I_{2,0}^{(1)} + O(\Phi_\epsilon). \tag{A.53}$$

where we used (A.44). We remark that in the expansion of $\mathbb{E}\hat{I} = \mathbb{E}I_{2,0}^{(1)}$ the only term with one off-diagonal entry is $\mathbb{E}\hat{I}_{2,0}^{(1)}$. All the other terms contain at least two off-diagonal entries.

Remark A.8 (Power counting 2)**.** Comparing (A.5) and (A.46), we have $\hat{I}_{r',u'} = (I_{2,0}^{(1)})_{r',u'}$. Consider the terms with $u' = 0$. As in (A.37) we organize the terms according to the number of off-diagonal Green function entries. For $r' \ge 2$,

$$w_{\hat{I}_{r',0}}\hat{I}_{r',0} = \sum_{l=0}^{n} w_{\hat{I}_{r',0}^{(l+1)}}\hat{I}_{r',0}^{(l+1)} = \sum_{l=0}^{n} w_{\hat{I}_{r',0}^{(l+1)}}(I_{2,0}^{(1)})_{r',u'}. \tag{A.54}$$

By a simple power counting as in Remark A.6, we get

$$\begin{aligned}|\mathbb{E}\hat{I}_{r',0}^{(1)}| &\le \frac{N^\epsilon}{q_t^{r'}}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)^{1/2}|P|^{2D-1}\Big], \\ |\mathbb{E}\hat{I}_{r',0}^{(l+1)}| &\le \frac{N^\epsilon}{q_t^{r'}}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)|P|^{2D-1}\Big], \qquad (l \ge 1),\end{aligned} \tag{A.55}$$

for N sufficiently large. Here, we used that each term contains a factor $\kappa_t^{(3)}\kappa_t^{(r'+1)} \le CN^{-2}q_t^{-r'}$. We conclude that all terms in (A.55) with $r' \ge 2$ are negligible, yet we remark that $|\mathbb{E}\hat{I}_{2,0}^{(1)}|$ is the leading error term in $|\mathbb{E}I_{2,0}^{(1)}|$, which is explicitly listed on the right side of (A.40).

Remark A.9 (Power counting 3)**.** Consider the terms $\hat{I}_{r',u'}$, with $1 \le u' \le r'$. For $u' = 1$, note that $\partial_{i_1 i_2}(P^{D-1}\overline{P}^D)$ contains two off-diagonal Green function entries. Explicitly,

$$\begin{aligned}\hat{I}_{r',1} = -2(D-1)\frac{N\kappa_t^{(r'+1)}N\kappa_t^{(3)}}{N^3}&\sum_{\substack{1\le i_1,i_2\le N\\ N+1\le\alpha\le N+M}}(\partial_{i_1i_2}^{r'-1}(G_{i_2\alpha}G_{i_1i_1}G_{\alpha\alpha}))\Big(\frac{1}{N}\sum_{i_3=1}^{N}G_{i_3i_1}G_{i_2i_3}\Big)P'P^{D-2}\overline{P^D}\\ -2D\frac{N\kappa_t^{(r'+1)}N\kappa_t^{(3)}}{N^3}&\sum_{\substack{1\le i_1,i_2\le N\\ N+1\le\alpha\le N+M}}(\partial_{i_1i_2}^{r'-1}(G_{i_2\alpha}G_{i_1i_1}G_{\alpha\alpha}))\Big(\frac{1}{N}\sum_{i_3=1}^{N}\overline{G_{i_3i_1}G_{i_2i_3}}\Big)P'P^{D-2}\overline{P^D},\end{aligned}\tag{A.56}$$

where the summation index i_3 is generated from $\partial_{i_1i_2}P$. For $r' \ge 1$, using Lemma A.1 we get

$$|\mathbb{E}\hat{I}_{r',1}| \le \frac{2N^\epsilon}{q_t^{r'}}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)^{3/2}|P'||P|^{2D-2}\Big] \le 2\Phi_\epsilon, \tag{A.57}$$

for N sufficiently large, where we used that $\partial_{i_1i_2}^{r'-1}(G_{i_2\alpha}G_{i_1i_1}G_{\alpha\alpha}), r' \ge 1$, contains at least one off-diagonal Green function entry.

For $2 \le u' \le r'$, we first note that, for N sufficiently large,

$$\begin{aligned}|\mathbb{E}\hat{I}_{r',u'}| &\le \frac{N^\epsilon}{q_t^{r'}}\Big|\mathbb{E}\Big[\frac{1}{N^3}\sum_{i_1,i_2,\alpha}(\partial_{i_1i_2}^{r'-u'}(G_{i_2\alpha}G_{i_1i_1}G_{\alpha\alpha}))(\partial_{i_1i_2}^{u'}(P^{D-1}\overline{P^D}))\Big]\Big|\\ &\le \frac{N^\epsilon}{q_t^{r'}}\Big|\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)^{1/2}\frac{1}{N^2}\sum_{1\le i_1,i_2\le N}|\partial_{i_1i_2}^{u'}(P^{D-1}\overline{P^D})|\Big].\end{aligned}\tag{A.58}$$

Since $u' \le 2$, the partial derivative in $\partial_{i_1i_2}^{u'}(P^{D-1}\overline{P^D})$ acts on P and $\overline{P}$ (and on their derivatives) more than once. For example, for $u' = 2$,

$$\begin{aligned}\partial_{i_1i_2}^2 P^{D-1} &= \frac{4(D-1)(D-2)}{N}\Big(\sum_{i_3}G_{i_3i_2}G_{i_1i_3}\Big)^2(P')^2P^{D-3}\\ &+\frac{2(D-1)}{N}\Big(\sum_{i_3}G_{i_3i_2}G_{i_1i_3}\Big)^2P''P^{D-2}-\frac{2(D-1)}{N}\sum_{i_3}G_{i_3i_2}G_{i_1i_3}P'P^{D-2},\end{aligned}\tag{A.59}$$

where $\partial_{i_1i_2}$ acted twice on P, respectively P', to produce the first two terms. More generally, for $u' \ge 2$, consider a resulting term containing

$$P^{D-u_1'}\overline{P^{D-u_2'}}(P')^{u_3'}(\overline{P'})^{u_4'}(P'')^{u_5'}(\overline{P''})^{u_6'}(P''')^{u_7'}(\overline{P'''})^{u_8'} \tag{A.60}$$

with $1 \le u_1' \le D, 0 \le u_2' \le D$ and $\sum_{n=1}^{8} u_n' \le u'$. Note that $P^{(4)}$ is constant hence we do not list it. We find that such a term above was generated from $P^{D-1}\overline{P^D}$ by letting the partial derivative $\partial_{i_1i_2}$ act $u_1'-1$-times on P and u_2'-times on $\overline{P}$, which implies that $u_1'-1 \ge u_3'$ and $u_2' \ge u_4'$. If $u_1'-1 > u_3'$, then $\partial_{i_1i_2}$ acted on the derivatives of $P, \overline{P}$ directly $(u_1'-1-u_3')$-times, and a similar argument holds for $\overline{P'}$. Whenever $\partial_{i_1i_2}$ acted on $P, \overline{P}$ or their derivatives, it generated a term $2N^{-1}\sum_{i_l=1}G_{i_1i_l}G_{i_li_2}$, with $i_l, l \ge 3$, a fresh summation index. For each fresh summation index, we apply Lemma A.1 to gain a factor $(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2})$. The total number of fresh summation indicies in a term corresponding to (A.60) is

$$u_1'+u_2'+(u_1'-u_3')+(u_2'-u_4') = 2u_1'+2u_2'-u_3'-u_4' = 2\widetilde{s}_0-\widetilde{s}-2, \tag{A.61}$$

with $\widetilde{u}_0 := u_1'+u_2'$ and $\widetilde{u} := s_3+s_4$. We note this numbers do not decrease when $\partial_{i_1i_2}$ acts on off-diagonal Green functions entries later. Thus we conclude, upon using $|G_{\alpha i}|, |P''(m)|, |P'''(m)|$, $|P^{(4)}(m)| \prec 1$, that, for $2 \le u' \le r'$,

$$|\mathbb{E}\hat{I}_{r',u'}| \le \frac{N^{\epsilon}}{q_t^{r'}}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)^{1/2}\frac{1}{N^2}\sum_{i_1,i_2}|\partial_{i_1 i_2}^{u'}(P^{D-1}\overline{P^D})|\Big]$$
$$\le \frac{N^{2\epsilon}}{q_t^{r'}}\sum_{\widetilde{u}_0=2}^{2D}\sum_{\widetilde{s}=1}^{\widetilde{u}_0-2}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)^{1/2+2\widetilde{u}_0-\widetilde{u}-2}|P'|^{\widetilde{s}}|P|^{2D-\widetilde{u}_0}\Big]$$
$$+\frac{N^{2\epsilon}}{q_t^{r'}}\sum_{\widetilde{u}_0=2}^{2D}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)^{1/2+2\widetilde{u}_0-1}|P'|^{\widetilde{s}_0-1}|P|^{2D-\widetilde{s}_0}\Big], \tag{A.62}$$

for N sufficiently large. Here the last term on the right corresponds to $\widetilde{u}=\widetilde{u}_0-1$. Thus, we conclude the form (A.62) and the definition of Φ_ϵ in (A.1) that $\mathbb{E}[\hat{I}_{r',u'}], 2\le u'\le r'$, is negligible.

To sum up, we have established that all terms $\mathbb{E}[\widetilde{I}_{r',u'}]$ with $1\le u'\le r'$ are negligible.

