DGA-structure on additive higher Chow groups

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We construct a graded-commutative differential graded algebra structure on additive higher Chow groups of a smooth projective variety over a perfect field. We show that these groups are equipped with Frobenius and Verschiebung operators, that turn the collection into a Witt-complex.

1 Introduction

The objective of this paper is to show that additive higher Chow groups of modulus \(m \geq 1\) of a smooth projective variety form a graded-commutative differential graded algebra (DGA), and that the projective system of the DGAs over all \(m \geq 1\) forms a Witt-complex over \(k\). Its zero-dimensional case was studied by [Rülling, 2007].

The additive higher Chow theory can be seen as an attempt to understand motivic cohomology of non-reduced schemes. Even when the underlying reduced spaces are smooth, such schemes generally have non-trivial relative Quillen \(K\)-groups that the usual higher Chow groups in [Bloch, 1986] cannot capture. This theory hopes to describe such relative \(K\)-groups in terms of algebraic cycles. The first attempt was the simplicial additive higher Chow group in [Bloch and Esnault, 2003a], where it is shown that the additive higher Chow groups of zero-cycles for the reduced space \(\text{Spec}(k)\) form the absolute Kähler differentials \(\Omega^*_k/\mathbb{Z}\). A cubical version was proposed in [Bloch and Esnault, 2003b] for zero-cycles over the reduced space \(\text{Spec}(k)\) with a certain \textit{modulus condition} for which one had the same result, and Rülling computed the group of zero-cycles with higher modulus \(m \geq 1\) in [Rülling, 2007] in terms of the big de Rham-Witt forms. The higher dimensional cases were subsequently defined and studied in [Park, 2007], [Krishna and Levine, 2008], [Park, 2009], [Krishna and Park, 2012a], and [Krishna and Park, 2012b].

An interesting point is that, all of the above three calculations of zero-cycles in [Bloch and Esnault, 2003a], [Bloch and Esnault, 2003b], [Rülling, 2007] gave collections of cycle groups that are always certain differential forms. In fact, the \(K\)-theory of the infinitesimal deformations of the base field \(k\) is expressed in terms of the modules of absolute Kähler differentials and the absolute de Rham-Witt complex of Hesselholt-Madsen, as shown for example in [Hesselholt, 2005]. These de Rham-Witt complexes have the structure of a graded-commutative DGA and they are initial objects in the category of so called Witt-complexes (see [Hesselholt and Madsen, 2001], [Rülling, 2007]).

This paper is along the line of extending these results to higher dimensional cases, and the following results summarize the outcomes of our attempts:

\textbf{Theorem 1.0.1.} Let \(X\) be a smooth projective variety over a perfect field \(k\) of characteristic \(\neq 2\). Then, for each integer \(m \geq 1\), the additive higher Chow groups \(\text{TCH}(X;m) := (\text{TCH}^q(X,n;m),\wedge,\delta)\) of \(X\) of modulus \(m\) form a DGA with the following operators: the graded-commutative associative product

\[
\wedge : \text{TCH}^q(X,n_1;m) \otimes \mathbb{Z} \text{TCH}^{p}(X,n_2;m) \rightarrow \text{TCH}^q(X,n;m),
\]

where \(q = q_1 + q_2 - 1\), \(n = n_1 + n_2 - 1\) and the graded derivation \(\delta\) for the product \(\wedge\) satisfying the Leibniz rule: for \(\xi \in \text{TCH}^q(X,n_1; m)\) and \(\eta \in \text{TCH}^p(X,n_2; m)\),

\[
\delta(\xi \wedge \eta) = \delta(\xi) \wedge \eta + (-1)^{n_2 \wedge -1} \xi \wedge \delta(\eta).
\]
The product and derivation commute with the pull-back maps of additive Chow groups, and the push-forward maps satisfy a projection formula. Furthermore, the projective system \( \{ \text{CH}(X; m) \}_{m \geq 1} \) of DGAs is equipped with certain operators \( \text{d}, R, F, \text{tr} \) \((r \geq 1)\), and they form a restricted Witt-complex over \( k \). This Witt-complex structure is natural with respect to pull-back maps.

If we let \( \text{CH}(X; m)_0 = \oplus_{n \geq 1} \text{CH}^n(X; m) \), then it follows that \( \{ \text{CH}(X; m)_{n,}\} \) is a differential graded sub-algebra of \( \text{CH}(X; m) \) and the projective system \( \{ \text{CH}(X; m)_0 \}_{m \geq 1} \) is a sub-Witt-complex. If \( X = \text{Spec}(k) \), this coincides with the de Rham-Witt complex description in [Rülling, 2007].

The product \( \cdot \) in (1.0.1) is given by the formula \( \Delta \circ \mu \circ \varepsilon \); here, \( \varepsilon \) is the concatenation of two cycles on \( X \times G_m \times \square^{n-1} \), \( i = 1,2 \), and the concatenated cycle lies on \( X \times X \times G_m \times G_m \times \square^{n-1} \). We apply the “product” \( \mu \) induced by the multiplication \( \mu : G_m \times G_m \to G_m \), and take the pull-back via the diagonal \( \Delta : X \to X \times X \). Showing that these operations \( \mu \) and \( \Delta^* \) make sense on additive higher Chow groups is a part of the goals of this paper, and by definition on the level of cycles this product pairing \( \cdot \) is nontrivial, and it isn’t obtained trivially by “extension by zero”. For more discussion on nontriviality, see Remark 3.5.2.

The entire paper is devoted to achieving this goal. Section 2 recalls the definition of additive higher Chow cycles and discusses some results on modulus conditions, and Section 3 defines the product structure \( \Lambda \). Section 4 defines the differential operator \( \delta \) and proves the Leibniz rule. The Witt-complex structure is discussed in Section 5. Establishing Theorem 1.0.1 required us to introduce numerous concrete additive higher Chow cycles, and we hope they play useful roles for other future works in the subject. Some calculations require a technical result, that we call the normalization theorem (see Theorem 1.0.3 below).

We couldn’t yet extend Theorem 1.0.1 to smooth affine (or, more generally quasi-projective) varieties, due to the current absence of the moving lemma as in [Krishna and Park, 2012a, Theorem 4.1] for such varieties (See Remarks 3.0.3, 4.3.10). We leave a memorandum that follows from the arguments of this paper:

Scholium 1.0.2. If the moving lemma for additive higher Chow groups holds for smooth affine (or, quasi-projective) varieties, too, then Theorem 1.0.1 extends to them, as well.

The Appendix (Section 6) proves the normalization theorem used in the proof of Theorem 1.0.1:

**Theorem 1.0.3.** Let \( X \) be a smooth quasi-projective variety. Then, the natural inclusion \( Tz^q_N(X; \bullet; m) \to Tz^q(X; \bullet; m) \) of complexes is a quasi-isomorphism, where \( Tz^q(X; \bullet; m) \) \((Tz^q_N(X; \bullet; m))\) is the (normalized) additive cycle complex in Definition 2.2.1 (Definition 6.1.1, respectively).

For higher Chow complexes, a similar result was known by [Bloch, online note, Theorem 4.4.2] and such results help us in proving the moving lemma for higher Chow groups of smooth quasi-projective varieties as in [Bloch, 1994]. We hope Theorem 1.0.3 will lead us to an eventual resolution of Scholium 1.0.2 in the future.

**Conventions:** In this paper, \( k \) is always a fixed perfect field of characteristic \( \neq 2 \). A \( k \)-scheme, or a scheme over \( k \), is a separated scheme of finite type over \( k \). A \( k \)-variety is a reduced \( k \)-scheme. We frequently work with equidimensional \( k \)-schemes to use codimensions of cycles, but one can work with more general \( k \)-schemes by using dimensions of cycles. All products \( X \times X \) of schemes are assumed to be \( \times_k \), unless specified otherwise. For each \( k \)-scheme \( X \), we denote by \( X^N \) the normalization of \( X_{\text{red}} \).

## 2 Additive higher Chow groups

We recall some definitions regarding additive higher Chow complexes from [Krishna and Park, 2012a]. Let \( X \) be an equidimensional \( k \)-scheme. Let \( A^1 = \text{Spec } k[t], G_m = \text{Spec } k[t, t^{-1}], \mathbb{P}^1 = \text{Proj } k[Y_0, Y_1] \), and \( \square^n = (\mathbb{P}^1 \setminus \{1\})^n \). For \( n \geq 1 \), let \( B_n = G_m \times \square^{n-1}, B_n = A^1 \times (\mathbb{P}^1)^{n-2} \) and \( B_n = \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \supset B_n \). Let \( (t, y_1, \ldots, y_{n-1}) \) be the coordinates on \( B_n \).

On \( B_n \), define the Cartier divisors \( F_{1,n} := \{ y_i = 1 \} \) for \( 1 \leq i \leq n-1 \), \( F_{0,n} := \{ t = 0 \} \), which is contained in \( B_n \), and let \( F_n := \sum_{i=1}^{n-1} F_{1,n} \). A face of \( B_n \) is a closed subscheme defined by a set of equations of the form \( y_{i_1} = e_1, \ldots, y_{i_{n-1}} = e_{n-1} \). For \( \varepsilon = 0, \infty \), and \( i = 1, \ldots, n-1 \), let \( t_{n,i} : B_{n-1} \to B_n \) be the inclusion \((t, y_1, \ldots, y_{n-2}) \mapsto (t, y_1, \ldots, y_{i-1}, \varepsilon, y_{i+1}, \ldots, y_{n-2})\). Its image is identified with a codimension one face.

Recall that the (cubical) higher Chow complex (see [Bloch, 1986] for the simplicial version), from which the additive one is derived, is defined as follows: let \( z^q(X, n) \) be the subgroup of \( \mathbb{Z}^q(X \times \square^n) \) of cycles that intersect all faces of \( \square^n \) properly, where faces are defined in the same way as the above except we ignore the \( G_m \)-factors. The group \( z^q(X, n) \) is \( z^q(X, n) \modul{o} \) the degenerate cycles. This gives a cubical object, and the associated complex is the higher Chow complex. Its homology groups are the higher Chow groups \( \text{CH}^q(X, n) \). See [Totaro, 1992].

The additive higher Chow complex is defined similarly using the spaces \( B_n \) instead of \( \square^n \), but we impose an additional condition called the modulus conditions, that control how the cycles should behave at “infinity”.
2.1 Modulus conditions

We recall all modulus conditions that have been used in the literature in studying additive higher Chow groups.

**Definition 2.1.1.** Let \( X \) be a \( k \)-scheme, and let \( V \) be an integral closed subscheme of \( X \times B_n \). Let \( \overline{V} \) denote the Zariski closure of \( V \) in \( X \times \hat{B}_n \) and let \( \nu: \overline{V}^N \rightarrow \overline{V} \subset X \times \hat{B}_n \) be the normalization of \( \overline{V} \). Fix an integer \( m \geq 1 \) and let \( n \geq 1 \). When \( n = 1 \), the notations \( F^1_{1} \) and \( F^1_{1,i} \) below are regarded as the zero divisors.

(i) We say that \( V \) satisfies the *modulus \( m \) condition \( M_{\text{sum}} \) on \( X \times B_n \), if as Weil divisors on \( \overline{V}^N \) we have

\[
(m + 1)[\nu^*(F^1_{n,0})] \leq [\nu^*(F^1_{n})].
\]

(ii) We say that \( V \) satisfies the *modulus \( m \) condition \( M_{\text{sup}} \) on \( X \times B_n \), if as Weil divisors on \( \overline{V}^N \) we have

\[
(m + 1)[\nu^*(F^1_{n,0})] \subseteq \sup_{1 \leq i \leq n-1} [\nu^*(F^1_{n,i})].
\]

(iii) We say that \( V \) satisfies the *modulus \( m \) condition \( M_{\text{ssup}} \) on \( X \times B_n \) if there exists an integer \( 1 \leq i \leq n-1 \) such that as Weil divisors on \( \overline{V}^N \), we have

\[
(m + 1)[\nu^*(F^1_{n,0})] \leq [\nu^*(F^1_{n,i})].
\]

If \( V \) is a cycle on \( X \times B_n \), we say that \( V \) satisfies the modulus condition if all of its irreducible components satisfy the modulus condition.

When no confusion arises, we simply say that \( V \) satisfies the modulus condition \( M \) without mentioning the integer \( m \). Note that since \( V \) is contained in \( X \times B_n \), its closure \( \overline{V} \) intersects all the Cartier divisors \( F^1_F \) and \( F^1_{1,i} \) (\( 1 \leq i \leq n - 1 \)) properly in \( X \times \hat{B}_n \). In particular, their pull-backs of \( F^1_{n,0} \) and \( F^1_{1,i} \) are all effective Cartier divisors on \( \overline{V}^N \).

