**Robin Hartshorne’s Algebraic Geometry Solutions**  
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**CHAPTER II SECTION 7, PROJECTIVE MORPHISMS**

**7.1.**

**7.9.** Let $r + 1$ be the rank of $E$.

(a). There are several ways to prove it.

Proof 1 We assume the following result from Chow group theory: (See Appendix A section 2 A11 and section 3. The group $A(X)$ is here $CH(X)$.)

\[
CH^*(\mathbb{P}(E)) \cong \left( \mathbb{Z}[\xi]/\sum_{i=0}^{r} (-1)^i c_i(E) \xi^{r-i} \right) \otimes_{\mathbb{Z}} CH^*(X)
\]

as graded rings. If we look at the grade 1 part, as $\mathbb{Z}$-modules,

\[
CH^1(\mathbb{P}(E)) \cong (\mathbb{Z} \otimes_{\mathbb{Z}} CH^0(X)) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}} CH^1(X))
\]

and $CH^1(\mathbb{P}(X)) = Pic(-)$ so that $Pic(\mathbb{P}(E)) \cong \mathbb{Z} \oplus Pic(X)$ as desired.

Proof 2 We can use the Grothendieck groups, i.e. $K$-theory to do so. Note that

\[
K(\mathbb{P}(E)) \cong \left( \mathbb{Z}[\xi]/\sum_{i=0}^{r} (-1)^i c_i(E) \xi^{r-i} \right) \otimes_{\mathbb{Z}} K(X)
\]

as rings. For the detail, see Yuri Manin *Lectures on the K-functor in Algebraic Geometry*, Russian Mathematical Surveys, 24 (1969) 1-90, in particular, p. 44, from Prop (10.2) to Cor. (10.5).

Proof 3 Here we give a direct proof. In fact, it adapts a way from Proof 1. It can also use the method from Proof 2. Totally your choice.

Define a map $\phi : \mathbb{Z} \oplus Pic(X) \to Pic(\mathbb{P}(E))$ by $(n, L) \mapsto (\pi^*L)(n) := (\pi^*L) \otimes O_{\mathbb{P}(E)}(n)$.

**Claim.** This map is injective.

Assume that $\phi(n, L) = O_{\mathbb{P}(E)}$, i.e. $\pi^*L \otimes O_{\mathbb{P}(E)}(n) \cong O_{\mathbb{P}(E)}$. Apply $\pi_*$ to it. From II (7.11), recall the fact

\[
\pi_*(O_{\mathbb{P}(E)}(n)) = \begin{cases} 
0 & n < 0 \\
O_X & n = 0 \\
\text{Sym}^n(E) & n > 0
\end{cases}.
\]

So, by applying the projection formula (Ex. II (5.1)-(d)), we obtain, $L \otimes \pi_*O_{\mathbb{P}(E)}(n) \cong O_X$, i.e.

\[
\pi_*O_{\mathbb{P}(E)} \cong L^{-1}.
\]

Note that it is a line bundle and $\text{rk}(\text{Sym}^n(E)) \geq r + 1 \geq 2$ if $n > 0$ by the given assumption, so that the only possible choice for $n$ is $n = 0$. Then, it implies that $L \cong O_X$. Hence $\phi$ is injective.

**Claim.** This map is surjective.

In case $E$ is a trivial bundle, then $\mathbb{P}(E) \cong X \times \mathbb{P}^r$ so that we already know the result.
In general, choose an open subset $U \subset X$ over which $\mathcal{E}$ is trivial and let 
$Z = X - U$. Then, we have a closed immersion $\mathbb{P}(\mathcal{E}|_Z) \hookrightarrow \mathbb{P}(\mathcal{E})$ and an open immersion 
$\mathbb{P}(\mathcal{E}|_U) \hookrightarrow \mathbb{P}(\mathcal{E}|_U) \cong U \times \mathbb{P}^r$. Let $m = \dim X$. Then we have

\[
CH_{m+r-1}(\mathbb{P}(\mathcal{E}|_Z)) \xrightarrow{\phi_Z} \text{Pic}(\mathbb{P}(\mathcal{E})) \xrightarrow{\phi_X} \text{Pic}(\mathbb{P}(\mathcal{E}|_U)) \xrightarrow{\phi_U} 0.
\]

By induction on the dimension, $\phi_Z$ is surjective and we already know that $\phi_U$ is an isomorphism. Hence, by a simple diagram chasing, we have the surjectivity of $\phi_X$.

\(b\). Let $\pi : \mathbb{P}(\mathcal{E}) \to X$, $\pi' : \mathbb{P}^r(\mathcal{E}') \to X$ be the structure morphisms and let $\phi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ be the given isomorphism over $X$:

\[
\begin{array}{ccc}
\mathbb{P}(\mathcal{E}) & \xrightarrow{\phi} & \mathbb{P}(\mathcal{E}') \\
\downarrow \pi & & \downarrow \pi' \\
X & & X
\end{array}
\]

$\phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is an invertible sheaf on $\mathbb{P}(\mathcal{E})$ so that by part (a), we have

\[
(1) : \phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cong \pi'^*\mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}')}({n}')
\]

for some $\mathcal{L}' \in \text{Pic}X$ and $n' \in \mathbb{Z}$. Similarly, $\phi^{-1}$ being a morphism, we have

\[
(2) : \phi^{-1*}\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \cong \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)
\]

for some $\mathcal{L} \in \text{Pic}X$ and $n \in \mathbb{Z}$. By applying $\phi^*$ to (2), we have

\[
\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \cong \phi^*\phi^{-1*}\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \cong \phi^*\pi^*\mathcal{L} \otimes \phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \cong \pi'^*\mathcal{L}' \otimes (\phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \otimes^n 
\]

\[
\cong \pi'^*\mathcal{L}' \otimes (\pi'^*\mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}')}({n}')) \otimes^n \cong \pi'^*\left(\mathcal{L} \otimes \mathcal{L}'\otimes^n\right) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(nn').
\]

Recall that

\[
\pi'_*\left(\mathcal{O}_{\mathbb{P}(\mathcal{E}')}({n})\right) = \begin{cases} 
0 & m < 0 \\
\mathcal{O}_X & m = 0 \\
\text{Sym}^m\mathcal{E}' & m > 0
\end{cases}
\]

so that if we apply $\pi'_*$ to the above, then by the projection formula, we will have

\[
\mathcal{O}_X \cong \mathcal{L} \otimes \mathcal{L}'\otimes^n \otimes \pi'_* \left(\mathcal{O}_{\mathbb{P}(\mathcal{E}')}({nn}')\right).
\]

Since $\mathcal{O}_X$, $\mathcal{L} \otimes \mathcal{L}'\otimes^n$ are invertible sheaves, it makes sense only when $nn' = 1$. Hence we have either $(n,n') = (-1,-1)$ or $(n,n') = (1,1)$.

If $(n,n') = (-1,-1)$, then, we have $\mathcal{L} \cong \mathcal{L}'$ and (2) becomes $\phi^{-1*}\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \cong \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$. $\phi$ being an isomorphism, $\phi^{-1*} = \phi_*$, so that $\pi'_* = \pi_*\phi_* = \pi_*\phi^{-1*}$ and the projection formula gives $\mathcal{E}' \cong \mathcal{L} \otimes 0 \cong 0$ which is not possible. Hence $(n,n') = (1,1)$.

Hence, we have (2): $\phi^{-1*}\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \cong \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and as above, noting that $\phi^{-1*} = \phi_*$, applying $\pi_*$ and using the projection formula, we will have $\mathcal{E}' \cong \mathcal{L} \otimes \mathcal{E}$ as desired.