From (A.44), (A.46), (A.53), (A.55), (A.57) and (A.62) We find that

$$|1+\widetilde{m}||\mathbb{E}I_{2,0}^{(1)}| \le \frac{N^{\epsilon}}{q_t^2}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)^{1/2}|P|^{2D-1}\Big]+\Phi_\epsilon, \tag{A.63}$$

for N sufficiently large. Since $|1+\widetilde{m}|>c$ as proved in (3.23), we obtain that $|\mathbb{E}I_{2,0}^{(1)}|\le\Phi_\epsilon$. This shows (A.40). □

Summarizing, we showed in (A.39) and (A.40) that

$$|\mathbb{E}I_{2,0}|\le\Phi_\epsilon, \tag{A.64}$$

for N sufficiently large, i.e., all terms in $\mathbb{E}I_{2,0}$ are negligible.

A.5. **Estimate on $I_{r,0}, r\ge 4$.** For $r\ge 5$ we use the bounds $|G_{\alpha\alpha}|, |G_{\alpha i}|\prec 1$ to get

$$|\mathbb{E}I_{r,0}| = \Big|N\kappa^{(r+1)}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}(\partial_{\alpha i}^r G_{\alpha i})P^{D-1}\overline{P}^D\Big]\Big|$$
$$\le \frac{N^{\epsilon}}{q_t^4}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}|P|^{2D-1}\Big]\le\frac{N^{\epsilon}}{q_t^4}\frac{1}{d}\mathbb{E}[|P|^{2D-1}]\le\Phi_\epsilon, \tag{A.65}$$

for N sufficiently large. For $r=4$, $\partial_{\alpha i}^r G_{\alpha i}$ contains at least one off-diagonal term $G_{\alpha i}$, thus

$$|\mathbb{E}I_{4,0}| = \Big|N\kappa^{(5)}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}(\partial_{\alpha i}^4 G_{\alpha i})P^{D-1}\overline{P}^D\Big]\Big|$$
$$\le \frac{N^{\epsilon}}{q_t^3}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}|G_{\alpha i}||P|^{2D-1}\Big]$$
$$\le \frac{N^{\epsilon}}{q_t^3}\frac{1}{d}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)^{1/2}|P|^{2D-1}\Big]\le\Phi_\epsilon, \tag{A.66}$$

for N sufficiently large, where we used Lemma A.1 to get last line. We conclude that all terms $\mathbb{E}I_{r,0}$ with $r\le 4$ are negligible. This shows the fourth estimate in (A.28).

A.6. **Estimate on $I_{r,u}, r\ge 2, u\ge 1$.** For $r\ge 2$ and $u=1$, we have

$$\mathbb{E}I_{r,1} = N\kappa_t^{(r+1)}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}(\partial_{\alpha i}^{r-1}G_{\alpha i})\partial_{\alpha i}(P^{D-1}\overline{P^D})\Big]. \tag{A.67}$$

Note that each term in $\mathbb{E}I_{r,1}, r\ge 2$, contains at least two off-diagonal Green function entries. For the terms with at least three off-diagonal Green function entries, we use the bound $|G_{\alpha i}|, |G_{\alpha\alpha}|\prec 1$ and

$$N\kappa_t^{r+1}\mathbb{E}\Big[\frac{1}{N^3}\sum_{i_1,i_2,\alpha}|G_{\alpha i_1}G_{i_1 i_2}G_{i_2\alpha}||P'||P|^{2D-2}\Big]\le N^{\epsilon}\frac{s^{(r+1)}}{q_t}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)^{3/2}|P'||P|^{2D-2}\Big]$$
$$\le N^{\epsilon}s^{(r+1)}\mathbb{E}\Big[\sqrt{\operatorname{Im} m}\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)\Big(\frac{1}{N\eta}+q_t^{-2}\Big)|P'||P|^{2D-2}\Big], \tag{A.68}$$

for N sufficiently large, where we used Lemma A.1. The right side of (A.68) is negligible since $\operatorname{Im} m \prec 1$.

Denoting the terms with two off-diagonal Green function entries in $\mathbb{E}I_{r,1}$ by $\mathbb{E}I_{r,1}^{(2)}$, we have

$$\mathbb{E}I_{r,1}^{(2)} = N\kappa_t^{(r+1)}\mathbb{E}\Big[\frac{2(D-1)}{N^2}\sum_{i_1,\alpha} G_{\alpha\alpha}^{r/2}G_{i_1i_1}^{r/2}\Big(\frac{1}{N}\sum_{i_2=1}^{N} G_{\alpha i_2}G_{i_2i_1}\Big)P'P^{D-2}\overline{P^D}\Big]$$
$$+ N\kappa_t^{(r+1)}\mathbb{E}\Big[\frac{2D}{N^2}\sum_{i_1,\alpha} G_{\alpha\alpha}^{r/2}G_{i_1i_1}^{r/2}\Big(\frac{1}{N}\sum_{i_2=1}^{N} \overline{G_{\alpha i_2}G_{i_2i_1}}\Big)P'P^{D-2}\overline{P^D}\Big], \qquad (A.69)$$

where i_2 is a fresh summation index and where we noted that r is necessarily even in this case. Using Lemma A.1, we get the upper bound

$$|\mathbb{E}I_{r,1}^{(2)}| \le \frac{N^\epsilon}{q_t^{r-1}}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)|P'||P|^{2D-2}\Big], \qquad (A.70)$$

which is negligible for $r > 2$. For $r = 2$, we need to gain an additional factor q_t^{-1}. This can be done as in the proof of Lemma A.7 by considering the off-diagonal entries $G_{\alpha i_2}G_{i_2i_1}$, generated from $\partial_{\alpha i}P(m)$, since the index α appears an odd number of times.

Lemma A.10. *For any (small) $\epsilon > 0$, we have*

$$|\mathbb{E}I_{2,1}^{(2)}| \le \frac{N^\epsilon}{q_t^{2}}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)|P'||P|^{2D-2}\Big] + \Phi_\epsilon, \qquad (A.71)$$

uniformly on $\mathcal{E}$, for N sufficiently large. In particular, the term $\mathbb{E}I_{2,1}$ is negligible.

Proof. We start with the first term on the right side of (A.69). Using (A.43), we write

$$zN\kappa_t^{(3)}\mathbb{E}\Big[\frac{1}{N^3}\sum_{i_1,i_2,\alpha} G_{\alpha i_2}G_{\alpha\alpha}G_{i_1i_1}G_{i_1i_2}P'P^{D-2}\overline{P^D}\Big]$$
$$= N\kappa_t^{(3)}\mathbb{E}\Big[\frac{1}{N^3}\sum_{i_1,i_2,\alpha_1,\alpha_2} H_{\alpha_1\alpha_2}G_{\alpha_2 i_2}G_{\alpha_1\alpha_1}G_{i_1i_1}G_{i_1i_2}P'P^{D-2}\overline{P^D}\Big]. \qquad (A.72)$$

As in the proof of Lemma A.7, we apply the cumulant expansion to the right side. The leading terms of the expansion is

$$N\kappa_t^{(3)}\mathbb{E}\Big[\frac{1}{N^3}\sum_{i_1,i_2,\alpha_1} mG_{\alpha_1 i_2}G_{\alpha_1\alpha_1}G_{i_1i_1}G_{i_1i_2}P'P^{D-2}\overline{P^D}\Big], \qquad (A.73)$$

and, thanks to the additional factor of q_t^{-1} from the cumulant $\kappa_t^{(3)}$, all other terms in the cumulant expansion are negligible, as can be checked by power counting as in the proof of Lemma A.7. Replacing m by $\widetilde{m}$ in (A.73), we then get

$$|1+\widetilde{m}||N\kappa_t^{(3)}|\Big|\mathbb{E}\Big[\frac{1}{N^3}\sum_{i_1,i_2,\alpha_1} mG_{\alpha_1 i_2}G_{\alpha_1\alpha_1}G_{i_1i_1}G_{i_1i_2}P'P^{D-2}\overline{P^D}\Big]\Big| \le C\Phi_\epsilon,$$

for N sufficiently large; see (A.63). Since $|1+\widetilde{m}| \ge c$ as in (3.23), we conclude that the first term on the right side of (A.69) is negligible. In the same way, we can also show that the second term is negligible as well. We omit the detail. □

We conclude from (A.68) and (A.71) that $\mathbb{E}_{r,1}$ is negligible for all $r \ge 2$.