**Remark 2.1.2.** For \( n \geq 1 \), let \( V \subset X \times B_n \) be an irreducible closed subvariety satisfying a modulus condition in Definition 2.1.1. The modulus condition says that the closure of \( V \) in \( X \times \hat{B}_n \) can intersect \( \{ t = 0 \} \) only along the divisor \( F^1_n \). In particular, the closure of \( V \) in \( X \times \mathbb{A}^1 \times \square^{n-1} \) does not intersect \( \{ t = 0 \} \). Furthermore, for \( n = 1 \), that the closure of \( V \) in \( X \times \mathbb{A}^1 \) does not intersect \( \{ t = 0 \} \) is equivalent to that \( V \) satisfies any one of the above modulus conditions.

Here, we present another property that shows how nontrivial the modulus conditions are.

**Lemma 2.1.3.** Let \( V \subset X \times B_n \) be an irreducible closed subvariety satisfying a modulus condition above. Then,

(i) \( V \) does not contain a nonempty subset of the form \( \mathbb{G}_m \times W \), where \( W \subset X \times \square^{n-1} \).

(ii) The composition \( V \twoheadrightarrow X \times B_n \overset{pr}{\longrightarrow} X \times \square^{n-1} \) is quasi-finite.

**Proof.** (i) If \( V \supseteq \mathbb{G}_m \times W \), then the Zariski closure \( \overline{V} \) in \( X \times \mathbb{A}^1 \times \square^{n-1} \) contains \( \mathbb{A}^1 \times W \) so that \( \overline{V} \cap \{ t = 0 \} \neq \emptyset \). Contradiction. (ii) Given a point \( p \), the fiber \( V \cap pr^{-1}(p) \) is a *proper* closed subscheme of \( \mathbb{G}_m \times \{ p \} \) by (1). Hence, it is finite.

**Remark 2.1.4.** The modulus condition \( M_{\text{sum}} \) was used in [Bloch and Esnault, 2003b] and [Rülling, 2007]. The modulus condition \( M_{\text{sup}} \) was used in [Krisha and Levine, 2008] and [Park, 2009]. The modulus condition \( M_{\text{ssup}} \) was recently introduced in [Krisha and Park, 2012a]. However, some experts seem to think that \( M_{\text{ssup}} \) may not be as suitable as \( M_{\text{sum}} \) owing to its stringent behavior.

2.2 Additive cycle complexes

We recall the definition of additive cycle complexes based on the above modulus conditions from [Krisha and Park, 2012a, Definition 2.5]. (N.B. We correct a minor error in *loc.cit.* on part (0) below.)

**Definition 2.2.1.** Let \( M \) be a modulus condition in Section 2.1. Let \( X \) be a \( k \)-scheme, and let \( r, m, n \) be integers with \( m, n \geq 1 \).

(0) \( T_{Z_r}(X, 1; m)_M \) is the free abelian group on integral closed subschemes \( Z \) of \( X \times \mathbb{G}_m \) of dimension \( r \), satisfying the modulus condition.

For \( n > 1 \), \( T_{Z_r}(X, n; m)_M \) is the free abelian group on integral closed subschemes \( Z \) of \( X \times B_n \) of dimension \( r + n - 1 \) such that:

(1) For each face \( F \) of \( B_n \), \( Z \) intersects \( Z \times F \) properly on \( X \times B_n \).
(2) \( Z \) satisfies the modulus \( m \) condition \( M \) on \( X \times B_n \).

For each \( i = 1, \cdots, n-1 \) and \( \epsilon = 0, \infty \), let \( \partial_i^\epsilon(Z) \) := \( [(\text{Id}_X \times \iota_{n,i,\epsilon})^*(Z)] \). The proper intersection with faces ensures that \( \partial_i^\epsilon(Z) \) are well-defined.

If \( M \) is the modulus condition \( M_{sup} \), then we further suppose that

\[ \partial_i^\epsilon(Z) \text{ lies in } T_{Zr}(X, n-1; m)_{M}. \]

Algebraic cycles on \( X \times B_n \) that belong to \( T_{Zr}(X, n; m)_M \) are called the admissible cycles (additive higher Chow cycles, or additive cycles) with respect to the modulus condition \( M \). When the scheme \( X \) is equidimensional of dimension \( d \) over \( k \), we write for \( q \geq 0 \), \( T_{Zh}(X, n; m)_M := T_{Zd+1-q}(X, n; m)_M \). \( \square \)

From the above, we have the cubical abelian group \( (n \mapsto T_{Zh}(X, n; m)_M) \) in the sense of [Krishna and Levine, 2008, §1.1]. As seen in [Krishna and Park, 2012a], using the containment lemma [Krishna and Park, 2012a, Proposition 2.4], for \( M_{sum} \) and \( M_{sup} \), the condition (3) is implied from (1) and (2). For \( M_{sup} \) this isn’t necessarily the case.

**Definition 2.2.2.** Let \( M \) be a modulus condition. The additive higher Chow complex, or just the additive cycle complex, \( T_{Zh}(X, \bullet; m)_M \) of \( X \) in codimension \( q \) and with the modulus \( m \) condition \( M \) is the nondegenerate complex associated to the cubical abelian group \( (n \mapsto T_{Zh}(X, n; m)_M) \), i.e.,

\[ T_{Zh}(X, n; m)_M := T_{Zh}(X, n; m)_M/T_{Zh}(X, n; m)_{M,\text{degn}}. \]

The boundary map of this complex at level \( n \) is given by \( \partial := \sum_{i=1}^{n-1} (-1)^i \partial_i^\infty - \partial_i^0 \), and it satisfies \( \partial^2 = 0 \).

The homology \( T_{CHh}(X, n; m)_M := H_n(T_{Zh}(X, \bullet; m)_M) \) for \( n \geq 1 \) is the additive higher Chow group of \( X \) with modulus \( m \) condition \( M \). \( \square \)

**Remark 2.2.3.** The modulus conditions in Section 2.1 satisfy the implications \( M_{sup} \Rightarrow M_{sup} \Rightarrow M_{sum} \), so that we have the natural homomorphisms of abelian groups \( T_{CHh}(X, n; m)_{M_{sup}} \rightarrow T_{CHh}(X, n; m)_{M_{sup}} \rightarrow T_{CHh}(X, n; m)_{M_{sum}} \). Whether these additive Chow groups are identical is not known in general. However for \( X = \text{Spec}(k) \), where \( \text{char}(k) \neq 2 \), the additive higher Chow groups of zero-cycles are all isomorphic using the arguments in [Rülling, 2007] without modification.

**Remark 2.2.4.** Earlier works used a bit different notations. The following shows how they are related to the notation of this paper:

(i) \( THh^p_M(k, n) \) in [Bloch and Esnault, 2003b] \( \simeq T_{CH^n}(\text{Spec}(k), n; 1)_{M_{sum}} \),
(ii) \( CH^h(A_k(n+1), n-1) \) in [Rülling, 2007] \( \simeq T_{CH^n}(\text{Spec}(k), n; m)_{M_{sum}} \),
(iii) \( ACH^h(X, n-1; n; m) \) in [Park, 2009] \( \simeq T_{CH^n}(X, n; m)_{M_{sup}} \),
(iv) \( TCH^h(X, n; m) \) in [Krishna and Levine, 2008] \( \simeq T_{CH^n}(X, n; m)_{M_{sup}} \),
(v) \( TH^h(X, n; m) \) in [Krishna and Park, 2012a] \( \simeq T_{CH^n}(X, n; m) \).

**Conventions.** We study the additive higher Chow groups based on only the modulus condition \( M_{sum} \) in this paper. Henceforth, we drop the subscript \( M \) from the notations of additive higher Chow groups.

### 2.3 Double modulus conditions

The modulus condition in Definition 2.1.1 have multivariate analogues. Their principal usages will appear in a forthcoming paper [Krishna and Park, in progress] that studies some multivariate analogues of de Rham-Witt complexes in terms of algebraic cycles. Although this paper is about the “univariate” case, some considerations of the multiple modulus conditions simplify our discussions. Below, we consider only the double modulus condition for \( M_{sum} \):

**Definition 2.3.1.** Let \( n \geq 2 \) and let \( X \) be a \( k \)-scheme.

(i) Let \( B_{2,n} := (\mathbb{G}_m)^2 \times \Delta^{n-2} \) and \( \overline{B}_{2,n} = (\mathbb{A}^1)^2 \times (\mathbb{P}^1)^{n-2} \).
(ii) Let \( (t_1, t_2, y_1, \cdots, y_{n-2}) \in \overline{B}_n \) be the coordinates. The faces of \( B_{2,n} \) are similarly defined by a sequence of equations of the form \( y_i = \epsilon \) with \( \epsilon \in \{0, \infty\} \).
(iii) Let $V$ be an integral closed subscheme of $X \times B_{2,n}$. Let $\overline{V}$ denote the Zariski closure of $V$ in $X \times \overline{B}_{2,n}$, and let $\nu : \overline{V}^N \rightarrow \overline{V} \subset X \times \overline{B}_{2,n}$ be the normalization of $\overline{V}$. Let $m_1, m_2 \geq 1$ be integers.

We say that $V$ satisfies the modulus $(m_1, m_2)$ condition on $X \times B_{2,n}$, if as Weil divisors on $\overline{V}^N$ we have

$$
\sum_{i=1}^{2} (m_i + 1)[\nu^*\{t_i = 0\}] \leq \sum_{i=1}^{n-2} [\nu^*\{y_i = 1\}].
$$

(2.3.1)

Here, when $n = 2$ we interpret the right hand side to be zero so that we have $[\nu^*\{t_1 = 0\}] = [\nu^*\{t_2 = 0\}] = 0$, in particular. We say that a cycle on $X \times B_{2,n}$ satisfies the modulus $(m_1, m_2)$ condition if each irreducible component of $V$ satisfies it.

\[\textit{Remark 2.3.2.} \text{ Since } (m_1 + 1)[\nu^*\{t_i = 0\}] \leq \sum_{i=1}^{2} (m_i + 1)[\nu^*\{t_i = 0\}], \text{ we deduce that } V \text{ satisfies the modulus } m_i \text{ conditions on } X \times \mathbb{G}_m \times B_{n-1}. \]

\[\textit{Lemma 2.3.3.} \text{ Let } X \text{ and } Y \text{ be } k\text{-schemes, and let } V_1 \text{ and } V_2 \text{ be cycles satisfying the modulus } m_1 \text{ and } m_2 \text{ conditions on } X \times B_{n_1}, \text{ and } Y \times B_{n_2}, \text{ respectively. Then, } V_1 \times V_2 \text{ satisfies the modulus } (m_1, m_2) \text{ condition on } X \times Y \times \overline{B}_{2,n_1+n_2}. \]

\[\textit{Proof.} \text{ We may assume that } V_1, V_2 \text{ are irreducible. Let } W \subset V_1 \times V_2 \text{ be an irreducible component. It is enough to show that } W \text{ satisfies the modulus } (m_1, m_2) \text{ condition. Let } t_1 : \overline{V}_1 \subset X \times \overline{B}_{n_1} \text{ and } t_2 : \overline{V}_2 \subset Y \times \overline{B}_{n_2} \text{ be the Zariski closures of } V_1 \text{ and } V_2, \text{ respectively. Take their normalizations } \nu_{\overline{V}_i} : \overline{V}_i^N \rightarrow \overline{V}_i \text{ for } i = 1, 2. \text{ By }[\text{Krishna and Levine, 2008, Lemma 3.1}], \text{ the product of reduced normal } k\text{-schemes is again normal over perfect fields. Thus, the morphism } \nu = \nu_{\overline{V}_1} \times \nu_{\overline{V}_2} : \overline{V}_1^N \times \overline{V}_2^N \rightarrow \overline{V}_1 \times \overline{V}_2 = \overline{V}_1 \cup \overline{V}_2 \text{ is the normalization under the identification } X \times Y \times \overline{B}_{n_1+n_2} \simeq X \times \overline{B}_{n_1} \times Y \times \overline{B}_{n_2}. \text{ Thus, we regard } \overline{V}_1^N \times \overline{V}_2^N \text{ as } \overline{V}_1 \times \overline{V}_2^N. \text{ This gives a commutative diagram}
\]

\[
\begin{array}{ccc}
\overline{V}_1^N \times \overline{V}_2^N & \xrightarrow{\nu} & \overline{V}_1 \times \overline{V}_2 \\
\overline{V}^N \xrightarrow{\nu_W} & & \overline{V}_1 \times \overline{V}_2 \xrightarrow{t_1 \times t_2} X \times Y \times \mathbb{A}^2 \times (\mathbb{P}^1)^{n-1}.
\end{array}
\]

Here $\overline{W}$ is the Zariski closure of $W$, $\nu_W$ is the normalizations of $\overline{W}$, and $t^N$ is the natural inclusion. Here, $n = n_1 + n_2 - 1$.