Next, consider the terms

$$\mathbb{E}I_{r,u} = N\kappa_t^{(r+1)}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}(\partial_{\alpha i}^{r-u}G_{\alpha i})\partial_{\alpha i}^{u}(P^{D-1}\overline{P^D})\Big], \qquad (A.74)$$

with $2 \le u \le r$. We proceed in a similar way as in Remark A.8. Note that each term in $\partial_{\alpha i}^{r-u}G_{\alpha i}$ contains at least one off-diagonal Green function entries when $r-u$ is even, yet when $r-u$ is odd there is a term with no off-diagonal entries. Since $u \ge 2$, the partial derivative $\partial_{\alpha i}^{u}$ acts on P or $\overline{P}$ (or their derivatives) more than once in total; *cf.* Remark A.8. Consider such a term with

$$P^{D-u_1}\overline{P^{D-u_2}}(P')^{u_3}(\overline{P'})^{u_4},$$

for $1 \le u_1 \le D$ and $0 \le u_2 \le D$. Since $P''(m), P'''(m), P^{(4)}(m) \prec 1$ and $P^{(5)} = 0$, we do not include derivatives of order two of higher here. We see that such a term was generated from $P^{D-1}\overline{P^D}$ by letting the partial derivative $\partial_{\alpha i}$ act (u_1-1)-times on P and u_2-times on $\overline{P}$, which implies that $u_3 \le u_1 - 1$ and $u_4 \le u_2$. If $u_3 < u_1 - 1$, then $\partial_{\alpha i}$ acted on P' as well $[(u_1-1)-u_3]$-times, and a similar argument holds for $\overline{P'}$. Whenever $\partial_{\alpha i}$ acts on P or $\overline{P}$ (or their derivatives), it generates a fresh summation index i_l, $l \ge 3$, with a term $2N^{-1}\sum_{i_l} G_{\alpha i_l} G_{i_l i}$. The total number of fresh summation indices in this case is

$$(u_1-1) + u_2 + [(u_1-1)-u_3)] + [u_2 - u_4] = 2u_1 + 2u_2 - u_3 - u_4 - 2.$$

Assume first that $r = u$ so that $\partial_{\alpha i}^{r-u} = G_{\alpha i}$. Then applying Lemma A.1 $(2u_1+2u_2-u_3-u_4-2)$-times and letting $u_0 = u_1 + u_2$ and $u' = u_3 + u_4$, we obtain an upper bound for $r = u \ge 2$,

$$|\mathbb{E} I_{r,r}| \le \frac{N^\epsilon}{q_t^{r-1}} \sum_{u_0=2}^{2D}\sum_{u'=1}^{u_0-1} \mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)^{1/2}\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)^{2u_0-u'-2} |P'|^{u'}|P|^{2D-u_0}\Big] \le \Phi_\epsilon, \quad (A.75)$$

for N sufficiently large, *i.e.* $\mathbb{E} I_{r,r}$ is negligible for $r \ge 2$.

Next, assume that $2 \le u < r$. Then applying Lemma A.1 $(2u_1+2u_2-u_3-u_4-2)$-times, we get

$$\begin{aligned} |\mathbb{E} I_{r,u}| \le \frac{N^\epsilon}{q_t^{r-1}} \sum_{u_0=2}^{2D}\sum_{u'=1}^{u_0-1} \mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)^{2u_0-u'-2} |P'|^{u'}|P|^{2D-u_0}\Big] \\ + \frac{N^\epsilon}{q_t^{r-1}} \sum_{u_0=2}^{2D} \mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)^{u_0-1} |P'|^{u_0-1}|P|^{2D-u_0}\Big], \end{aligned} \quad (A.76)$$

for N sufficiently large with $2 \le u < r$. In particular, $|\mathbb{E} I_{r,u}| \le \Phi_\epsilon$, $2 \le u < r$. In (A.76) the second term bounds the terms corresponding to $u_0 - 1 = u'$ obtained by acting on $\partial_{\alpha i}$ exactly (u_1-1)-times on P and u_2-times on $\overline{P}$ but never on their derivatives.

To sum up, we showed that $\mathbb{E} I_{r,u}$ is negligible for $1 \le u < r$. This proves (A.30) for $1 \le u < r$.

A.7. **Estimate on $I_{3,0}$.** We notice that $I_{3,0}$ contains terms with zero, two or four off-diagonal Green function entries and we split accordingly

$$w_{I_{3,0}} I_{3,0} = w_{I_{3,0}^{(0)}} I_{3,0}^{(0)} + w_{I_{3,0}^{(2)}} I_{3,0}^{(2)} + w_{I_{3,0}^{(4)}} I_{3,0}^{(4)}.$$

When there are two off-diagonal entries, we can use Lemma A.1 to get the bound

$$\begin{aligned} |\mathbb{E} I_{3,0}^{(2)}| &= \Big| N\kappa_t^{(4)} \mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha} G_{ii}G_{\alpha\alpha}(G_{\alpha i})^2 P^{D-1}\overline{P}^D\Big]\Big| \\ &\le \frac{N^\epsilon}{q_t^2}\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta} + \frac{N-M}{N^2}\Big)|P|^{2D-1}\Big] \le \Phi_\epsilon, \end{aligned}$$

for N sufficiently large. A similar estimate holds for $|\mathbb{E} I_{3,0}^{(4)}|$. The only non-negligible term is $I_{3,0}^{(0)}$. Let

$$S_N \equiv S_N(z) := \frac{1}{N}\sum_{i=1}^{N}(G_{ii}(z))^2, \quad S_M \equiv S_N(z) := \frac{1}{N}\sum_{\alpha=N+1}^{M+N}(G_{\alpha\alpha}(z))^2. \quad (A.77)$$

Lemma A.11. *We have*

$$w_{I_{3,0}^{(0)}} I_{3,0}^{(0)} = -N\kappa_t^{(4)} \mathbb{E}\big[S_N^2 P^{D-1}\overline{P^D}\big]. \quad (A.78)$$

Proof. Recalling the definition of $I_{r,s}$ in (A.5), we get

$$w_{I_{3,0}} I_{3,0} = \frac{N\kappa_t^{(4)}}{3!}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}(\partial_{\alpha i}^3 G_{\alpha i})P^{D-1}\overline{P}^D\Big].$$

We then note that the terms without off-diagonal entries in $\partial_{\alpha i}^3 G_{\alpha i}$ are of the form $-G_{\alpha\alpha}G_{ii}G_{\alpha\alpha}G_{ii}$. We only have to determine the weight $w_{I_{3,0}^{(0)}}$. With regard to the indices, taking the third derivative corresponds to distributing the indices αi or $i\alpha$ thrice each. In this sense, the very first α and the very last i are from the original $G_{\alpha i}$. The choice of αi or $i\alpha$ must be exact in the sense that the connected indices in the following diagram must have been put at the same time:

$$\alpha\underbrace{\alpha\quad i}\underbrace{i\quad \alpha}\underbrace{\alpha\quad i}i.$$

Hence the only combinatorial factor we have to count is the order of distributing the indices. In this case, we have three connected indices, so the number of terms is $3! = 6$. Thus, $w_{I_{3,0}}I_{3,0} = 1$ and (A.78) holds. ☐

Lemma A.12. *For any (small) $\epsilon > 0$, we have, for all $z \in \mathcal{E}$,*

$$N\kappa_t^{(4)}\mathbb{E}\big[S_N S_M P^{D-1}\overline{P^D}\big] = N\kappa_t^{(4)}\mathbb{E}[m^2(zm+1-\frac{1}{d})^2 P^{D-1}\overline{P^D}] + O(\Phi_\epsilon). \tag{A.79}$$

Proof. Fix $\epsilon > 0$. We first claim that

$$N\kappa_t^{(4)}\mathbb{E}\big[S_N S_M P^{D-1}\overline{P^D}\big] = N\kappa_t^{(4)}\mathbb{E}[m^2 S_M P^{D-1}\overline{P^D}] + O(\Phi_\epsilon). \tag{A.80}$$

Using the resolvent identity (A.43) and Lemma 2.11 we get

$$\begin{aligned}\mathbb{E}\Big[zmS_N S_M P^{D-1}\overline{P}^D\Big] &= -\mathbb{E}\Big[S_N S_M P^{D-1}\overline{P}^D\Big] + \mathbb{E}\Big[\frac{1}{N}\sum_{i,\alpha} X_{\alpha i}G_{\alpha i}S_N S_M P^{D-1}\overline{P}^D\Big]\\ &= -\mathbb{E}\Big[S_N S_M P^{D-1}\overline{P}^D\Big] + \sum_{r=1}^{\ell'}\frac{N\kappa_t^{(r+1)}}{r!}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}\partial_{\alpha i}^r\big(G_{\alpha i}S_N S_M P^{D-1}\overline{P}^D\big)\Big]\\ &\quad + \mathbb{E}\Big[\Omega_{\ell'}\Big(\frac{1}{N}\sum_{i,\alpha}X_{\alpha i}G_{\alpha i}S_N S_M P^{D-1}\overline{P}^D\Big)\Big],\end{aligned} \tag{A.81}$$

for arbitrary $\ell' \in \mathbb{N}$. Using the resolvent identity (A.43) once more, we write

$$zS_N = \frac{1}{N}\sum_{i=1}^N zG_{ii}G_{ii} = -\frac{1}{N}G_{ii} + \frac{1}{N}\sum_{i,\alpha}X_{\alpha i}G_{\alpha i}G_{ii}.$$

Thus, using Lemma 2.11, we also have

$$\begin{aligned}\mathbb{E}\Big[zmS_N S_M P^{D-1}\overline{P}^D\Big] &= -\mathbb{E}\Big[m^2 S_M P^{D-1}\overline{P}^D\Big] + \mathbb{E}\Big[\frac{1}{N}\sum_{i,\alpha} X_{\alpha i}G_{\alpha i}G_{ii}mS_M P^{D-1}\overline{P}^D\Big]\\ &= -\mathbb{E}\Big[m^2 S_M P^{D-1}\overline{P}^D\Big] + \sum_{r=1}^{\ell'}\frac{N\kappa_t^{(r+1)}}{r!}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha}\partial_{\alpha i}^r\big(G_{\alpha i}G_{ii}mS_M P^{D-1}\overline{P}^D\big)\Big]\\ &\quad + \mathbb{E}\Big[\Omega_{\ell'}\Big(\frac{1}{N}\sum_{i,\alpha}X_{\alpha i}G_{\alpha i}G_{ii}S_M P^{D-1}\overline{P}^D\Big)\Big],\end{aligned} \tag{A.82}$$

for arbitrary $\ell' \in \mathbb{N}$. By Corollary A.3 and Remark A.4, The two error terms $\mathbb{E}[\Omega_{\ell'}(\cdot)]$ in (A.81) and (A.82) are negligible for $\ell' \geq 8D$.