\[\text{Let } (t_1, t_2, y_1, \ldots, y_{n-1}) \in \mathbb{A}^2 \times (\mathbb{P}^1)^{n-1} \text{ be the coordinates. Then, for } D^1 := \sum_{i=1}^{n_1-1} \{y_i = 1\} - (m_1 + 1)\{t_1 = 0\}, \text{ and } D^2 := \sum_{i=n_1}^{n-1} \{y_i = 1\} - (m_2 + 1)\{t_2 = 0\}, \text{ and } n - 1 = (n_1 - 1) + (n_2 - 1), \text{ we have } ((t_1 \times 1) \circ (1 \times t_2))^*D^1 \geq 0 \text{ and } ((1 \times \nu_{\overline{V}_1}) \circ (1 \times \nu_{\overline{V}_2}))^*D^2 \geq 0. \text{ These hold by the modulus conditions of } V_1 \times Y \times \overline{B}_{n_2} \text{ and } X \times B_{n_1} \times V_2, \text{ regarding } V_1 \times V_2 \text{ as a cycle in } T_2(X \times Y \times \overline{B}_{n_2}, n_1; m), \text{ similarly for } X \times B_{n_1} \times V_2. \text{ Thus, we have } \nu^*(t_1 \times t_2)^*(D^1 + D^2) \geq 0 \text{ on } \overline{V}_1^N \times \overline{V}_2^N \text{ so that } (t^N)^*\nu^*(t_1 \times t_2)^*(D^1 + D^2) \geq 0 \text{ on } \overline{W}^N. \text{ By the commutativity, this means } \nu_W^*(t_1 \times t_2)^*(D^1 + D^2) \geq 0 \text{ on } \overline{W}^N, \text{ which is the desired modulus } (m_1, m_2) \text{ condition of } W. \]

2.4 \textit{A remark on effective divisors}

The following result, that refines [Krishna and Levine, 2008, Lemma 3.2] and [Krishna and Park, 2012a, Lemma 2.2], is used often in this paper:

\[\textit{Lemma 2.4.1.} \text{ Let } f : Y \rightarrow X \text{ be a dominant map of normal integral } k\text{-schemes. Let } D \text{ be a Cartier divisor on } X \text{ such that the generic points of } \text{Supp}(D) \text{ are contained in } f(Y). \text{ Suppose that } f^*(D) \geq 0 \text{ on } Y. \text{ Then, } D \geq 0 \text{ on } X. \]

\[\textit{Proof.} \text{ Note that } f(Y) \text{ is a constructible (by Chevalley) dense subset of } X, \text{ so that by }[\text{Hartshorne, 1977, Exercise II-3.18(b)}], \text{ there is a nonempty open subset } U \subset X \text{ contained in } f(Y). \text{ Since the generic points of } \text{Supp}(D) \text{ are all contained in } f(Y), \text{ we may assume that } U \text{ also contains the generic points of } \text{Supp}(D). \text{ Then, } D|_U \geq 0 \text{ on } U \text{ if and only if } D \geq 0 \text{ on } X. \text{ Thus, we may replace } X \text{ by } U, Y \text{ by } f^{-1}(U), \text{ and } D \text{ by } D|_U, \text{ i.e., we may assume that } f : Y \rightarrow X \text{ is surjective. Then, by }[\text{Krishna and Park, 2012a, Lemma 2.2}] \text{ we are done.} \]

3 Product structure

In this section, we construct a graded-commutative associative algebra structure on the additive higher Chow groups of a smooth projective variety, which is compatible with the module structure of [Krishna and Levine, 2008, Theorem 4.10] on these groups over the ordinary Chow ring of the variety. We deduce the algebra structure in Theorem 3.5.1 from the following result, whose proof will occupy most of this section.

**Proposition 3.0.2.** Let \( X \) and \( Y \) be smooth quasi-projective varieties over \( k \). Then there exists an external product on the additive higher Chow groups

\[
\times_{\mu} : \text{TCH}^{q_1}(X, n_1; m) \otimes_{\mathbb{Z}} \text{TCH}^{q_2}(Y, n_2; m) \to \text{TCH}^{q}(X \times Y, n; m),
\]

(3.0.1)

where \( q = q_1 + q_2 - 1 \), \( n = n_1 + n_2 - 1 \), and \( q_i, n_i, m \geq 1 \) for \( i = 1, 2 \). Furthermore, one has

\[
\langle \xi \rangle \times_{\mu} \eta = (-1)^{(n_1-1)(n_2-1)} \cdot \eta \times_{\mu} \xi
\]

(3.0.2)

for all classes \( \xi \in \text{TCH}^{q_1}(X, n_1; m) \) and \( \eta \in \text{TCH}^{q_2}(Y, n_2; m) \).

Once we have the above external product, in case \( X \) is smooth projective, the internal product structure essentially comes via the diagonal pull-back \( \Delta_X^*: \text{TCH}^q(X \times X, n; m) \to \text{TCH}^q(X, n; m) \) which exists due to the general functoriality in [Krishna and Park, 2012a, Theorem 7.1], as in the classical intersection theory. This functoriality result is based on the moving lemma of [Krishna and Park, 2012a, Theorem 4.1].

**Remark 3.0.3.** If one can extend [Krishna and Park, 2012a, Theorem 4.1] to smooth quasi-projective varieties, then the functoriality result [Krishna and Park, 2012a, Theorem 7.1] also extends, and we have the internal product for all smooth quasi-projective varieties. At present, this argument works only for the smooth projective case.

3.1 External product

Let \( X \) and \( Y \) be equidimensional \( k \)-schemes. The external product \( \times_{\mu} \) is based on the product map \( \mu: \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \). Consider the diagram, where \( n = n_1 + n_2 - 1 \):

\[
\begin{array}{ccc}
X \times \mathbb{G}_m \times \square^{n_1-1} \times Y \times \mathbb{G}_m \times \square^{n_2-1} & \xrightarrow{\tau} & X \times Y \times \mathbb{G}_m \times \mathbb{G}_m \times \square^{n-1} \\
\mu & & \downarrow 1 \times 1 \times \mu \times 1 \\
X \times Y \times \mathbb{G}_m \times \square^{n-1} & & 
\end{array}
\]

(3.1.1)

Here, \( \tau \) is the transposition map \((x, t, y, x', t', y') \mapsto (x, x', t, t', y, y')\). We denote the composite map also by \( \mu \).

For a given pair of admissible cycles \( V_1 \in \text{TCH}^{q_1}(X, n_1; m) \) and \( V_2 \in \text{TCH}^{q_2}(Y, n_2; m) \), we want to define an admissible cycle \( V_1 \times \mu V_2 \).

**Lemma 3.1.1.** The map \( \mu \) in (3.1.1) restricted to \( V_1 \times V_2 \) is quasi-finite. In particular, \( \text{codim}_{X \times Y \times B_n} (\mu(V_1 \times V_2)) = q \), or, \( \text{dim}(\mu(V_1 \times V_2)) = \text{dim}(V_1 \times V_2) \).

**Proof.** Regard \( V_1 \) as a subvariety of \( \mathbb{G}_m \times X \times \square^{n_1-1} \). By Lemma 2.1.3, each fiber of the composition \( \pi_1: V_1 \hookrightarrow \mathbb{G}_m \times X \times \square^{n_1-1} \overset{pr}{\to} X \times \square^{n_1-1} \) is finite, where \( pr \) is the projection. Similarly, the fibers of \( \pi_2: V_2 \hookrightarrow \mathbb{G}_m \times Y \times \square^{n_2-1} \to Y \times \square^{n_2-1} \) are also finite. Then, the lemma follows from :

**Claim:** Let \( \pi_i: V_i \hookrightarrow Y_i \times X_i \overset{pr_i}{\to} X_i \) be the compositions for \( i = 1, 2 \), that are quasi-finite. Let \( f: Y_1 \times Y_2 \to Y_3 \) be any morphism. Then, the composition

\[
\pi: V_1 \times V_2 \hookrightarrow Y_1 \times X_1 \times Y_2 \times X_2 \cong Y_1 \times Y_2 \times X_1 \times X_2 \xrightarrow{\tilde{f}} Y_3 \times X_1 \times X_2
\]

is quasi-finite, too, where \( \tilde{f} = f \times \text{Id}_{X_1 \times X_2} \).

Indeed, for each point \( p \in Y_3 \times X_1 \times X_2 \), let \( x_i = pr_i(p) \) for projections \( pr_i: Y_3 \times X_1 \times X_2 \to X_i \). Then, since \( \pi^{-1}(p) \subset \pi_1^{-1}(x_1)(x_2) \) and the latter set is finite by assumption, the fiber \( \pi^{-1}(p) \) is finite. Apply this to \( X_1 = X \times \square^{n_1-1}, X_2 = Y \times \square^{n_2-1}, Y_1 = Y_2 = Y_3 = \mathbb{G}_m, \) and \( f = \mu \). From this, \( \mu: V_1 \times V_2 \to \mu(V_1 \times V_2) \) is surjective with finite fibers. Hence, \( \text{dim}(V_1 \times V_2) = \text{dim}(\mu(V_1 \times V_2)) \).

A form of the following lemma was suggested by an anonymous referee:

**Lemma 3.1.2.** The map \( \mu \) in (3.1.1) restricted on \( V_1 \times V_2 \) is a projective (and hence finite) morphism.
Proof. We show that $\mu|_{V_1 \times V_2}$ is projective. Let $\Gamma_\mu \subset (\mathbb{G}_m)^2 \times \mathbb{G}_m$ be the graph of $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$, and let $\Gamma_\mu$ be its Zariski closure in $(\mathbb{P}^1)^2 \times \mathbb{P}^1$. The projection onto the third factor $pr_3 : \Gamma_\mu \to \mathbb{P}^1$ is projective, so that its pull-back along the natural inclusion $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$

$$\Gamma_\mu^\ast : = \Gamma_\mu \times_{\mathbb{P}^1} \mathbb{G}_m = \Gamma_\mu \cup (\{0\} \times \mathbb{G}_m) \cup (\{\infty\} \times \{0\} \times \mathbb{G}_m) \to \mathbb{G}_m$$

is also projective.

Let $V \subset X \times Y \times \Gamma_\mu \times \Box^{n-1}$ be the pull-back of $V_1 \times V_2$ along the isomorphism $\Gamma_\mu \simeq \mathbb{G}_m \times \mathbb{G}_m$ (up to a suitable permutation of factors), and let $V$ be its Zariski closure in $X \times Y \times \Gamma_\mu \times \Box^{n-1}$. The map $\nabla \to X \times Y \times \mathbb{P}^1 \times \Box^{n-1}$ induced by $pr_3$ is projective so that its pull-back along the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$

$$\nabla \cap \left( X \times Y \times \Gamma_\mu \times \Box^{n-1} \right) \to X \times Y \times \mathbb{G}_m \times \Box^{n-1} \quad (3.1.2)$$

is projective. Let $\nabla_1$ be the closure of $V_1$ in $X \times \mathbb{P}^1 \times \Box^{n-1}$. Let

$$\pi_1 : X \times Y \times \Gamma_\mu \times \Box^{n-1} \to X \times \mathbb{P}^1 \times \Box^{n-1}$$

be the composition of the projection to $X \times \Gamma_\mu \times \Box^{n-1}$ with the map $pr_1 : \Gamma_\mu \to \mathbb{P}^1$, which is the restriction of the projection $(\mathbb{P}^1)^2 \to \mathbb{P}^1$ to the first factor. Then, $\pi_1^{-1}(\nabla_1)$ is closed and it contains $V$. Hence, it contains $\nabla$. On the other hand, the modulus condition for $V_1$ implies that $\nabla_1 \cap (X \times \{0\} \times \Box^{n-1}) = \emptyset$. Hence, $\nabla_1 \cap (X \times Y \times (\{0\} \times \mathbb{G}_m) \times \Box^{n-1}) = \emptyset$. The same argument for $V_2$ shows that $\nabla \cap (X \times Y \times (\{\infty\} \times \{0\} \times \mathbb{G}_m) \times \Box^{n-1}) = \emptyset$. Hence, by (3.1.2), $\nabla \cap (X \times Y \times \mathbb{G}_m \times \Box^{n-1}) = V \simeq V_1 \times V_2$ is projective over $X \times Y \times \mathbb{G}_m \times \Box^{n-1}$ via $\mu$. This shows that the map $\mu|_{V_1 \times V_2}$ is projective. The map $\mu|_{V_1 \times V_2}$ is also quasi-finite by Lemma 3.1.1 and we deduce that it is finite.

Definition 3.1.3. (i) For any irreducible closed subvariety $V \subset X \times Y \times \mathbb{G}_m \times \mathbb{G}_m \times \Box^{n-1}$ such that $\mu|_V : V \to \mu(V)$ is finite, define $\mu_\ast(V)$ as the push-forward $\deg(V/\mu(V)) \cdot [\mu(V)]$. This is a cycle on $X \times Y \times \mathbb{G}_m \times \Box^{n-1}$. We can extend it $Z$-linearly to the subgroup of cycles spanned by such cycles.

(ii) For $V_1$ and $V_2$ as before, we define the external product as $V_1 \times_\mu V_2 : = \mu_\ast(V_1 \times V_2)$, where $V_1 \times V_2$ is seen as the cycle associated to the scheme $V_1 \times V_2$. It is a sum over irreducible components of $V_1 \times V_2$, and it is a well-defined codimension $q_1 + q_2 - 1$ cycle on $X \times Y \times \mathbb{G}_m \times \Box^{n-1}$. We can extend it $Z$-bilinearly for all admissible cycles $V_1$ and $V_2$.