With the extra factor $N\kappa^{(4)}$, we write

$$\begin{aligned}N\kappa_t^{(4)}\mathbb{E}\Big[zmS_N S_M P^{D-1}\overline{P}^D\Big] &= -N\kappa_t^{(4)}\mathbb{E}\Big[S_N S_M P^{D-1}\overline{P}^D\Big] + \sum_{r=1}^{\ell'}\sum_{s=0}^{r} w_{\widetilde{I}_{r,s}}\mathbb{E}\widetilde{I}_{r,s} + O(\Phi_\epsilon),\\ N\kappa_t^{(4)}\mathbb{E}\Big[zmS_N S_M P^{D-1}\overline{P}^D\Big] &= -N\kappa_t^{(4)}\mathbb{E}\Big[m^2 S_M P^{D-1}\overline{P}^D\Big] + \sum_{r=1}^{\ell'}\sum_{s=0}^{r} w_{\widetilde{\widetilde{I}}_{r,s}}\mathbb{E}\widetilde{\widetilde{I}}_{r,s} + O(\Phi_\epsilon),\end{aligned} \tag{A.83}$$

with

$$\begin{aligned}\widetilde{I}_{r,s} &:= N\kappa_t^{(4)}N\kappa_t^{(r+1)}\frac{1}{N^2}\sum_{i,\alpha}\big(\partial_{\alpha i}^{r-s}(G_{\alpha i}S_N S_M)\big)\big(\partial_{\alpha i}^s(P^{D-1}\overline{P}^D)\big),\\ \widetilde{\widetilde{I}}_{r,s} &:= N\kappa_t^{(4)}N\kappa_t^{(r+1)}\frac{1}{N^2}\sum_{i,\alpha}\big(\partial_{\alpha i}^{r-s}(G_{\alpha i}G_{ii}S_M)\big)\big(\partial_{\alpha i}^s(P^{D-1}\overline{P}^D)\big),\end{aligned} \tag{A.84}$$

and $w_{\widetilde{I}_{r,s}} = w_{\widetilde{\widetilde{I}}_{r,s}} = \frac{1}{r!(r-s)!}$.

For $r=1, s=0$, we find that

$$\begin{aligned}\mathbb{E}\widetilde{I}_{1,0} &= -N\kappa_t^{(4)}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha} G_{\alpha\alpha}G_{ii}S_N S_M P^{D-1}\overline{P}^D\Big] + O(\Phi_\epsilon)\\ &= -N\kappa_t^{(4)}\mathbb{E}\Big[\frac{1}{d}m\big(zm+1-\frac{1}{d}\big)S_N S_M P^{D-1}\overline{P}^D\Big] + O(\Phi_\epsilon),\end{aligned} \tag{A.85}$$

and similarly,

$$\begin{aligned}\mathbb{E}\widetilde{\widetilde{I}}_{1,0} &= -N\kappa_t^{(4)}\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha} G_{\alpha\alpha}G_{ii}^2 S_N S_M P^{D-1}\overline{P}^D\Big] + O(\Phi_\epsilon)\\ &= -N\kappa_t^{(4)}\mathbb{E}\Big[\frac{1}{d}m\big(zm+1-\frac{1}{d}\big)S_N S_M P^{D-1}\overline{P}^D\Big] + O(\Phi_\epsilon),\end{aligned} \tag{A.86}$$

where we used (A.77). We hence conclude that $\mathbb{E}\widetilde{I}_{1,0} = \mathbb{E}\widetilde{\widetilde{I}}_{1,0}$ up to negligible error. Following the ideas in Subsection A.4, also we can bound

$$\begin{aligned}|\mathbb{E}\widetilde{I}_{1,1}| &\le \frac{N^\epsilon}{q_t^2}\Big|\mathbb{E}\Big[\frac{1}{N^2}\sum_{i,\alpha} G_{\alpha i}S_N S_M\big(\partial_{\alpha i}\big(P^{D-1}\overline{P}^D\big)\Big]\Big|\\ &\le \frac{N^\epsilon}{q_t^2}\Big|\mathbb{E}\Big[\Big(\frac{\operatorname{Im} m}{N\eta}+\frac{N-M}{N^2}\Big)^{3/2}|P'||P|^{2D-2}\Big] \le \Phi_\epsilon,\end{aligned}$$

and similarly $|\mathbb{E}\widetilde{\widetilde{I}}_{1,1}| \le \Phi_\epsilon$, for N sufficiently large. In fact, for $r\le 2, u\le 0$ we can use, with small notational modifications the power counting outlined in Remark A.8 and Remark A.9 to conclude that

$$\mathbb{E}\widetilde{I}_{r,u} \le O(\Phi_\epsilon), \quad \mathbb{E}\widetilde{\widetilde{I}}_{r,u} \le O(\Phi_\epsilon).$$

Therefore the only non-negligible terms on the right hand side of (A.83) are $N\kappa_t^{(4)}\mathbb{E}[S_N S_M P^{D-1}\overline{P}^D]$, $N\kappa_t^{(4)}\mathbb{E}[m^2 S_M P^{D-1}\overline{P}^D]$ as well as $\widetilde{I}_{1,0}, \widetilde{\widetilde{I}}_{1,0}$. Since, by (A.85) and (A.86), the latter agree up to negligible error terms, we conclude that the former two must be equal up to do negligible error terms. Thus (A.80) holds. Similarly, expanding the term $\mathbb{E}[zm^3 S_M P^{D-1}\overline{P}^D]$ in two different ways to above, we get

$$N\kappa_t^{(4)}\mathbb{E}\Big[m^2 S_N S_M P^{D-1}\overline{P}^D\Big] = N\kappa^{(4)}\mathbb{E}\Big[m^2\big(zm+1-\frac{1}{d}\big)^2 P^{D-1}\overline{P}^D\Big]. \tag{A.87}$$

Together with (A.80) this shows (A.79) and concludes the proof of the lemma. □

Finally, from Lemma A.11 and Lemma A.12, we conclude that

$$w_{I_{3,0}}\mathbb{E}I_{3,0} = -\frac{s^{(4)}}{q_t^2}\mathbb{E}\Big[m^2\big(zm+1-\frac{1}{d}\big)^2 P^{D-1}\overline{P}^D\Big] + O(\Phi_\epsilon). \tag{A.88}$$

This proves the third estimate in (A.28).

Proof of Lemma A.5. The estimates in (A.28) and (A.30) were obtained in (A.32), (A.64), (A.65), (A.66) and (A.88). Estimate (A.30) were obtained in (A.34), (A.68), (A.75) and (A.76). □

B. Proof of Lemma 4.2 and Lemma 4.4

In this Section we prove Lemma 4.2 and Lemma 4.4.

B.1. **Proof of Lemma 4.2.**

Proof of Lemma 4.2. Recall the definition of α_1, α_2 and β in (4.2) and the definition of Λ_t in (4.3), where we for simplicity omit the z-dependence. Recall that $\Lambda_t \prec 1$ on $\mathcal{E}$ by Proposition 2.13 and that $\alpha_1 \le C|\alpha_2|$ by Lemma 3.1.

Let $D > 10$ and choose any small $\epsilon > 0$. For brevity, N is implicitly assumed to be sufficiently large. From Lemma 3.1, we can easily see that $C \operatorname{Im} \widetilde{m} > \eta$, thus

$$\begin{aligned} \operatorname{Im} m_t + \frac{N-M}{N^2} &\le \frac{1}{N\eta}\big(\operatorname{Im} m_t + (1-\frac{1}{d})\eta\big) \\ &\le \beta\big(\operatorname{Im}(m_t - \widetilde{m}_t) + \operatorname{Im}\widetilde{m}_t + (1-\frac{1}{d})\eta\big) \\ &\le C\beta(\alpha_1 + \Lambda_t). \end{aligned} \tag{B.1}$$

Using (B.1) and applying Young's inequality we get for the first term on the right side of (4.1) that

$$\begin{aligned} &N^\epsilon\Big(\frac{\operatorname{Im} m_t}{N\eta} + \frac{N-M}{N^2} + q_t^{-4}\Big)|P(m_t)|^{2D-1} \le N^\epsilon \frac{\alpha_1+\Lambda_t}{N\eta}|P(m_t)|^{2D-1} + N^\epsilon q_t^{-4}|P(m_t)|^{2D-1} \\ &\le \frac{N^{(2D+1)\epsilon}}{2D} C^{2D}\beta^{2D}(\alpha_1+\Lambda_t)^{2D} + \frac{N^{(2D+1)\epsilon}}{2D} q_t^{-8D} + \frac{2(2D-1)}{2D} N^{-\frac{\epsilon}{2D-1}}|P(m_t)|^{2D}. \end{aligned} \tag{B.2}$$

For the second term on the right hand side of (4.1), we have

$$N^{-\epsilon/8} q_t^{-1}\Lambda_t^2 |P(m_t)|^{2D-1} \le \frac{N^{-(D/4-1)\epsilon}}{2D} q_t^{-2D}\Lambda_t^{4D} + \frac{2D-1}{2D} N^{-\frac{\epsilon}{2D-1}}|P(m_t)|^{2D}. \tag{B.3}$$

Taylor expanding $P'(m_t)$ around $\widetilde{m}_t$, we get

$$|P'(m_t) - P'(\widetilde{m}_t) - P''(\widetilde{m}_t)(m_t - \widetilde{m}_t)| \le C q_t^{-2}\Lambda_t^2, \tag{B.4}$$

and $|P'(m_t)| \le |\alpha_2| + 3((1+\sqrt{1/d})^2+1)\Lambda_t \le |\alpha_2| + 15\Lambda_t$, for all $z \in \mathcal{E}$ with high probability. We note that, for any fixed $s \ge 2$,

$$\begin{aligned} (\alpha_1+\Lambda_t)^{2s-u'-2}(|\alpha_2|+15\Lambda_t)^{u'} &\le N^{\epsilon/2}(\alpha_1+\Lambda_t)^{s-1}(|\alpha_2|+15\Lambda_t)^{s-1} \\ &\le N^\epsilon(\alpha_1+\Lambda_t)^{s/2}(|\alpha_2|+15\Lambda_t)^{s/2} \end{aligned}$$

with high probability uniformly on $\mathcal{E}$, since $\alpha_1 \le C|\alpha_2| \le C$ and $\Lambda_t \prec 1$. In the third term of (4.1), note that $2s - u' - 2 \ge s$ since $u' \le s-2$. Hence, for $2 \le s \le 2D$,

$$\begin{aligned} &N^\epsilon q_t^{-1}\Big(\frac{\operatorname{Im} m_t}{N\eta} + \frac{N-M}{N^2}\Big)^{2s-u'-2}|P'(m_t)|^{u'}|P(m_t)|^{2D-s} \\ &\le N^\epsilon q_t^{-1}\beta^s(\alpha_1+\Lambda_t)^{2s-u'-2}(|\alpha_2|+15\Lambda_t)^{u'}|P(m_t)|^{2D-s} \\ &\le N^{2\epsilon} q_t^{-1} C^s\beta^s(\alpha_1+\Lambda_t)^{s/2}(|\alpha_2|+15\Lambda_t)^{s/2}|P(m_t)|^{2D-s} \\ &\le N^{2\epsilon} q_t^{-1}\frac{s}{2D} C^{2D}\beta^{2D}(\alpha_1+\Lambda_t)^D(|\alpha_2|+15\Lambda_t)^D + N^{2\epsilon}q_t^{-1}\frac{2D-s}{2D}|P(m_t)|^{2D} \end{aligned} \tag{B.5}$$

uniformly on $\mathcal{E}$ with high probability. For the last term, we note that

$$\frac{1}{N\eta} + q_t^{-1}\Big(\frac{\operatorname{Im} m_t}{N\eta}\Big)^{1/2} + q_t^{-2} \prec \beta \tag{B.6}$$

uniformly on $\mathcal{E}$ with high probability. Hence we find that, for $2 \le s \le 2D$,

$$\begin{aligned} &N^\epsilon\Big(\frac{1}{N\eta} + q_t^{-1}\Big(\frac{\operatorname{Im} m_t}{N\eta} + \frac{N-M}{N^2}\Big)^{1/2} + q_t^{-2}\Big)\Big(\frac{\operatorname{Im} m_t}{N\eta} + \frac{N-M}{N^2}\Big)^{s-1}|P'(m_t)|^{s-1}|P(m_t)|^{2D-s} \\ &\le N^{2\epsilon}C^s\beta\cdot\beta^{s-1}(\alpha_1+\Lambda_t)^{s/2}(|\alpha_2|+15\Lambda_t)^{s/2}|P(m_t)|^{2D-s} \\ &\le \frac{s}{2D}\Big(N^{2\epsilon}N^{\frac{(2D-s)\epsilon}{4D^2}}\Big)^{2D/s} C^{2D}\beta^{2D}(\alpha_1+\Lambda_t)^D(|\alpha_2|+15\Lambda_t)^D \\ &\quad + \frac{2D-s}{2D}\Big(N^{-\frac{(2D-s)\epsilon}{4D^2}}\Big)^{2D/(2D-s)}|P(m_t)|^{2D} \\ &\le N^{(2D+1)\epsilon}C^{2D}\beta^{2D}(\alpha_1+\Lambda_t)^D(|\alpha_2|+15\Lambda_t)^D + N^{-\epsilon/2D}|P(m_t)|^{2D}, \end{aligned} \tag{B.7}$$

for all $z \in \mathcal{E}$, with high probability. We thus get

$$\mathbb{E}[|P(m_t)|^{2D}] \leq CN^{(2D+1)\epsilon}\mathbb{E}[\beta^{2D}(\alpha_1+\Lambda_t)^D(|\alpha_2|+15\Lambda_t)^D] + \frac{N^{(2D+1)\epsilon}}{2D}q_t^{-8D} + \frac{N^{-(D/4-1)\epsilon}}{2D}q_t^{-2D}\mathbb{E}[\Lambda_t^{4D}] + CN^{-\epsilon/2D}\mathbb{E}[|P(m_t)|^{2D}], \tag{B.8}$$

for all $z \in \mathcal{E}$. Note that the last term on the right hand side of (B.8) may be absorbed into the left hand side of (B.8). Therefore,

$$\begin{aligned}&\mathbb{E}[|P(m_t)|^{2D}]\\ &\leq CN^{(2D+1)\epsilon}\mathbb{E}[\beta^{2D}(\alpha_1+\Lambda_t)^D(|\alpha_2|+15\Lambda_t)^D] + C\frac{N^{(2D+1)\epsilon}}{2D}q_t^{-8D} + C\frac{N^{-(D/4-1)\epsilon}}{2D}q_t^{-2D}\mathbb{E}[\Lambda_t^{4D}]\\ &\leq N^{3D\epsilon}\beta^{2D}|\alpha_2|^{2D} + N^{3D\epsilon}\beta^{2D}\mathbb{E}[\Lambda_t^{2D}] + N^{3D\epsilon}q_t^{-8D} + N^{-D\epsilon/8}q_t^{-2D}\mathbb{E}[\Lambda_t^{4D}],\end{aligned} \tag{B.9}$$

uniformly on $\mathcal{E}$, where we used that $D > 10$ and the inequality

$$(a+b)^{\mathrm{p}} \leq 2^{\mathrm{p}-1}(a^{\mathrm{p}}+b^{\mathrm{p}}), \tag{B.10}$$

for any $a, b \geq 0$ and $p \geq 1$, to get the second line.

From the third order Taylor expansion of $P(m_t)$ around $\widetilde{m}_t$, we get

$$\left|P(m_t) - \alpha_2(m_t-\widetilde{m}_t) - \frac{P''(\widetilde{m}_t)}{2}(m_t-\widetilde{m}_t)^2\right| \leq Cq_t^{-2}\Lambda_t^3 \tag{B.11}$$

since $P(\widetilde{m}_t) = 0$ and $P'''(\widetilde{m}_t) = 8\mathrm{e}^{-t}q_t^{-2}s^{(4)}(z^2\widetilde{m}_t + z(z\widetilde{m}_t + 1 - \frac{1}{d}))$. Then using $\Lambda_t \prec 1$ and $P''(\widetilde{m}_t) = 2z + O(q_t^{-2})$ we get

$$\Lambda_t^2 \prec 2|\alpha_2|\Lambda_t + 2|P(m_t)|, \quad (z \in \mathcal{E}). \tag{B.12}$$

Taking the $2D$-power of the inequality and using (B.10), we get after taking the expectation

$$\mathbb{E}[\Lambda_t^{4D}] \leq 4^{2D}N^{\epsilon/2}|\alpha_2|^{2D}\mathbb{E}[\Lambda_t^{2D}] + 4^{2D}N^{\epsilon/2}\mathbb{E}[|P(m_t)|^{2D}]. \quad (z \in \mathcal{E}) \tag{B.13}$$

Replacing from (B.9) for $\mathbb{E}[|P(m_t)|^{2D}|]$, for N sufficiently large,

$$\mathbb{E}[\Lambda_t^{4D}] \leq N^{\epsilon}|\alpha_2|^{2D}\mathbb{E}[\Lambda_t^{2D}] + N^{(3D+1)\epsilon}\beta^{2D}|\alpha_2|^{2D} + N^{(3D+1)\epsilon}\beta^{4D} + N^{(3D+1)\epsilon}q_t^{-8D} + N^{-D\epsilon/8+\epsilon}q_t^{-2D}\mathbb{E}[\Lambda_t^{4D}] \tag{B.14}$$

uniformly on $\mathcal{E}$. Applying the Schwarz inequality to the first and the third term on the right side, absorbing the terms $o(\mathbb{E}[\Lambda_t^{4D}])$ into the left side and using $q_t^{-2} \leq \beta$ in the fourth term, we arrive at

$$\mathbb{E}[\Lambda_t^{4D}] \leq N^{2\epsilon}|\alpha_2|^{4D} + N^{(3D+2)\epsilon}\beta^{2D}|\alpha_2|^{2D} + N^{(3D+2)\epsilon}\beta^{4D} \tag{B.15}$$

uniformly on $\mathcal{E}$. Feeding back, we obtain, for any $D > 10$ and small $\epsilon > 0$,

$$\begin{aligned}\mathbb{E}[|P(m_t)|^{2D}] &\leq N^{3D\epsilon}\beta^{2D}|\alpha_2|^{2D} + N^{3D\epsilon}\beta^{2D}\mathbb{E}[\Lambda_t^{2D}] + N^{(3D+1)\epsilon}\beta^{4D} + q_t^{-2D}|\alpha_2|^{4D}\\ &\leq N^{5D\epsilon}\beta^{2D}|\alpha_2|^{2D} + N^{5D\epsilon}\beta^{4D} + q_t^{-2D}|\alpha_2|^{4D},\end{aligned} \tag{B.16}$$

uniformly on $\mathcal{E}$, for n sufficiently large, where we used Schwarz inequality to get second line.