Remark 3.1.4. When $X = \text{Spec} (k)$ and $V_1, V_2$ are 0-cycles, this definition coincides with [Rülling, 2007, Definition-Proposition 3.9 (ii)].

Lemma 3.1.5. Let $m_1, m_2 \geq 1$. Let $V_1 \in \mathcal{T}_2^{\text{flat}}(X, n_1; m_1)$, $V_2 \in \mathcal{T}_2^{\text{flat}}(Y, n_2; m_2)$. Then, the cycle $V_1 \times_\mu V_2$ intersects properly with all faces of $X \times Y \times \mathbb{P}^1$.

Proof. We may assume that $V_1, V_2$ are irreducible. Let $F \subset \mathbb{P}^1$ be a face. We need to prove that $\dim(Z \cap (X \times Y \times F)) \leq \dim Z - \text{codim}_{X \times Y \times \mathbb{P}^1}(X \times Y \times F)$. Since $\partial^i_i(V_1 \times \mu \times V_2) = \partial^i_\mu(V_1) \times \mu \times V_2$ if $1 \leq i \leq n - 1$, and $V_1 \times_\mu \partial^i_\mu(V_2)$ if $n_1 \leq i \leq n$, by induction on the codimension of $X \times Y \times \mathbb{P}^1$, it is enough to show the lemma when $F$ are codimension one faces. We do it for $1 \leq i \leq n_1 - 1$ as the other case $n_1 \leq i \leq n - 1$ is exactly the same. In this case, $\dim(\partial^i_\mu(V_1 \times \mu \times V_2)) = \dim(\partial^i_\mu(V_1) \times \mu \times V_2) = \dim(\partial^i_\mu(V_1) \times \mu \times V_2) - \dim(V_1 - 1 + \dim V_2 = 1)$ $\dim(V_1 \times_\mu V_2) = 1$, where $\uparrow$ and $\downarrow$ hold by Lemmas 2.3.3 and 3.1.1. The inequality holds because $V_1$ intersects with the faces properly. This finishes the proof.

Proposition 3.1.6. Let $m_1, m_2 \geq 1$. For any cycle $V$ on $X \times \mathbb{B}_{2,n+1}$ satisfying the modulus $(m_1, m_2)$ condition such that $V \to \mu_\ast(V)$ is finite, the cycle $\mu_\ast(V)$ satisfies the modulus $\min\{m_1, m_2\}$ condition on $X \times \mathbb{P}^1$.

Proof. Let $m := \min\{m_1, m_2\}$. We may assume that $V$ is irreducible. Since $\mu_\ast(V) = \deg(V/\mu(V)) \cdot [\mu(V)]$, it is enough to show that $\mu(V)$ satisfies the modulus $m$ condition. Consider the commutative diagram

$$
\begin{array}{ccc}
\nabla^N \xrightarrow{\nu\nu} \nabla^\ast & \xrightarrow{i} & X \times \mathbb{A}^2 \times (\mathbb{P}^1)^{n-1} \\
\square & \uparrow & \square \\
\mu(V)^N \xrightarrow{\nu_\ast(V)} \mu(V)^\ast & \xrightarrow{i_{\mu(V)}} & X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1}.
\end{array}
$$
Here, $\overline{V}$ and $\overline{\mu(V)}$ are Zariski closures of $V$ and $\mu(V)$, and $\nu_V$ and $\nu_{\mu(V)}$ are the normalizations of the closures. The map $\overline{\mu}^N$ is given by the universal property of normalization.

Let $(t_1, t_2, y_1, \ldots, y_{n-1}) \in \mathbb{A}^2 \times (\mathbb{P}^1)^{n-1}$ and $(w, y_1, \ldots, y_{n-1}) \in \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1}$ be the coordinates. Let $D := \sum_{i=1}^{n-1} \{y_i = 1\} - \sum_{j=1}^{2} (m_j + 1) \{t_j = 0\}$ on $X \times Y \times \mathbb{A}^2 \times (\mathbb{P}^1)^{n-1}$. Since $V$ satisfies the modulus $(m_1, m_2)$ condition, we have $\nu_{\mu(V)}\iota^*(D) \geq 0$.

On the other hand, consider the Cartier divisor $D' := \sum_{i=1}^{n-1} \{y_i = 1\} - (m + 1) \{w = 0\}$ on $X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1}$. On $X \times \mathbb{A}^2 \times (\mathbb{P}^1)^{n-1}$, we have $\mu^* D' = \sum_{i=1}^{n-1} \{y_i = 1\} - \sum_{j=1}^{2} (m + 1) \{t_j = 0\} \geq D$. Thus, by pullback via $\iota \circ \nu_V$, we have $\nu_{V}^* \iota^* \mu^* D' \geq \nu_{V}^* \iota^*(D) \geq 0$. This inequality is equivalent to $(\overline{\mu}^N)^* \nu_{\mu(V)}^* \iota^*(D) \geq 0$ by the commutativity of the diagram. Applying Lemma 2.4.1 to $\overline{\mu}^N$, we get $\nu_{\mu(V)}^* \iota^* (D') \geq 0$. This is the modulus $m$ condition for $\mu(V)$.

**Corollary 3.1.7.** Let $X$ and $Y$ be equidimensional $k$-schemes. Let $V_1 \in \mathbb{T}^2_1 (X, n_1; m)$ and $V_2 \in \mathbb{T}^2_2 (Y, n_2; m)$. Then $\mu_1 (V_1 \times V_2) = V_1 \times_{\mu} V_2$ is in $\mathbb{T}^2_3 (X \times Y, n; m)$.

From the associativity of $\mu$ we immediately have:

**Corollary 3.1.8.** The external product $\times_{\mu}$ is associative on the level of cycles.

### 3.2 Some cycle computations

Before we proceed further, we collect some concrete cycle calculations that are used in several parts of this paper. Let $\sigma$ be a permutation on $\{1, 2, \cdots, n-1\}$ acting naturally on $\mathbb{A}^{n-1}$ of $X \times \mathbb{G}_m \times \mathbb{A}^{n-1}$ via $\sigma \cdot (x, t, y_1, \cdots, y_{n-1}) := (x, t, y_{\sigma(1)}, \cdots, y_{\sigma(n-1)})$, where $X$ is a $k$-scheme. This action on the space naturally induces an action on the group of admissible cycles. Note that if $Z$ is an admissible cycle, then for any permutation $\sigma$, the cycle $\sigma \cdot Z$ is admissible.

#### 3.2.1 The cycles $Z\{r\}$

For $r \geq 1$ and $n \geq 2$, consider the morphism $\chi_{n,r}: X \times \mathbb{G}_m \times \mathbb{A}^{n-1} \rightarrow X \times \mathbb{G}_m \times \mathbb{A}^{n-1}$ given by $(x, t, y_1, \cdots, y_{n-1}) \mapsto (x, t, y_1 \cdot y_{r+1}, \cdots, y_{n-1})$. Given an irreducible closed subvariety $Z \subset X \times \mathbb{G}_m \times \mathbb{A}^{n-1}$, let $\chi_{n,r}(Z)$ denote its image under the finite (and flat) map $\chi_{n,r}$ and define $Z\{r\}$ to be the cycle

$$Z\{r\} := (\chi_{n,r})_*(Z) = (k(Z), k(\chi_{n,r}(Z))): [\chi_{n,r}(Z)].$$

Notice that $\chi_{n,r}(Z) \subset X \times \mathbb{G}_m \times \mathbb{A}^{n-1}$ is closed as $\chi_{n,r}$ is a finite morphism. We can extend this definition $Z$-linearly so that $Z\{r\}$ is defined for all cycles $Z$.

**Lemma 3.2.1.** Let $Z \in \mathbb{T}^2(q)(X, n; m)$. Then, $\chi_{n,r}(Z)$ (and hence $Z\{r\}$) is also in $\mathbb{T}^2(q)(X, n; m)$.

**Proof.** We may assume that $Z$ is irreducible. We first show that $\chi_{n,r}(Z)$ intersects properly with all faces. We do it by induction on $n \geq 1$. For codimension 1 faces, note that $\partial_F^r (\chi_{n,r}(Z)) = r \partial_F (Z)$, and $\partial_F (\chi_{n,r}(Z)) = (\partial_F (\chi_{n,r}(Z)))$ for $i \geq 2$. Consequently, for general faces $F$, we have $F \cdot (\chi_{n,r}(Z)) = r (F \cdot Z)$ if $F$ involves $\{y_i = c\}$, and $F.(\chi_{n,r}(Z)) = (\chi_{n,r}(F \cdot Z))$ otherwise. Since the intersection $F \cdot Z$ is proper, we deduce that so does $\chi_{n,r}(F \cdot Z)$ by the induction hypothesis. This shows that the intersection $F \cdot \chi_{n,r}(Z)$ is proper.

Now, we show that $\chi_{n,r}(Z)$ satisfies the modulus condition. For simplicity, let $W: \rightarrow = \chi_{n,r}(Z)$. Let $\overline{Z}$ and $\overline{W}$ be the Zariski closures of $Z$ and $W$ in $X \times \mathbb{G}_m$ and let $\iota_Z: Z^N \rightarrow Z$ and $\iota_W: W^N \rightarrow W$ be their normalizations. Thus, we have the commutative diagram

$$\begin{align*}
Z^N & \xrightarrow{\nu_Z} Z \xrightarrow{\iota_Z} X \times \mathbb{G}_m \\
W^N & \xrightarrow{\nu_W} W \xrightarrow{\iota_W} X \times \mathbb{G}_m.
\end{align*}$$

Here, we have

$$(m + 1)[\nu_Z^* t_Z^* (F_{n,0})] \leq \sum_{i=1}^{n-1} [\nu_Z^* t_Z^* (F_{n,i})] \leq r [\nu_Z^* t_Z^* (F_{n,1})] + \sum_{i=2}^{n-1} [\nu_Z^* t_Z^* (F_{n,i})].$$

where the inequality $\dagger$ follows from the modulus condition of $Z$. By the definition of $\chi_{n,r}$, note that $\chi_{n,r}^* F_{n,0} = F_{n,0} = F_{n,1}$, $\chi_{n,r}^* F_{n,1} = F_{n,1}$, and $\chi_{n,r}^* F_{n,i} = F_{n,i}$ for $i \geq 2$. Hence, the inequality (3.2.2) is equivalent to $(m + 1)[\nu_Z^* t_Z^* (\chi_{n,r}^* F_{n,0})] \leq \sum_{i=1}^{n-1} [\nu_Z^* t_Z^* (\chi_{n,r}^* F_{n,i})] = [\nu_Z^* t_Z^* (\chi_{n,r}^* F_{n})]$. By the commutativity of the diagram, this is $[(\chi_{n,r}^{(n)})^* (\nu_{\mu(V)}^* t_{\mu(V)}^*) (F_{n})] - (m + 1)[\nu_W^* t_W^* (F_{n,0})] \geq 0$. By Lemma 2.4.1, this implies $[\nu_Z^* t_Z^* (F_{n})] - (m + 1)[\nu_W^* t_W^* (F_{n,0})] \geq 0$, which is the modulus condition for $W = \chi_{n,r}(Z)$. Hence, $\chi_{n,r}(Z)$ is an admissible cycle.
3.2.2 The cycles $\gamma'_Z$

We consider somewhat mutated analogues $V'_X$ for $i \geq 1$ of $W'_n$ in [Levine, 1994, Lemma 4.1]. The case $i = 1$ is used in Section 3.3 to prove Lemma 3.3.1, and the general cases $i \geq 1$ are used in Section 5.4 in the discussion of Witt-complex structures.

Let $X$ be a smooth quasi-projective $k$-variety. Let $n \geq 2$. Let $(x, t, y_1, \cdots, y_{n-1}, y, \lambda)$ be the coordinates of $X \times \hat{B}_{n+2}$. Consider the closed subscheme $\iota: V'_X \hookrightarrow X \times B_{n+2}$ given by the equation

$$(1 - y)(1 - \lambda) = \left\{ \begin{array}{ll}
1 - y_1, & \text{if } i = 1, \\
(1 - y_1)(1 + y_1 + \cdots + y_1^{-1} - \lambda(1 + y_1 + \cdots + y_1^{-2})), & \text{if } i \geq 2.
\end{array} \right. \tag{3.2.3}$$

Let $\hat{V}'_X$ be the Zariski-closure of $V'_X$ in $X \times \hat{B}_{n+2}$. Let $\pi_1: X \times \hat{B}_{n+2} \rightarrow X \times \hat{B}_{n+1}$ be the projection that ignores the coordinate $y_1$, and let $\pi'_1$ be the restriction $\pi_1|\hat{V}'_X$.