By the Markov inequality, we thus obtain that for fixed $z \in \mathcal{E}$, $|P(m_t)| \prec |\alpha_2|\beta + \beta^2 + q_t^{-1}|\alpha_2|^2$. From the Taylor expansion of $P(m_t)$ around $\widetilde{m}_t$ we get

$$|\alpha_2(m_t-\widetilde{m}_t) + z(m_t-\widetilde{m}_t)^2| \prec \beta\Lambda_t^2 + |\alpha_2|\beta + \beta^2 + q_t^{-1}|\alpha_2|^2, \tag{B.17}$$

for each fixed $z \in \mathcal{E}$, where we used that $q_t^{-2} \leq \beta$. To achieve a uniform bound on $\mathcal{E}$, we choose $18N^8$ lattice points $z_1, \ldots, z_{18N^8}$ in $\mathcal{E}$ such that, for any $\widetilde{z} \in \mathcal{E}$, there exists z_n satisfying $|\widetilde{z} - z_n| \leq N^{-4}$. Since

$$|m_t(\widetilde{z}) - m_t(z_n)| \leq |\widetilde{z} - z_n| \sup_{z\in\mathcal{E}}\left|\frac{\partial m_t(z)}{\partial z}\right| \leq |\widetilde{z} - z_n| \sup_{z\in\mathcal{E}}\frac{1}{(\operatorname{Im} z)^2} \leq N^{-2} \tag{B.18}$$

and since a similar estimate holds for $|\widetilde{m}_t(\widetilde{z}) - \widetilde{m}_t(z)|$, a union bound gives that (B.17) holds uniformly on $\mathcal{E}$ with high probability. This completes the prove of Lemma 4.2. □

B.2. **Proof of Lemma 4.4.** We start with the upper bound on the largest eigenvalue $\lambda_1^{X^\dagger X}$.

Lemma B.1. *Let X_0 satisfy Assumption 2.6 with $\phi > 0$. Let L_t be deterministic number defined in Lemma 3.1. Then,*

$$\lambda_1^{X^\dagger X} - L_t \prec \frac{1}{q_t^4} + \frac{1}{N^{2/3}}, \tag{B.19}$$

uniformly in $t \in [0, 6\log N]$.

Proof. Fix $t \in [0, 6\log N]$. Recall first the deterministic z- and t-dependent parameters α_1, α_2 and β from (4.2). We further introduce the z-independent quantity

$$\widetilde{\beta} := \Big(\frac{1}{q_t^4} + \frac{1}{N^{2/3}}\Big)^{1/2}. \tag{B.20}$$

We mostly drop the z- and t-dependence for brevity.

Fix a small $\epsilon > 0$ and define the domain $\mathcal{D}_\epsilon$ by

$$\mathcal{D}_\epsilon := \Big\{z = E + \mathrm{i}\eta : N^{4\epsilon}\widetilde{\beta}^2 \le \varkappa_t \le q_t^{-1/3}, \eta = \frac{N^\epsilon}{N\sqrt{\varkappa_t}}\Big\}, \tag{B.21}$$

where $\varkappa_t \equiv \varkappa_t(E) = E - L_t$. Note that on $\mathcal{D}_\epsilon$ for any sufficiently small ϵ,

$$N^{-1+\epsilon} \ll \eta \le \frac{N^{-\epsilon}}{N\widetilde{\beta}}, \quad \varkappa_t \ge N^{5\epsilon}\eta. \tag{B.22}$$

In particular we have $N^\epsilon\widetilde{\beta} \le (N\eta)^{-1}$, hence $N^\epsilon q_t^{-2} \le C(N\eta)^{-1}$ so that q_t^{-2} is negligible when compared to $(N\eta)^{-1}$ and β on $\mathcal{D}_\epsilon$. Note moreover that

$$\begin{aligned} |\alpha_2| &\sim \sqrt{\varkappa_t + \eta} \sim \sqrt{\varkappa_t} = \frac{N^\epsilon}{N\eta} \sim N^\epsilon\beta, \\ \alpha_1 &= \operatorname{Im}\widetilde{m}_t \sim \frac{\eta}{\sqrt{\varkappa_t + \eta}} \sim \frac{\eta}{\sqrt{\varkappa_t}} \le N^{-5\epsilon}\sqrt{\varkappa_t} \sim N^{-5\epsilon}|\alpha_2| \sim N^{-4\epsilon}\beta. \end{aligned} \tag{B.23}$$

In particular we have $\alpha_1 \ll |\alpha_2|$ on $\mathcal{D}_\epsilon$.

We next claim that

$$\Lambda_t := |m_t - \widetilde{m}_t| \ll \frac{1}{N\eta} \tag{B.24}$$

with high probability on the domain $\mathcal{D}_\epsilon$.

Since $\mathcal{D}_\epsilon \subset \mathcal{E}$, we find from Proposition 2.12 that $\Lambda_t \le N^{\epsilon'}\beta$ for any $\epsilon' > 0$ with high probability. Fix $0 < \epsilon' < \epsilon/9$. From (B.9) we get

$$\begin{aligned} &\mathbb{E}[|P(m_t)|^{2D}] \\ &\le CN^{(4D-1)\epsilon'}\mathbb{E}[\beta^{2D}(\alpha_1 + \Lambda_t)^D(|\alpha_2| + 15\Lambda_t)^D] + \frac{N^{(2D+1)\epsilon'}}{D}q_t^{-8D} + \frac{N^{-(D/4-1)\epsilon'}}{D}q_t^{-2D}\mathbb{E}[\Lambda_t^{4D}] \\ &\le C^{2D}N^{6D\epsilon'}\beta^{4D} + \frac{N^{(2D+1)\epsilon'}}{D}q_t^{-8D} + \frac{N^{4D\epsilon'}}{D}q_t^{-2D}\beta^{4D} \\ &\le C^{2D}N^{6D\epsilon'}\beta^{4D} \end{aligned}$$

for N sufficiently large, where we used that $\Lambda_t \le N^{\epsilon'}\beta \ll N^\epsilon\beta$ with high probability and, by (B.23), $\alpha_1 \ll |\alpha_2|, |\alpha_2| \le CN^\epsilon\beta$ on $\mathcal{D}_\epsilon$. Applying the Markov inequality and a simple lattice argument combined with a union bound, we get $|P(m_t)| \le CN^{4\epsilon'}\beta^2$ uniformly on $\mathcal{D}_\epsilon$ with high probability. From the Taylor expansion of $P(m_t)$ around $\widetilde{m}_t$ in (B.11), we then get that

$$|\alpha_2|\Lambda_t \le 5\Lambda_t^2 + CN^{4\epsilon'}\beta^2, \tag{B.25}$$

uniformly on $\mathcal{D}_\epsilon$ with high probability, where we also used that $\Lambda_t \ll 1$ on $\mathcal{D}_\epsilon$ with high probability.

Since $\Lambda_t \le N^{\epsilon'}\beta \le CN^{\epsilon'-\epsilon}|\alpha_2|$ with high probability on $\mathcal{D}_\epsilon$, we have $|\alpha_2|\Lambda_t \ge CN^{\epsilon-\epsilon'}\Lambda_t^2 \gg |z|\Lambda_t^2$. Thus the first term on the right side of (B.25) can be absorbed into the left side and we conclude that

$$\Lambda_t \le CN^{4\epsilon'}\frac{\beta}{|\alpha_2|}\beta \le CN^{4\epsilon'-\epsilon}\beta,$$

must hold with high probability on $\mathcal{D}_\epsilon$. Hence, using that $0 < \epsilon' < \epsilon/9$, we obtain that

$$\Lambda_t \le N^{-\epsilon/2}\beta \le 2\frac{N^{-\epsilon/2}}{N\eta},$$

with high probability on $\mathcal{D}_\epsilon$. This proves the claim that $\Lambda_t \ll (N\eta)^{-1}$ on $\mathcal{D}_\epsilon$ with high probability. Moreover, this also shows that

$$\operatorname{Im} m_t \le \operatorname{Im} \widetilde{m}_t + \Lambda_t = \alpha_1 + \Lambda_t \ll \frac{1}{N\eta}, \tag{B.26}$$

on $\mathcal{D}_\epsilon$ with high probability, where we used (B.23).

Now we prove the estimate (B.19). If $\lambda_1 \in [E-\eta, E+\eta]$ for some $E \in [L_t + N^\epsilon(q_t^{-4} + N^{-2/3}), L_t + q^{-1/3}]$ with $z = E + \mathrm{i}\eta \in \mathcal{D}_\epsilon$,

$$\operatorname{Im} m_t(z) \ge \frac{1}{N}\operatorname{Im}\frac{1}{\lambda_1 - E - \mathrm{i}\eta} = \frac{1}{N}\eta(\Lambda_1 - E)^2 + \eta^2 \ge \frac{1}{5N\eta}, \tag{B.27}$$

which contradicts the high probability bound $\operatorname{Im} m_t \ll (N\eta)^{-1}$ in (B.26). The size of each interval $[E-\eta, E+\eta]$ is at least $N^{-1+\epsilon}q_t^{1/6}$. Thus, considering $O(N)$ such intervals, we can conclude that $\lambda_1 \notin [L_t + N^\epsilon(q_t^{-4} + N^{-2/3}), L_t + q^{-1/3}]$ with high probability. From (2.56), we find that $\lambda_1 - L_t \prec q_t^{-1/3}$ with high probability, hence we conclude that (B.19) holds for fixed $t \in [0, 6\log N]$. Then we can obtain (B.19) uniformly in $t \in [0, 6\log N]$ by using a lattice argument and the continuity of the Dyson matrix flow. □