**Lemma 3.2.2.** The map $\pi'_1: V'_X \rightarrow X \times B_{n+1}$ is an isomorphism for $i = 1$, and a surjection for $i \geq 2$. \hfill \Box

**Proof.** When $i = 1$, we define the inverse $\theta: X \times B_{n+1} \rightarrow V'_X$, given by

$$(x, t, y_2, \cdots, y_{n-1}, y, \lambda) \mapsto (x, t, -(1 - y)(1 - \lambda) + 1, y_2, \cdots, y_{n-1}, y, \lambda).$$

That is this is inverse to $\pi'_1$ is immediate. For $i \geq 2$, that $\pi'_1$ is a surjection is obvious. \hfill \blacksquare

**Definition 3.2.3.** For $i \geq 1$ and $n \geq 2$, and an irreducible closed subvariety $Z \subset X \times G_m \times \boxtimes^{n-1}$, define $\gamma'_Z := \pi'_1(V'_X \cdot (Z \times \boxtimes^2))$ as an abstract algebraic cycle. For a general cycle $Z$, we can define $\gamma'_Z$ by extending the above $Z$-linearly.

**Remark 3.2.4.** The cycle $\gamma'_Z$ can be described as follows, too: for $i \geq 1$ and $n \geq 2$, consider the rational maps $\nu^i: X \times G_m \times \boxtimes^{n-1} \times \boxtimes \rightarrow X \times G_m \times \boxtimes^n$ given by

$$\nu^i: (x, t, y_1, \cdots, y_{n-1}) \times y \mapsto (x, t, y_2, y_3, \cdots, y_{n-1}, y, \frac{y - y_1^i}{y - y_1^{-1}}). \tag{3.2.4}$$

Then, $\gamma'_Z$ is the Zariski closure of $\nu^i(Z \times \boxtimes)$. Indeed, if we let $(y - y_1^i)/(y - y_1^{-1}) = \lambda$, we get

$$(1 - y)(1 - \lambda) = (1 - y_1^i) - \lambda (1 - y_1^{-1}).$$

After pulling out the factor $(1 - y_1)$ from the right hand side, we obtain the defining equation (3.2.3) of $V'_X$. \hfill \Box

**Lemma 3.2.5.** Let $Z \in \mathcal{T}_{2^n}(X, n; m)$. Then, $\gamma'_Z$ is in $\mathcal{T}_{2^n}(X, n + 1; m)$. \hfill \Box

**Proof.** We may assume that $Z$ is irreducible. We first prove that $\gamma'_Z$ intersects properly with all faces of $X \times B_{n+1}$. Write $\gamma'_{Z,n} = \gamma'_Z$ to keep track of $n$. Let $\epsilon \in (0, \infty)$.

If a given face $F$ of $X \times B_{n+1}$ involves a face $\{y_j = \epsilon\}$ for $j \in \{n-1, n\}$, then $\gamma'_{Z,n}$ intersects $F$ properly. Indeed, (see (3.2.4)) we see that $\partial_n^{\infty}(\gamma'_{Z,n}) = \sigma \cdot Z$, $\partial_n^i(\gamma'_{Z,n}) = \sigma \cdot (Z\{i\})$ for the cyclic permutation $\sigma = (1, 2, \cdots, n)$, and $\partial_n^{\infty}(\gamma'_{Z,n}) = 0$, $\partial_n^{\infty}(\gamma'_{Z,n}) = \sigma \cdot (Z\{i\})$. Since $Z$ is admissible, so are $Z\{i\}$ and $Z\{i-1\}$ by Lemma 3.2.1, thus, in particular all of $\sigma \cdot Z$, $\sigma \cdot (Z\{i\})$, and $\sigma \cdot (Z\{i-1\})$ intersect properly with all faces. In particular $\gamma'_{Z,n}$ intersects $F$ properly.

We now show that $\gamma'_{Z,n}$ intersects properly with all faces $F$ that do not involve $\{y_j = \epsilon\}$ for $j \in \{n-1, n\}$. This follows by induction on $n \geq 2$. Indeed, from (3.2.4), for $j < n$ and $\epsilon = 0, \infty$, we have $\partial_j^i(\gamma'_{Z,n}) = \gamma'_{Z,n-1}$ so that the dimension of $\partial_j^i(\gamma'_{Z,n})$ is at least one less by the induction hypothesis. Repeatedly applying this argument for the remaining defining equations $\{y_j = \epsilon\}$ of the face $F$, we are done.

We now show that $\gamma'_Z$ satisfies the modulus condition. Let $Z'$ be an irreducible component of $V'_X \cdot (Z \times \boxtimes^2)$ and let $W' = \pi'_1(Z')$. Any irreducible component of $\gamma'_Z$ is of this form by Definition 3.2.3. Thus, it is enough to show that $W'$ satisfies the modulus condition. Since $Z \times \boxtimes^2$ satisfies the modulus condition, by the containment lemma in [Krishna and Park, 2012a, Proposition 2.4], $Z'$ also satisfies the modulus condition. In fact, it satisfies a stronger condition (3.2.5) below: let $Z'$ be the Zariski closure of $Z'$ in $X \times \hat{B}_{n+2}$ and let $\nu_{Z'}: Z'^N \rightarrow Z' \rightarrow \hat{V}'_X$.
be the normalization composed with the inclusion. Similarly, $\mathcal{W}$ is the Zariski closure of $W'$ and $\nu: \mathcal{W}_{\lambda} \to \mathcal{W}$ is the normalization. Consider the diagram

$$
\begin{array}{ccc}
\mathcal{W}^N & \xrightarrow{\nu} & \mathcal{W}'^N \\
\downarrow{\pi_i^N} & & \downarrow{\pi_i^N} \\
X \times \hat{B}_{n+2} & \xrightarrow{\pi_1^N} & X \times \hat{B}_{n+1}.
\end{array}
$$

We use $(x, t, y_1, \cdots , y_{n-1}, y, \lambda)$ and $(x, t, y_2, \cdots , y_{n-1}, y, \lambda)$ for the coordinates of $X \times \hat{B}_{n+2}$ and $X \times \hat{B}_{n+1}$, respectively. From the modulus condition of $Z$, we have

$$(m+1)|v^* \tau^* \{ t = 0 \}| \leq |v^* \tau^* \sum_{j=1}^{n-1} \{ y_j = 1 \} |.$$ (3.2.5)

Note that we did not include the divisors $\{ y = 1 \}$ and $\{ \lambda = 1 \}$. But, $V_*^\epsilon$ is an effective divisor on $X \times B_{n+2}$ given by the equation $(1-y)(*') = (1-y)(1-\lambda)$ for some polynomial $(*)$ as in (3.2.3) so that $|v^* \tau^* \{ y = 1 \}| \leq |v^* \tau^* \{ y = 1 \} | + |v^* \tau^* \{ \lambda = 1 \} |$ (The equality holds if $i = 1$). By the commutativity of the diagram, we thus deduce from (3.2.5) the inequality $(m+1)|v^* \tau^* \{ t = 0 \}| \leq [ (\pi^N_i)^* v^* \tau^* \{ y = 1 \}) + \{ y = 1 \} + \{ \lambda = 1 \} ]$. Since $\pi^N_i$ is dominant by Lemma 3.2.2, using Lemma 2.4.1 we deduce

$$(m+1)|v^* \tau^* \{ t = 0 \}| \leq |v^* \tau^* \sum_{j=2}^{n-1} \{ y_j = 1 \} + \{ y = 1 \} + \{ \lambda = 1 \} |.$$ But, this is the modulus condition for $W'$. Hence $\gamma^i_Z$ satisfies the modulus condition.

### 3.3 Graded commutativity

We now come back to the algebra structure on additive higher Chow groups. We use cycles considered in Section 3.2.2 for the case of $i = 1$.

**Lemma 3.3.1.** Let $X$ be a smooth quasi-projective $k$-variety. Let $n \geq 3$. Let $Z \in T^q_X(X; n; m)$ be a cycle such that $\partial^\epsilon(Z) = 0$ for all $1 \leq i \leq n-1$ and $\epsilon \in \{ 0, \infty \}$. Let $\sigma$ be a permutation on $\{ 1, 2, \cdots , n-1 \}$. Then, there exists a cycle $\gamma^2_Z \in T^q_X(X; n+1; m)$ such that

$$Z = (\text{sgn}(\sigma))(\sigma \cdot Z) + \partial(\gamma^2_Z).$$ (3.3.1)

**Proof.** Case 1. We first consider the case when $\sigma$ is the transposition $\tau = (p, p+1)$ for $1 \leq p \leq n - 2$. We do it for $\tau = (1, 2)$ only, for the other cases are similar. For the unique permutation $\xi$ on $\{ 1, \cdots , n \}$ determined by $\xi : (x, t, y_1, y_2, \cdots , y_n) \mapsto (x, t, y_{n-1}, y_1, y_2, \cdots , y_{n-2})$, (more precisely, $\xi = (2,3)(n-1, n-2, \cdots , 2,1)^2$), consider the cycle $\gamma^2_Z := \xi \cdot \gamma^2_Z$, where $\gamma^2_Z$ is as in Definition 3.2.3 and (3.2.4). (N.B. The components of this cycle can be described also as follows: consider the morphism $\nu: X \times \mathbb{G}_m \times \square^{n-1} \times \square \to X \times \mathbb{G}_m \times \square^n$ given by

$$\nu(x, t, y_1, \cdots , y_{n-1}, y) = (x, t, y_2, y_3, \cdots , y_{n-1}, y_1) .$$ (3.3.2)

This is obtained from (3.2.4) with $i = 1$ by permuting the coordinates of $\square^n$ so that set-theoretically $\gamma^2_Z$ is the closure of $v(Z \times \square)$.) Being a permutation of the cycle in Lemma 3.2.5, this cycle $\gamma^2_Z$ is also admissible. Here, we have $\partial^\epsilon_\nu(\gamma^2_Z) = 0$, $\partial^\epsilon_\tau(\gamma^2_Z) = \tau \cdot Z$ and $\partial^\epsilon_\sigma(\gamma^2_Z) = 0$, $\partial^\epsilon_\xi(\gamma^2_Z) = Z$. On the other hand, for $\epsilon \in \{ 0, \infty \}$, $\partial^\epsilon_\xi(\gamma^2_Z)$ is a permutation of $\gamma^2_Z$ which is 0 because $\partial^\epsilon_\xi(Z) = 0$ by the given assumptions. Similarly, for $j \geq 4$ and $\epsilon \in \{ 0, \infty \}$, we have $\partial^\epsilon_\xi(\gamma^2_Z)$ is a permutation of $\gamma^2_Z$ which is again 0. In particular, $\partial(\gamma^2_Z) = Z + \tau \cdot Z$, so we are done for this case.

Case 2. Now let $\sigma$ be any permutation on $\{ 1, \cdots , n - 1 \}$. By a basic fact from group theory, we can write $\sigma$ as a product $\sigma = \tau_1 \tau_{n-1} \cdots \tau_1$ where each $\tau_i$ is a transposition of the form $(p, p+1)$ considered in Case 1. Let $\sigma_0 := \text{Id}$ and $\sigma_i := \tau_i \tau_{i-1} \cdots \tau_1$ for $1 \leq i \leq r$. Then, for each $1 \leq i \leq r$, by Case 1, we have

$$(-1)^{l-1} \sigma_{i-1} \cdot Z + (-1)^{l-1} \tau_1 \cdot \sigma_{i-1} \cdot Z = \partial \left( (-1)^{l-1} \tau_1 \gamma^2_{\sigma_{i-1}} \right).$$ (3.3.3)

Since $\tau_1 \cdot \sigma_{i-1} = \sigma_i$, by taking the sum of (3.3.3) over $1 \leq i \leq r$, after cancellations we have $Z + (-1)^r \sigma \cdot Z = \partial(\gamma^2_Z)$, where $\gamma^2_Z := \sum_{i=1}^n (-1)^{l-1} \gamma^2_{\sigma_{i-1}}$. Since $(-1)^r = \text{sgn}(\sigma)$, we now obtain (3.3.1).
Lemma 3.3.2. Let $X$ and $Y$ be smooth quasi-projective $k$-varieties. For two cycles $\xi \in T^q(X,n_1;m)$ and $\eta \in T^r(Y,n_2;m)$ all of whose codimension 1 faces are trivial, we have

\[
\xi \times_\mu \eta = (-1)^{(n_1-1)(n_2-1)} \eta \times_\mu \xi + \partial(\gamma^{r,s})
\]

for an admissible cycle $\gamma^{r,s}$, where $r = n_1 - 1$, $s = n_2 - 1$.

Proof. Let $n := n_1 + n_2 - 1$. By the given assumptions, all codimension one faces of $\xi \times_\mu \eta$ are also trivial. Thus, the lemma follows from Lemma 3.3.1 applied to $X$ and $Y$ by taking $\gamma^{r,s} = \gamma^{\sigma_{\xi \times_\mu \eta}}$, where $\sigma$ is the permutation on $\{1, \cdots, r+s\}$ given by $\sigma(i) = r + i$ for $i < s$ and $\sigma(i) = i - s$ for $i \geq s + 1$.

3.4 External product on homology

We now show that $\times_\mu$ on the cycle complexes descends onto the homology groups. Note first that from the definition of $\partial$, we immediately have:

Lemma 3.4.1. Let $X,Y$ be equidimensional $k$-schemes. For two cycles $\xi \in T^r(X,n_1;m)$ and $\eta \in T^s(Y,n_2;m)$, we have

\[
\partial(\xi \times_\mu \eta) = \partial \xi \times_\mu \eta + (-1)^{n_1-1} \xi \times_\mu \partial \eta.
\]

Proof. (of Proposition 3.0.2) The external product in Definition 3.1.3 is

\[
\times_\mu : T^r(X,n_1;m) \otimes_\mathbb{Z} T^s(Y,n_2;m) \rightarrow T^q(X \times Y,n;m).
\]

Suppose that $\xi \in T^r(X,n_1;m)$ and $\eta \in T^s(Y,n_2;m)$. If $\partial \xi = 0$ and $\partial \eta = 0$, then by Lemma 3.4.1, we have $\partial(\xi \times_\mu \eta) = 0$. Hence, $\times_\mu$ maps a pair of cocycles to a cocycle. If $\partial \xi = 0$, then $\xi \times_\mu \partial \eta = (-1)^{n_1-1} \partial(\xi \times_\mu \eta)$. Similarly, if $\partial \eta = 0$, then $\partial \xi \times_\mu \eta = \partial(\xi \times_\mu \eta)$. Hence, $\times_\mu$ maps a pair consisting of a cocycle and a boundary to a boundary. Hence, the map $\times_\mu$ of (3.4.2) descends to

\[
\times_\mu : TCH^q(X,n_1;m) \otimes_\mathbb{Z} TCH^s(Y,n_2;m) \rightarrow TCH^q(X \times Y,n;m).
\]

The associativity comes from Corollary 3.1.8. To prove the graded-commutativity, by the normalization theorem (Theorem 6.1.2 in Appendix I), we know that every pair of classes $\xi \in TCH^q(X,n_1;m)$ and $\eta \in TCH^s(Y,n_2;m)$ can be represented by a pair of cycles (also denoted by $\xi$, $\eta$) all of whose codimension one faces are trivial. Then, by Lemma 3.3.2, we have (3.3.4). This completes the proof of Proposition 3.0.2.

3.5 Internal product

We now prove the first main result of the paper:

Theorem 3.5.1. Let $X$ be a smooth projective variety over $k$. Then there exists an internal product on the additive higher Chow groups of $X$

\[
\wedge_X : TCH^q(X,n_1;m) \otimes_\mathbb{Z} TCH^s(X,n_2;m) \rightarrow TCH^q(X,n;m),
\]

where $q_i, n_i, m_i \geq 1$ for $i = 1, 2$ and $q = q_1 + q_2 - 1$, $n = n_1 + n_2 - 1$. This is associative and satisfies the equation

\[
\xi \wedge_X \eta = (-1)^{(n_1-1)(n_2-1)} \eta \wedge_X \xi,
\]

for all classes $\xi \in TCH^q(X,n_1;m)$ and $\eta \in TCH^s(X,n_2;m)$.

This internal product is natural with respect to the pull-back maps of additive higher Chow groups and satisfies the projection formula

\[
f_*(a \wedge_X f^*(b)) = f_*(a) \wedge_Y b
\]

for a morphism $f : X \rightarrow Y$ of smooth projective varieties.
We show that the internal product \( \Delta_X^* \) of \( X \times_X \mu \) via \( \text{TCH}^2(X \times X, n; m) \), where \( \times_X \mu \) is well-defined by Proposition 3.0.2, and the pull-back \( \Delta_X^* \) exists for additive higher Chow groups by [Krishna and Park, 2012a, Theorem 7.1], which requires that \( X \) is smooth projective. Associativity follows from Corollary 3.1.8, and (3.5.2) follows from Lemma 3.3.2.

The rest of the proof concerns the naturality and the projection formula. Observe that if \( X \xrightarrow{\phi} Y \rightarrow Z \) are morphisms of smooth projective varieties, then from the contravariance functoriality of the additive higher Chow groups [Krishna and Park, 2012a, Theorem 7.1], one deduces the naturality of the internal products easily. Note also that if the projection formulas for \( f \) and \( g \) hold, then it holds for \( g \circ f \):

\[
(g \circ f)_* [a \wedge_X (g \circ f)^*(b)] = g_* [f_* [a \wedge_X (f)^*(b)]] = g_* [f_*(a) \wedge_Y (g)^*(b)] = (g \circ f)_* (a) \wedge_Z b = (g \circ f)_* (a) \wedge_Z b.
\]

We remark that the projection formula means

\[
f_* [\Delta_X^* \{ \mu_* (a \times f^*(b)) \}] = \Delta_Y^* [\mu_* \{ f_*(a) \times b \}].
\]

Since any given morphism \( f : X \to Y \) factors as the composite \( X \to X \times Y \to Y \), it only remains to prove the projection formula (3.5.4) for closed embeddings and projections, separately.

For any closed embedding \( f : X \to Y \), by the moving lemma [Krishna and Park, 2012a, Theorem 1.1], it is enough to prove (3.5.4) for the cycles \( a \in \text{Tz}^{n_1}(X, n_1; m), b \in \text{Tz}^{n_2}(Y, n_2; m) \) for which all pull-backs below make sense. Indeed,

\[
f_* [\Delta_X^* \{ \mu_* (a \times ((X) \cdot b)) \}] = \Delta_Y^* [\mu_* \{ (f \times f)_*(a \times ((X) \cdot b)) \}]
\]

where \( \dagger \) follows from [Serre, 1965, V-30, (11)] and \( \dagger \) follows from the left Cartesian diagram of below:

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_X} & X \times X \\
\downarrow f & & \downarrow f \times f \\
Y & \xrightarrow{\Delta_Y} & Y \times Y
\end{array}
\]

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\text{Id}_X \times \Delta_Y} & X \times Y \\
\downarrow f & & \downarrow f \times \text{Id}_Y \\
Y & \xrightarrow{\Delta_Y} & Y \times Y
\end{array}
\]  

(3.5.5)

Now, consider the projection \( f : X \times Y \to Y \). Let \( p : X \times Y \times X \to X \times Y \times Y \) be the obvious projection. Let \( Z := X \times Y \). Then for cycles \( a \in \text{Tz}^{n_1}(Z, n_1; m) \) and \( b \in \text{Tz}^{n_2}(Y, n_2; m) \), whenever all pull-backs below make sense, we have

\[
f_* [\Delta_Z^* \{ \mu_* (a \times f^*(b)) \}] = f_* [\Delta_X^* \{ \mu_* (a \times (X \times b)) \}] = f_* [\Delta_Y^* \{ p^* (\mu_* (a \times b)) \}]
\]

where \( \dagger \) follows from the right Cartesian square of (3.5.5) above. By the moving lemma again, the equality then holds for any additive higher Chow classes. This completes the proof of the projection formula (3.5.4).

**Remark 3.5.2.** One may question if this product structure \( \wedge_X \) is indeed nontrivial at the level of additive higher Chow groups, although may not be apparent from its definition. Here is a way to see it using the projection formula, just proven in Theorem 3.5.1: for a given smooth projective variety \( X \), choose a closed immersion \( \iota : X \to \mathbb{P}^N \) for some \( N > 0 \). Then, the projection formula \( \iota_* (a \times_X b) = \iota_* (a) \wedge b \) reduces the question of nontriviality to the case of projective spaces. But, for projective spaces, these groups satisfy the projective bundle formula (see [Krishna and Park, 2012a, Theorem 3.2] cf. [Krishna and Levine, 2008, Theorem 5.6]) so that the question further reduces to \( \text{Spec}(k) \), where we know the product is not trivial by [Rülling, 2007]. An explicit example on how this works is provided below.

**Example 3.1.** Let \( X \) be a smooth projective curve over a perfect field \( k \) of characteristic different from two. We show that the internal product \( \wedge_X : \text{TCH}^1(X, 1; 2) \otimes \text{TCH}^2(X, 1; 2) \to \text{TCH}^2(X, 1; 2) \) is nontrivial.

There is a finite and flat map \( f : X \to \mathbb{P}^1_k \). Since the map \( \text{CH}^1(X) \to \text{CH}^1(\mathbb{P}^1_k) \simeq \mathbb{Z} \) is the degree map, it is surjective. One can now check easily from the projection formula (cf. [Krishna and Levine, 2008, Theorem 4.10]) and the projective bundle formula that the map \( f_* : \text{TCH}^2(X, 1; 2) \to \text{TCH}^2(\mathbb{P}^1, 1; 2) \) is surjective.

Now, let \( \pi : X \to \text{Spec}(k) \) be the structure map. Since the push-forward map \( \text{TCH}^2(\mathbb{P}^1, 1; 2) \to \text{TCH}^1(k, 1; 2) \) is surjective (use the projective bundle formula), we conclude that the map \( \pi_* : \text{TCH}^2(X, 1; 2) \to \text{TCH}^1(k, 1; 2) \) is surjective. Using this, Theorem 3.5.1 and the isomorphism \( k \xrightarrow{\sim} \text{TCH}^1(k, 1; 2) \), one deduces easily that the pairing \( \text{TCH}^1(X, 1; 2) \otimes \text{TCH}^2(X, 1; 2) \to \text{TCH}^2(X, 1; 2) \) is nontrivial.
For a smooth projective variety $X$ over $k$, let $\text{TCH}_n(X;m) = \bigoplus_{i \geq 1} \text{TCH}^i(X,m+1;n)$ and let $\text{TCH}(X;m) = \bigoplus_{n \geq 0} \text{TCH}_n(X;m)$. Let $\text{CH}(X) = \bigoplus_n \text{CH}^n(X)$.

**Corollary 3.5.3.** For a smooth projective variety $X$, there is an internal product structure $\wedge: \text{TCH}(X;m) \otimes_{\text{CH}(X)} \text{TCH}(X;m) \to \text{TCH}(X;m)$, that makes $\text{TCH}(X;m)$ a graded-commutative ring and ‘$\wedge$’ is bilinear with respect to the $\text{CH}(X)$-module structure on $\text{TCH}(X;m)$. The unit of the ring structure is $[X \times \{1\}] \in \text{TCH}^1(X,1;m)$.

**Proof.** This follows from Theorem 3.5.1 once we know that the internal product $\wedge$ of Theorem 3.5.1 is bilinear over the ring $\text{CH}(X)$, where the $\text{CH}(X)$-module structure on $\text{TCH}(X;m)$ is given by [Krishna and Levine, 2008, §4]. For a (Chow) cycle $\alpha$, the action $\alpha \cdot (-)$ is $\Delta^\alpha_X(\alpha \times (-))$, if the pull-back is defined. (By the moving lemma, it is defined on a quasi-isomorphic subcomplex.) Since $\wedge = \Delta^\alpha_X \circ \times \mu$, where $\times \mu$ is as in Definition 3.1.3, to prove the bilinearity $\alpha \cdot (\xi \wedge) = (\alpha \cdot \xi) \wedge = \xi \wedge (\alpha \cdot \eta)$ for a Chow cycle $\alpha$ on $X$, we need to show that

$$
\Delta^\alpha_X[\alpha \times \Delta^\alpha_X(\mu_\ast(\xi \times \eta))] = \Delta^\alpha_X[\mu_\ast(\Delta^\alpha_X(\alpha \times \xi) \times \eta)] = \Delta^\alpha_X[[\mu_\ast(\xi \times \Delta^\alpha_X(\alpha \times \eta))]],
$$

(3.5.6)

whenever the pull-backs $\Delta^\alpha_X$ are defined on the level of cycles (again, which is true on quasi-isomorphic subcomplexes of cycles by the moving lemma).

Note that from the commutative diagram

$$
\begin{array}{c}
\begin{array}{ccc}
X \times X \times \mathbb{G}_m & \xleftarrow{\mu} & X \times X \times \mathbb{G}_m \\
\downarrow \Delta_X & & \downarrow \Delta_X \\
X \times \mathbb{G}_m & \xleftarrow{\mu} & X \times \mathbb{G}_m \times \mathbb{G}_m,
\end{array}
\end{array}
$$

we have the equality $\Delta^\alpha_X \circ \mu_\ast = \mu_\ast \circ \Delta^\alpha_X$. Thus, the second term of (3.5.6) is

$$
\Delta^\alpha_X(\mu_\ast((\Delta^\alpha_X \times \text{Id})^\ast(\alpha \times \xi \times \eta))) = \mu_\ast(\Delta^\alpha_X((\Delta^\alpha_X \times \text{Id})^\ast(\alpha \times \xi \times \eta))),
$$

(3.5.8)

and the third term of (3.5.6) is

$$
\Delta^\alpha_X(\mu_\ast((\text{Id} \times \Delta^\alpha_X)^\ast(\xi \times \alpha \times \eta))) = \mu_\ast((\text{Id} \times \Delta^\alpha_X)^\ast(\xi \times \alpha \times \eta))).
$$

(3.5.9)

Thus, the second equality of (3.5.6) holds via the transposition $\alpha \times \xi \times \eta \to \xi \times \alpha \times \eta$.