Proof of Lemma 4.4 and Theorem 2.9. Fix $t \in [0, \log N]$. For the largest eigenvalue $\lambda_1^{X_t^\dagger X_t}$, we already showed that $(L_t - \lambda_1^{X_t^\dagger X_t})_- \prec q_t^{-4} + N^{-2/3}$ in Lemma B.1. It thus suffices to consider $(L_t - \lambda_1^{X_t^\dagger X_t})_+$. By Lemma 3.1, there exists $c > 0$ such that $c(L_t - \lambda_1^{X_t^\dagger X_t})_+^{3/2} \le n_{\widetilde{\rho}_t}(\lambda_1^{X_t^\dagger X_t}, L_t)$. Hence by Corollary 2.8, we have the estimate

$$(L_t - \lambda_1^{X_t^\dagger X_t})_+^{3/2} \prec \frac{(L_t - \lambda_1^{X_t^\dagger X_t})_+}{q_t^2} + \frac{1}{N}, \tag{B.28}$$

so that $(L_t - \lambda_1^{X_t^\dagger X_t})_+ \prec q_t^{-4} + N^{-2/3}$. Thus $|\lambda_1^{X_t^\dagger X_t} - L_t| \prec q_t^{-4} + N^{-2/3}$. This proves (4.23) for fixed $t \in [0, 6\log N]$. Uniformity follows by the continuity of the Dyson matrix flow. □

C. Proof of Lemma 5.4

In this section, we prove Lemma 5.4. We begin by considering the case $r \ge 5$. In this case, we can see that $J_r = O(N^{\frac{2}{3}-\epsilon'})$, since it contains at least two off-diagonal entries in $\partial_{j\alpha}^r(F'(Y)G_{ij}G_{\alpha i})$ and $|J_r|$ is bounded by

$$N^3 N^{-1} q_t^{-4} N^{-2/3+2\epsilon} \ll N^{2/3-\epsilon'}$$

which can be checked by a simple power counting. Therefore, we only need to consider the cases $r = 2, 3, 4$.

C.1. Proof of Lemma 5.4 for $r = 2$.

Observe that

$$\partial_{j\alpha}(F'(Y)G_{ij}G_{\alpha i}) = F'(Y)\partial_{j\alpha}^2(G_{ij}G_{\alpha i}) + 2\partial_{j\alpha}F'(Y)\partial_{j\alpha}(G_{ij}G_{\alpha i}) + (\partial_{j\alpha}^2 F'(Y))G_{ij}G_{\alpha i}. \tag{C.1}$$

We first consider the expansion of $\partial_{j\alpha}^2(G_{ij}G_{\alpha i})$. We can estimate the terms with four off diagonal Green function entries, since, for example,

$$\sum_{i,j,\alpha} |\mathbb{E}[F'(Y)G_{ij}G_{\alpha j}G_{\alpha j}G_{\alpha i}]| \le N^{C\epsilon}\sum_{i,j,\alpha}|G_{ij}G_{\alpha j}G_{\alpha j}G_{\alpha i}| \le N^{C\epsilon}\Big(\frac{\operatorname{Im} m}{N\eta_0}\Big)^2 \le N^{-4/3+C\epsilon}, \tag{C.2}$$

where we used Lemma A.1. Thus, for sufficiently small ϵ and ϵ',

$$\frac{\mathrm{e}^{-t}q_t^{-1}}{N}\sum_{i,j,k}|\mathbb{E}[F'(Y)G_{ij}G_{\alpha j}G_{\alpha j}G_{\alpha i}]| \ll N^{2/3-\epsilon'}. \tag{C.3}$$

For the terms with three off-diagonal Green function entries, the bound we get from Lemma A.1 is

$$q_t^{-1}N^{-1}N^3N^{C\epsilon}\Big(\frac{\operatorname{Im} m}{N\eta_0}\Big)^{3/2} \sim q_t^{-1}N^{1+C\epsilon},$$

which is not sufficient. To gain an additional factor of q_t^{-1}, which makes the above bound $q_t^{-2}N^{1+C\epsilon} \ll N^{2/3-\epsilon'}$, we use Lemma 2.11 to expand in an unmatched index. For example, such a term is of the form

$$G_{ij}G_{\alpha j}G_{\alpha\alpha}G_{ji}$$

and we focus on the unmatched index α in $G_{\alpha j}$. We get

$$\begin{aligned}\frac{q_t^{-1}}{N}\sum_{i,j,\alpha}\mathbb{E}[F'(Y)G_{ij}G_{\alpha j}G_{\alpha\alpha}G_{ji}] &= \frac{q_t^{-1}}{N}\sum_{i,j,k,\alpha}\mathbb{E}[F'(Y)G_{ij}H_{\alpha k}G_{kj}G_{\alpha\alpha}G_{ji}] \\ &= \frac{q_t^{-1}}{N}\sum_{r'=1}^{\ell}\frac{\kappa^{(r'+1)}}{r'!}\sum_{i,j,k,\alpha}\mathbb{E}[\partial_{\alpha k}^{r'}(F'(Y)G_{ij}G_{kj}G_{\alpha\alpha}G_{ji})] + O(N^{2/3-\epsilon'}),\end{aligned} \tag{C.4}$$

for $\ell = 10$.

For $r' = 1$, we need to consider $\partial_{\alpha k}(F'(Y)G_{ij}G_{kj}G_{\alpha\alpha}G_{ji})$. When $\partial_{\alpha k}$ acts on $F'(Y)$ it creates a fresh summation index n, and we get a term

$$\begin{aligned}&\frac{q_t^{-1}}{N^2}\sum_{i,j,k,\alpha}\mathbb{E}[(\partial_{\alpha k}(F'(Y))G_{ij}G_{kj}G_{\alpha\alpha}G_{ji})] \\ &\quad = -\frac{2q_t^{-1}}{N^2}\int_{E_1}^{E_2}\sum_{i,j,k,n,\alpha}\mathbb{E}[G_{ij}G_{kj}G_{\alpha\alpha}G_{ji}F''(Y)\,\mathrm{Im}(G_{nk}(y+L+\mathrm{i}\eta_0)G_{\alpha n}(y+L+\mathrm{i}\eta_0))]\mathrm{d}y \\ &\quad = -\frac{2q_t^{-1}}{N^2}\int_{E_1}^{E_2}\sum_{i,j,k,n,\alpha}\mathbb{E}[G_{ij}G_{kj}G_{\alpha\alpha}G_{ji}F''(Y)\,\mathrm{Im}(\widetilde{G}_{nk}\widetilde{G}_{\alpha n})]\mathrm{d}y,\end{aligned} \tag{C.5}$$

where we abbreviate $\widetilde{G} \equiv G(y+L+\mathrm{i}\eta_0)$. Applying Lemma A.1 to the index n and $\widetilde{G}$, we get

$$\sum_{n=1}^{N}|\widetilde{G}_{kn}\widetilde{G}_{n\alpha}| \prec N^{-2/3+2\epsilon},$$

which also shows that

$$|\partial_{\alpha k}F'(Y)| \prec N^{-1/3+C\epsilon}. \tag{C.6}$$

Applying Lemma A.1 to the remaining off-diagonal Green function entries, we obtain that

$$\frac{q_t^{-1}}{N^2}\sum_{i,j,k,\alpha}|\mathbb{E}[(\partial_{\alpha k}(F'(Y))G_{ij}G_{kj}G_{\alpha\alpha}G_{ji})]| \le q_t^{-1}N^{-2}N^{-1/3+C\epsilon}N^4N^{-1+3\epsilon} = q_t^{-1}N^{2/3+C\epsilon}. \tag{C.7}$$

If $\partial_{\alpha k}$ acts on $G_{ij}G_{kj}G_{\alpha\alpha}G_{ji}$, then we always get four or more off-diagonal Green function entries with the only exception being

$$-G_{ij}G_{kk}G_{\alpha j}G_{\alpha\alpha}G_{ji}.$$

To the terms with four or more off-diagonal Green function entries, we apply Lemma A.1 and obtain a bound similar to (C.7) by power counting. For the term of the exception, we rewrite it as

$$\begin{aligned}&-\frac{q_t^{-1}}{N^2}\sum_{i,j,k,\alpha}\mathbb{E}[F'(Y)G_{ij}G_{kk}G_{\alpha j}G_{\alpha\alpha}G_{ji}] = -\frac{q_t^{-1}}{N}\sum_{i,j,\alpha}\mathbb{E}[mF'(Y)G_{ij}G_{\alpha j}G_{\alpha\alpha}G_{ji}] \\ &= -\widetilde{m}\frac{q_t^{-1}}{N}\sum_{i,j,\alpha}\mathbb{E}[F'(Y)G_{ij}G_{\alpha j}G_{\alpha\alpha}G_{ji}] + \frac{q_t^{-1}}{N}\sum_{i,j,\alpha}\mathbb{E}[(\widetilde{m}-m)F'(Y)G_{ij}G_{\alpha j}G_{\alpha\alpha}G_{ji}]\end{aligned} \tag{C.8}$$

Here, the last term is bounded by $q_t^{-1}N^{2/3+C\epsilon}$ as we can easily check with Proposition 2.12 and Lemma A.1. We thus arrive at

$$\begin{aligned}&\frac{q_t^{-1}}{N}(1+\widetilde{m})\sum_{i,j,\alpha}\mathbb{E}[F'(Y)G_{ij}G_{\alpha j}G_{\alpha\alpha}G_{ji}] \\ &\qquad = \frac{q_t^{-1}}{N}\sum_{r'=1}^{\ell}\frac{\kappa^{(r'+1)}}{r'!}\sum_{i,j,k,\alpha}\mathbb{E}[\partial_{\alpha k}^{r'}(F'(Y)G_{ij}G_{kj}G_{\alpha\alpha}G_{ji})] + O(N^{2/3-\epsilon'}).\end{aligned} \tag{C.9}$$

On the right side, the summation is from $r' = 2$, hence we have gained a factor $N^{-1}q_t^{-1}$ from $\kappa^{(r'+1)}$ and added a fresh summation index k, so the net gain is q_t^{-1}. Since $|1+\widetilde{m}| \sim 1$, this shows that

$$\frac{q_t^{-1}}{N}\sum_{i,j,\alpha}\mathbb{E}[F'(Y)G_{ij}G_{\alpha j}G_{\alpha\alpha}G_{ji}] = O(N^{2/3-\epsilon'}). \tag{C.10}$$

Together with (C.3), this takes care of the first term on the right side of (C.2). For the second term on the right side of (C.2), we focus on

$$\partial_{j\alpha}F'(Y) = -\int_{E_1}^{E_2}\sum_{n=1}^{N}[F''(Y)\,\mathrm{Im}(\widetilde{G}_{jn}\widetilde{G}_{n\alpha})]\mathrm{d}y \tag{C.11}$$

and apply the same argument to the unmatched index α in $\widetilde{G}_{n\alpha}$. For the third therm, we focus on $G_{ij}G_{\alpha i}$ and again apply the same argument with the index α in $G_{\alpha i}$.