Treating $\xi \times \eta$ as a higher Chow cycle on $X \times X \times \mathbb{G}_m \times \mathbb{G}_m$ and pulling back via $\Delta_X$, the first term of (3.5.6) is $\Delta^\alpha_X(\mu_\ast(\alpha \times \Delta^\alpha_X(\xi \times \eta)))$. Hence, by $\Delta^\alpha_X \circ \mu_\ast = \mu_\ast \circ \Delta^\alpha_X$, this first term of (3.5.6) is

$$
\mu_\ast((\text{Id} \times \Delta^\alpha_X)^\ast(\alpha \times \xi \times \eta)).
$$

(3.5.10)

Comparing (3.5.8) with (3.5.10), to prove the first equality of (3.5.6), it is enough to check that $\Delta^\alpha_X \circ (\text{Id} \times \Delta^\alpha_X)^\ast = \Delta^\alpha_X \circ (\Delta^\alpha_X \times \text{Id})^\ast$ as compositions of pull-backs of higher Chow cycles. This follows immediately from the diagram

$$
\begin{array}{c}
\begin{array}{ccc}
X \times X & \xrightarrow{\text{Id} \times \Delta^\alpha_X} & X \times X \\
\downarrow \Delta_X & & \downarrow \Delta_X \times \text{Id} \\
X & \xrightarrow{\Delta^\alpha_X} & X \times X.
\end{array}
\end{array}
$$

This finishes the proof of (3.5.6), thus the corollary.

As another consequence of Theorem 3.5.1, we get the following result:

**Corollary 3.5.4.** Let $X$ be a smooth projective $k$-variety with $\text{char}(k) \neq 2$. Then for any $q,n,m \geq 1$, the group $\text{TCH}^q(X,n;m)$ is a $\mathbb{W}_m(k)$-module, where $\mathbb{W}_m(k)$ is the big Witt ring of $k$ of length $m$. In particular, $\text{TCH}^q(X,n;m)$ is a $k$-vector space if $\text{char}(k) = 0$.

**Proof.** This follows from Theorem 3.5.1 by considering the composite map

$$
\begin{align*}
\text{TCH}^q(X,1;m) \otimes \mathbb{Z} \text{TCH}^q(X,n;m) & \xrightarrow{f \otimes \text{Id}} \text{TCH}^q(X,1;m) \otimes \mathbb{Z} \text{TCH}^q(X,n;m) \\
& \xrightarrow{\wedge} \text{TCH}^q(X,n;m),
\end{align*}
$$

where $f: X \to \text{Spec}(k)$ is the structure map, and using the isomorphism $\mathbb{W}_m(k) \cong \text{TCH}^1(k,1;m)$ of [Rülling, 2007]. That this gives a module structure also follows from Theorem 3.5.1. Finally, in characteristic zero, $\mathbb{W}_m(k)$ is itself a $k$-vector space.
Remark 3.5.5. In characteristic zero, the Witt ring $\mathcal{W}_m(k)$ has several $k$-vector structures so that $TCH^q(X,n;m)$ also inherits several $k$-vector space structures.

4 Differential operator

We saw in Section 3 that the additive higher Chow groups of a smooth projective variety form a graded-commutative algebra. Our goal in this section is to show (as Theorem 4.3.9) that this algebra is equipped with a differential operator, which turns this algebra into a differential graded algebra (DGA).

4.1 The operator $\delta$

Let $X$ be an equidimensional $k$-scheme. Let $\mathbb{G}_m^\times$ denote the variety $\mathbb{G}_m\setminus\{1\}$. We have natural inclusions of open sets $\mathbb{G}_m^\times \hookrightarrow \square \hookrightarrow \mathbb{P}^1$. For $n \geq 1$, define the map

$$\delta_n: X \times \mathbb{G}_m^\times \times \square^{n-1} \to X \times \mathbb{G}_m \times \square^n, \quad (x,t,y_1,\cdots,y_{n-1}) \mapsto (x,t,t^{-1},y_1,\cdots,y_{n-1}). \quad (4.1.1)$$

For any irreducible cycle $Z \subset X \times \mathbb{G}_m \times \square^{n-1}$, let $Z^\times$ denote its restriction to the open set $X \times \mathbb{G}_m \times \square^n$.

Lemma 4.1.1. For any irreducible admissible cycle $Z \in \mathbb{P}^2(X,n;m)$, $\delta_n(Z^\times)$ is closed in $X \times \mathbb{G}_m \times \square^n$.

Proof. It is enough to show that $\delta_n$ is a closed immersion. For this, it is enough to show that the map $\mathbb{G}_m^\times \to \mathbb{G}_m \times \square$, given by $t \mapsto (t,t^{-1})$, is a closed immersion. But this is obvious because its image is given by the equation $ty = 1$, where $(t,y) \in \mathbb{G}_m \times \square$.

Lemma 4.1.2. For $Z$ as in Lemma 4.1.1, the closed subvariety $V := \delta_n(Z^\times)$ satisfies the modulus condition.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{Z}^N & \xrightarrow{f} & X \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \\
\downarrow \delta_n & & \downarrow \delta_n \\
\mathbb{V}^N & \xrightarrow{g} & X \times \mathbb{P}^1 \times (\mathbb{P}^1)^n,
\end{array}$$

where $\mathbb{Z}$ and $\mathbb{V}$ are the Zariski closures of $Z$ and $V$, $\mathbb{Z}^N \to \mathbb{Z}$ and $\mathbb{V}^N \to \mathbb{V}$ are their normalizations, and $f$ and $g$ are the compositions. The map $\delta_n$ is induced by the composition $\mathbb{P}^1 \xrightarrow{\text{diag}} \mathbb{P}^1 \times (\mathbb{P}^1)^{1} \xrightarrow{\sigma \times \sigma} \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1}$, where $\sigma: \mathbb{P}^1 \to \mathbb{P}^1$ is the isomorphism induced by $k[y_0,y_1] \to k[y_0, y_1]$, $y_0 \to y_1$, $y_1 \to y_0$. This $\delta_n$ is projective. Here, the dominant map $Z^\times \to V$ gives $\overline{Z} \to \overline{V}$, and the induced dominant map $\delta_n$ is consequently projective and surjective.

From the definition of $\delta_n$, we have $\delta_n^*(F_{n+1,0}) = F_{n,0}$. First consider the case $n \geq 2$. Then, $\delta_n^*(F_{n+1,i}) = F_{n,i-1}$ for $i \geq 2$. Thus,

$$\langle \hat{\delta}_n^* g^*(F_{n+1,0} - (m+1)F_{n+1,0}) \rangle \geq \sum_{i=1}^{n} \langle \hat{\delta}_n^* g^*(F_{n+1,i}) \rangle - \langle \hat{\delta}_n^* g^*((m+1)F_{n+1,0}) \rangle \geq 0, \quad \text{(4.1.2)}$$

where $\geq^*$ follows by the modulus condition for $Z$. We deduce from Lemma 2.4.1 that $\langle g^*(F_{n+1} - (m+1)F_{n+1,0}) \rangle \geq 0$, which is the modulus condition for $V$.

In case $n = 1$, we have $\delta_1^* g^*(F_{2,0}) = f^* \delta_1^* g^*(F_{2,0}) = f^* g^*(F_{1,0}) = 0$ so that $\langle \delta_1^* g^*(F_{2,0} - (m+1)F_{2,0}) \rangle = \langle \delta_1^* g^*(F_{2,0}) \rangle \geq 0$. Hence again by Lemma 2.4.1 we have $g^* [F_{1} - (m+1)F_{2,0}] \geq 0$, which is the modulus condition for $V$.

Definition 4.1.3. For any irreducible admissible cycle $Z \in \mathbb{P}^2(X,n;m)$, we write $\delta_n(Z)$ for $\delta_n(Z^\times)$, or further $\delta(Z)$, if the reference to $n$ is apparent or unnecessary. We extend it $\mathbb{Z}$-linearly.
Proposition 4.1.4. If $Z \in Tz^q(X, n; m)$, then $\delta(Z) \in Tz^{q+1}(X, n+1; m)$. Furthermore, $\delta$ and $\partial$ satisfy the relation $\partial\delta + \delta\partial = 0$.

Proof. We may assume that $Z$ is irreducible. We first prove the following.

Claim: (i) $\partial n,i \circ \delta_n = 0$ for $\epsilon = 0$, $\infty$. (ii) $\partial n,i \circ \delta_n = \delta_{n-1} \circ \partial n,i-1$ for $i \geq 2$ and $\epsilon = 0$, $\infty$.

Here $\partial n,i$ is the $i$-th face $\partial_i$ on $B_{n+1}$. For (i), note from the definition of $\delta_n$ that $\delta_n(Z)$ does not intersect $\{y_1 = 0\}$ in $X \times B_{n+1}$ for otherwise it must intersect the divisor $\{t = \infty\}$ nontrivially in $X \times B_{n+1}$, but this divisor is empty. If the cycle $\delta_n(Z)$ intersects $\{y_1 = \infty\}$, then it must intersect the divisor $\{t = 0\}$ in $X \times B_{n+1}$. But, by Lemma 4.1.2, $\delta_n(Z)$ satisfies the modulus condition so that the cycle possibly intersects the divisor $\{t = 0\}$ only in $X \times k^1 \times (\mathbb{P}^1)^n$, not in $X \times G_m \times \mathbb{D} = X \times B_{n+1}$. Hence, $\partial n,i \circ \delta_n(Z) = 0$.

For (ii), observe from the definition of $\delta_n$ that for $i \geq 2$ and $\epsilon = 0$, $\infty$, the diagram

\[ \begin{array}{ccc} G_m \times \mathbb{D}^{n-2} & \xrightarrow{\delta_n-1} & G_m \times \mathbb{D}^{n-1} \\ \downarrow \delta_n-1 & & \downarrow \delta_n \\ G_m \times \mathbb{D}^{n-1} & \xrightarrow{\delta_n} & G_m \times \mathbb{D}^{n} \end{array} \]

is Cartesian. Hence $\delta_{n-1}(i^{n-1} \circ \delta_n(Z)) = (i^{n-1} \circ \delta_n(Z))$. This proves the Claim.

Using this and the proper intersection property of $Z$, we deduce that $\delta(Z)$ intersects properly with the faces. We also saw in Lemma 4.1.2 that $\delta(Z)$ satisfies the modulus condition. Since $\delta$ does not change the dimension of cycles, we now have $\delta(Z) \in Tz^{q+1}(X, n+1; m)$.

Finally, we have $\partial \circ \delta_n(Z) = \sum_{i=0}^{n} (-1)^i [\delta_n \circ \partial n,i \circ \delta_n(Z) - \delta_n \circ \partial n,i-1 \circ \delta_n(Z)]$

\[ = \sum_{i=2}^{n} (-1)^i [\delta_{n-1} \circ \partial n,i-1 \circ \delta_n(Z) - \delta_{n-1} \circ \partial n,i \circ \delta_n(Z)] \\
= -\sum_{i=1}^{n-1} (-1)^i [\delta_{n-1} \circ \partial n,i \circ \delta_n(Z) - \delta_{n-1} \circ \partial n,i-1 \circ \delta_n(Z)] \\
= -\delta_{n-1} \left( \sum_{i=1}^{n-1} (-1)^i [\partial n,i \circ \delta_n(Z) - \partial n,i-1 \circ \delta_n(Z)] \right) = -\delta_{n-1} \circ \partial(Z), \]

where the equality $\dagger$ follows from the Claim. This proves the proposition.

Corollary 4.1.5. For every $q \geq 1$, the above $\delta$ defines a chain map $\delta : Tz^q (X, \bullet; m) \to Tz^{q+1} (X, \bullet + 1; m)$.

Proof. It is clear from (4.1.1) that $\delta$ preserves the degenerate cycles. The corollary follows thus from Proposition 4.1.4.

Lemma 4.1.6. When $X = \text{Spec}(k)$, the operator $\delta$ restricted on the zero-cycles coincides with the differential operator $D$ of [Rülling, 2007, (3.9.3)] in $TCH^n(k, n; m)$.

Proof. The definition of $D$ is equivalent to the following: consider a rational map $\delta' : G_m \times \mathbb{D}^{n-1} \to G_m \times \mathbb{D}^{n}$ given by $(t, y_1, \cdots, y_{n-1}) \mapsto (t, \frac{y}{y_1}, \cdots, \frac{y}{y_{n-1}})$. Then, for a closed point $p$ on $G_m \times \mathbb{D}^{n-1}$, define $D(p) := -\delta'(p)$. So, it is enough to show that $\delta_n(p) + \delta'(p)$ is the boundary of an admissible 1-cycle.

We first consider the case where $p$ is a $k$-rational point, say $p = (s, z_1, \cdots, z_{n-1})$ with $s \neq 0, z_i \neq 0 \in k$. Then, we consider the parameterized admissible 1-cycle $\gamma : (y \mapsto (s, y, (sy - 1)/(y - 1), z_1, \cdots, z_{n-1}) \in G_m \times \mathbb{D}^{n+1}$, whose boundaries are

\[ \left\{ \begin{array}{ll} \partial n(\gamma) = (s, s, z_1, \cdots, z_{n-1}) = \delta(p), & \partial 1(\gamma) = 0, \\
\partial 1(\gamma) = 0, & \partial 1(\gamma) = (s, 1/s, z_1, \cdots, z_{n-1}) = \delta(p), \\
\partial 1(\gamma) = 0, & \text{for } i \geq 3, \epsilon \in \{0, \infty\}. \end{array} \right. \]

Hence, $\partial(-\gamma) = \delta'(p) + \delta_n(p)$, as desired.