C.2. **Proof of Lemma 5.4 for $r = 3$.** If $\partial_{j\alpha}$ acts on $F'(Y)$ at least once, then that term is bounded by

$$N^{\epsilon}N^{-1}q_t^{-2}N^3N^{-1/3+C\epsilon}N^{-2/3+2\epsilon} = q_t^{-2}N^{1+C\epsilon} \ll N^{2/3-\epsilon'},$$

where we used (C.6) and the fact that $G_{ij}G_{\alpha i}$ or $\partial_{jk}(G_{ij}G_{ki})$ contains at least two off-diagonal entries. Moreover, in the expansion $\partial_{jk}^2(G_{ij}G_{ki})$, the terms with three or more off-diagonal Green function entries can be bounded by

$$N^{\epsilon}N^{-1}q_t^{-2}N^3N^{C\epsilon}N^{-1+3\epsilon} = q_t^{-2}N^{1+C\epsilon} \ll N^{2/3-\epsilon'}.$$

Thus,

$$\begin{aligned}\frac{\mathrm{e}^{-t}s^{(4)}q_t^{-2}}{3!N}\sum_{i,j,\alpha}\mathbb{E}[\partial_{j\alpha}^3F'(Y)G_{ij}G_{\alpha i}] &= -\frac{4!}{2}\frac{\mathrm{e}^{-t}s^{(4)}q_t^{-2}}{3!}\sum_{i,j}\mathbb{E}[F'(Y)G_{ij}G_{jj}G_{ji}S_M]\\ &\quad -\frac{4!}{2}\frac{\mathrm{e}^{-t}s^{(4)}q_t^{-2}}{3!}\sum_{i,\alpha}\mathbb{E}[F'(Y)G_{i\alpha}G_{\alpha\alpha}G_{\alpha i}S_N] + O(N^{2/3-\epsilon'}),\end{aligned} \tag{C.12}$$

where the combinatorial factor $(4!/2)$ is computed as in Lemma A.11 and S_N, S_M are defined in (A.77). The first term on right side of (C.12) is computed by expanding

$$q_t^{-2}\sum_{i,j}\mathbb{E}[zmS_MF'(Y)G_{ij}G_{jj}G_{ji}]$$

in two different ways, respectively, as in Lemma A.12. Then we can obtain that

$$\begin{aligned}q_t^{-2}\sum_{i,j}\mathbb{E}[F'(Y)G_{ij}G_{jj}G_{ji}S_M] &= q_t^{-2}\sum_{i,j}\mathbb{E}\Big[\Big(zm+1-\frac{1}{d}\Big)^2F'(Y)G_{ij}G_{jj}G_{ji}\Big] + O(N^{2/3-\epsilon'})\\ &= q_t^{-2}\sum_{i,j}\mathbb{E}\Big[\frac{1}{d}\Big(1+\frac{1}{\sqrt{d}}\Big)^2F'(Y)G_{ij}G_{jj}G_{ji}\Big] + O(N^{2/3-\epsilon'}),\end{aligned} \tag{C.13}$$

where we used $m(z) = (-1-\frac{1}{d})^{-1} + O(N^{-1/3+\epsilon})$ with high probability. Next we consider

$$q_t^{-2}\sum_{i,j}\mathbb{E}[zF'(Y)G_{ij}G_{jj}G_{ji}] = \Big(1+\frac{1}{\sqrt{d}}\Big)^2q_t^{-2}\sum_{i,j}\mathbb{E}[F'(Y)G_{ij}G_{jj}G_{ji}] + O(N^{2/3-\epsilon'}). \tag{C.14}$$

Expanding the left hand side using the resolvent expansion (A.43), we obtain

$$q_t^{-2}\sum_{i,j}\mathbb{E}[zF'(Y)G_{ij}G_{jj}G_{ji}] = -q_t^{-2}\sum_{i,j}\mathbb{E}[F'(Y)G_{ij}G_{ji}] + q_t^{-2}\sum_{i,j}\mathbb{E}[F'(Y)X_{\alpha j}G_{ij}G_{\alpha j}G_{ji}].$$

Applying Lemma 2.11 to the second term on the right side, most of the terms are $O(N^{2/3-\epsilon'})$ either due to three (or more) off-diagonal entries, the partial derivative $\partial_{\alpha j}$ acting on $F'(Y)$, or higher cumulants. Thus we find that

$$-\frac{q_t^{-2}}{N^2}\sum_{i,j,\alpha}\mathbb{E}[F'(Y)G_{ij}G_{\alpha\alpha}G_{jj}G_{ji}]$$

is the only non-negligible term, which is generated when $\partial_{\alpha j}$ acts on $G_{\alpha j}$. From this argument we get

$$q_t^{-2}\sum_{i,j}\mathbb{E}[zF'(Y)G_{ij}G_{jj}G_{ji}] = -q_t^{-2}\sum_{i,j}\mathbb{E}[F'(Y)G_{ij}G_{ji}] - q_t^{-2}\sum_{i,j}\mathbb{E}\Big[\big(zm+1-\frac{1}{d}\big)F'(Y)G_{ij}G_{jj}G_{ji}\Big] + O(N^{2/3-\epsilon'}).$$

Combining with (C.13), (C.14) and the fact that $m(z) = (-1-\frac{1}{d})^{-1} + O(N^{-1/3+\epsilon})$ with high probability, we get

$$q_t^{-2}\sum_{i,j}\mathbb{E}[F'(Y)G_{ij}G_{jj}G_{ji}S_M] = q_t^{-2}\sum_{i,j}\mathbb{E}\Big[\Big(-\frac{1}{d}-\frac{1}{d\sqrt{d}}\Big)F'(Y)G_{ij}G_{ji}\Big] + O(N^{2/3-\epsilon'}). \tag{C.15}$$

For the second term of (C.12) we can apply similar argument, using an expansion of $q_t^{-2}\sum_{i,\alpha}\mathbb{E}[zmS_NF'(Y)G_{i\alpha}G_{\alpha\alpha}G_{\alpha i}]$ and $q_t^{-2}\sum_{i,\alpha}\mathbb{E}[F'(Y)G_{i\alpha}G_{\alpha\alpha}G_{\alpha i}]$. Then we obtain

$$\begin{aligned} q_t^{-2}\sum_{i,\alpha}\mathbb{E}[F'(Y)G_{i\alpha}G_{\alpha\alpha}G_{\alpha i}S_N] &= q_t^{-2}\sum_{i,\alpha}\mathbb{E}\Big[m^2F'(X)G_{i\alpha}G_{\alpha\alpha}G_{\alpha i}\Big] + O(N^{2/3-\epsilon'}) \\ &= q_t^{-2}\sum_{i,\alpha}\mathbb{E}[-(1+\sqrt{d})^{-1}F'(X)G_{i\alpha}G_{\alpha i}] + O(N^{2/3-\epsilon'}) \\ &= q_t^{-2}\sum_{i,j}\mathbb{E}\Big[-\Big(\frac{1}{\sqrt{d}}+\frac{1}{d}\Big)F'(Y)G_{ij}G_{ji}\Big] + O(N^{2/3-\epsilon'}), \end{aligned} \tag{C.16}$$

where we get the last line by the resolvent expansion (A.43) of $G_{\alpha i}$.

By combining (C.12), (C.15) and (C.16), we get

$$\frac{\mathrm{e}^{-t}s^{(4)}q_t^{-2}}{3!N}\sum_{i,j,\alpha}\mathbb{E}[\partial_{j\alpha}^3F'(Y)G_{ij}G_{\alpha i}] = 2\mathrm{e}^{-t}s^{(4)}q_t^{-2}\frac{1}{\sqrt{d}}\Big(1+\frac{1}{\sqrt{d}}\Big)^2\mathbb{E}[F'(Y)G_{ij}G_{ji}] + O(N^{2/3-\epsilon'}). \tag{C.17}$$

C.3. **Proof of Lemma 5.4 for $r = 4$.** We estimate the term as in the case $r = 2$ and one can get

$$\frac{q_t^-3}{N}\sum_{i,j,\alpha}\big|\mathbb{E}\big[\partial_{j\alpha}^4(F'(Y)G_{ij}G_{\alpha i})\big]\big| = O(N^{2/3-\epsilon'}). \tag{C.18}$$

We omit the proof.

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