Now suppose $p$ is not a $k$-rational point. In this case, $p$ may not be of the form $(s, z_1, \cdots, z_{n-1})$ over $k$. Nevertheless, by the good position assumption, the point $p$ lies in none of the codimension 1 face $\{y_1 = \epsilon\}$, where $\epsilon = 0, \infty$, so that we have a closed immersion $\iota : \text{Spec}(\kappa(p)) \to G_m \times (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-1} = G_m \times (G_m^{\times})^{n-1}$. This gives $n$ values $s, z_1, \cdots, z_{n-1} \in \kappa(p) \setminus \{0, 1\}$ with $\kappa(p) = k(s, z_1, \cdots, z_{n-1})$ such that there is a (not necessarily unique) factorization of $i$

\[ \text{Spec}(\kappa(p)) \xrightarrow{i} G_m^{\times, \kappa(p)} \times \kappa(p) (G_m^{\times, \kappa(p)})^{n-1} \xrightarrow{\pi} G_m \times (G_m^{\times})^{n-1}, \]
where \(\pi\) is induced by the map \(\pi: \text{Spec}(\kappa(p)) \to \text{Spec}(k)\), and \(\iota_p\) sends the singleton to the \(\kappa(p)\)-rational point \((s, z, \cdots, z_{n-1})\) (call it \(p'\)) of \(\mathbb{G}_{m, \kappa(p)}^x \times \mathbb{G}_{m, \kappa(p)}^y\). Then, by the first case, with the base field \(k\) replaced by \(\kappa(p)\), we have \(-\delta'(p') = \delta(p')\) in \(TCH^n(\kappa(p), n; m)\). Since \(\pi_*(p') = p\) as \(0\)-cycles and \(\pi_*\) obviously commutes with \(\delta_n\) and \(\delta'\), by the existence of the trace push-forward \(\pi_*: TCH^n(\kappa(p), n; m) \to TCH^n(k, n; m)\) of [Rülling, 2007, Lemma 3.18], we have \(-\delta'(p) = \delta_n(p)\) in \(TCH^n(k, n; m)\). \(\blacksquare\)

### 4.2 Computation of \(\delta^2\)

Now suppose that \(X\) is a smooth quasi-projective variety. Note that \(\delta^2(Z) = \delta(\delta(Z))\) is a nonzero cycle. Our next goal is to show that \(\delta^2\) is zero modulo a boundary. We consider certain cycles in \(z^2(G_m, 3)\) that are two dimensional variants of some 1-cycles in [Totaro, 1992]. For \(t \in G_m\), we let \(u := t^{-1}\).

For \(1 \leq j \leq 5\), and \(l = 1, 2\), let \(\Gamma_j^l \subset G_m \times \square^3\) be the 2-cycles defined by the rational maps \(\psi_j^l: G_m \times \square \to G_m \times \square^3\) given as follows: for \(t \in G_m \setminus \{1, -1\}\) and \(x \in \square \setminus \{0\}\)

\[
\psi_j^1(t, x) = \left( t, u', 1 - u' \right), \quad \psi_j^2(t, x) = \left( t, u', \frac{(1-u')x - (1-\ell')}{1-\ell'} \right), \\
\psi_j^3(t, x) = \left( t, x, u' - 1 \right), \quad \psi_j^4(t, x) = \left( t, x, \frac{u'x - 1}{x - 1} \right), \\
\psi_j^5(t, x) = \left( t, x, \frac{u'x - 1}{x - 1} \right).
\]

\[(4.2.1)\]

For an admissible irreducible 1-cycle \(\alpha \subset G_m \times \square^2\) defined by a rational map \(\varphi: G_m \to G_m \times \square^2\), \(\varphi(t) = (\varphi(t)(0), \varphi(t)(1), \varphi(t)(2))\), we often regard \(\alpha\) as the parameterization \((\varphi(t)(0), \varphi(t)(1), \varphi(t)(2))\) to simplify the notations.

It is easy to check from the definitions that all \(\Gamma_j^l\) are closed in \(G_m \times \square^3\) and they define admissible cycles in \(z^2(G_m, 3)\). One can also check straightforwardly that they have the following boundaries:

\[
\partial \Gamma_j^1 = (t, u', 1 - u') - (t, u', (1 - u')/(1 - \ell')) + (t, u', 1/(1 - \ell')),
\partial \Gamma_j^2 = (t, u', 1 - \ell') + (t, u', 1/(1 - \ell')),
\partial \Gamma_j^3 = - (t, u', 1 - \ell') - (t, \ell', 1 - \ell'),
\partial \Gamma_j^4 = (t, u', 1 - u'), \quad \partial \Gamma_j^5 = (t, \ell', 1 - \ell').
\]

\[(4.2.3)\]

Since \(u = t^{-1}\), note that \(\frac{1-u}{1-\ell} = \frac{t-1}{t-\ell} = \frac{t-1}{t-1} = -1\). Hence, we have \((t, u', -u') = (t, u', (1 - u')/(1 - \ell'))\). Hence, using (4.2.3) one checks that for \(l = 1, 2\),

\[(4.2.4)\]

As in (4.2.1), we consider two additional 2-cycles \(\Gamma_6^2\) and \(\Gamma_7^2 \subset G_m \times \square^3\) associated to the rational maps \(\psi_6^2\) and \(\psi_7^2: G_m \times \square \to G_m \times \square^3\) given by

\[
\psi_6^2(t, x) = \left( t, x, \frac{ux-u^2}{x-ux} , -u^2 \right), \quad \psi_7^2(t, x) = \left( t, u, x, \frac{ux+u^2}{x+ux} \right).
\]

\[(4.2.5)\]

and their boundaries are

\[
\partial \Gamma_6^2 = -2(t, u, -u^2) + (t, u^2 - u^2), \quad \partial \Gamma_7^2 = (t, u, u) - (t, u, -u^2) + (t, u, -u).
\]

\[(4.2.6)\]

Combining (4.2.4) and (4.2.6), we obtain in \(z^2(G_m, 2)\)

\[
2(t, u, u) = \partial \Gamma, \quad \Gamma := 2\Gamma_1 + \Gamma_2 - \Gamma_6^2 + 2\Gamma_7^2.
\]

\[(4.2.7)\]

Given an admissible cycle \(Z \in T\square^2(X, n; m)\), we can regard it as a higher Chow cycle in \(z^n(X \times G_m, n - 1)\). For any \(\Gamma_j^l \in z^2(G_m, 3)\), we get the exterior product \(\Gamma_j^l \times Z \in z^{n+2}(X \times G_m \times G_m, n + 2)\). Let

\[
\Delta_{G_m} : X \times G_m \times \square^{n+1} \to X \times G_m \times G_m \times \square^{n+1}
\]

\[(4.2.8)\]

be the diagonal map.

From the definitions of the cycles, \(\Gamma_j^l \times Z\) intersects \(X \times G_m \times \square^{n+1}\) properly under the closed embedding \(\Delta_{G_m}\). Since \(X\) is smooth, we can pull-back the cycles to define \(\Gamma_j^l \times Z := \Delta_{G_m}^*(\Gamma_j^l \times Z) \in z^{n+2}(X \times G_m, n + 2)\).
Lemma 4.2.1. The cycle $\Gamma_j^1 \star Z$ lies in $T_\mathbb{Q}^{q+2}(X, n+3; m)$ under the natural inclusion $T_\mathbb{Q}^{q+2}(X, n+3; m) \to z^{q+2}(X \times \mathbb{G}_m, n+2)$.

Proof. It is easy to check that $V' := |\Gamma_j^1 \star Z|$ is the closure of the image of $Z \times \square$ under the rational map

$$
\Psi_j^1 : X \times \mathbb{G}_m \times \square^n \to X \times \mathbb{G}_m \times \square^{n+2}
$$

(4.2.9)

$$
\Psi_j^1(x, t, y_1, \ldots, y_{n-1}) = (x, \psi_j^1(0), \psi_j^1(1), \psi_j^1(2), \psi_j^1(3), y_1, \ldots, y_{n-1}),
$$
in the notation of (4.2.2).

We only need to show that $V'$ satisfies the modulus condition. We follow the proof of Lemma 4.1.2 to prove the modulus condition for $V'$. Consider the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}^N \times \mathbb{P}^1 & \xrightarrow{f'} & X \times \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \\
\downarrow{\Phi_j^1} & & \downarrow{\Phi_j^1} \\
\mathbb{V}^N & \xrightarrow{g'} & X \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n+2}.
\end{array}
$$

Here $f' = f \times \text{Id}$, where $f : \mathbb{Z}^N \to X \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1}$ is the normalization map for $\mathbb{Z}$ as in (4.1.2). Note that the map $\Phi_j^1$ is defined since the rational maps $\psi_j^1$ naturally extend to morphisms $\psi_j^1 : \mathbb{P}^1 \times \mathbb{P}^1 \to (\mathbb{P}^1)^3$.

Since $Z$ satisfies the modulus condition, we have $[f^*(F_m^1) - (m+1)f^*\{t = 0\}] \geq 0$ on $\mathbb{Z}^N$. As $f'$ is identity on $\mathbb{P}^1$, this implies that $\sum_{i=3}^{n+1} f''^*[\{y_i = 1\} - (m+1)f'''^*[\{t = 0\}] \geq 0$ on $\mathbb{Z}^N$. Since $\Phi_j^1$ is identity on the last $(n-1)$-copies of $\mathbb{P}^1$, we conclude that $f^* \circ \Phi_j^1 \sum_{i=3}^{n+1} F_{n+i} - (m+1)F_{n+3,0} \geq 0$, which in turn yields $g^* \sum_{i=3}^{n+1} F_{n+i} - (m+1)F_{n+3,0} \geq 0$ on $\mathbb{Z}^N \times \mathbb{P}^1$. Now we deduce from Lemma 2.4.1 that $g^*[\sum_{i=3}^{n+1} F_{n+i} - (m+1)F_{n+3,0}] \geq 0$. In particular, $g^*[F_{n+3} - (m+1)F_{n+3,0}] \geq 0$. Hence, $V'$ (thus $\Gamma_j^1 \star Z$) satisfies the modulus condition.

Our main interest for $\delta^2$ is the following.

Proposition 4.2.2. Let $X$ be a smooth quasi-projective variety over a perfect field of char($k$) $\neq 2$ and let $\alpha \in T_\mathbb{Q}(X, n; m)$ be such that $\partial(\alpha) = 0$. Then $\delta^2(\alpha) = 0$ as a class in $TCH^{q+2}(X, n+2; m)$. In particular, $\delta$ descends to a map of additive higher Chow groups $\delta : TCH^q(X, n; m) \to TCH^{q+1}(X, n+1; m)$ such that $\delta^2 = 0$.

Proof. The last part of the proposition follows from Corollary 4.1.5 once we prove the first part. We begin with the following.

Claim: For any $\alpha \in T_\mathbb{Q}(X, n; m)$ and $\Gamma$ as in (4.2.7), one has $\partial(\Gamma \times \alpha) = \partial \Gamma \times \alpha - \Gamma \times \partial \alpha$ in $z^{q+2}(X \times \mathbb{G}_m \times \mathbb{G}_m, n+2)$.

This is easy; since $\Gamma$ is a $\mathbb{Z}$-linear combination of $\Gamma_j^1$’s, it suffices to prove the claim for each $\Gamma_j^1$. Note that for $c \in \{0, \infty\}$, $\partial_c^i(\Gamma_j^1 \times \alpha) = \partial_c^i(\Gamma_j^1) \times \alpha$ if $1 \leq i \leq 3$, and $\Gamma_j^1 \times \partial_{i-3}^c(\alpha)$ if $4 \leq i \leq n$. From this, we straightforwardly deduce $\partial(\Gamma_j^1 \times \alpha) = \partial(\Gamma_j^1) \times \alpha - \Gamma_j^1 \times \partial \alpha$. This proves the claim.

Now, from the definition of $\delta_n$ in (4.1.1), for any irreducible admissible cycle $Z \in T_\mathbb{Q}(X, n; m)$, $\delta^2(Z)$ is the image of $Z$ under the rational map

$$
X \times \mathbb{G}_m \times \square^{n-1} \to X \times \mathbb{G}_m \times \square^{n+1}
$$

(4.11)

$$(x, t, y_1, \ldots, y_{n-1}) \mapsto (x, t, t^{-1}, t^{-1}, y_1, \ldots, y_{n-1}) = (x, t, u, y_1, \ldots, y_{n-1}).$$

It follows from (4.11) and (4.2.8) that $\delta^2(Z)$ is an irreducible admissible cycle which is the scheme-theoretic inverse image of the irreducible cycle $(t, u) \times Z \subset X \times \mathbb{G}_m^2 \times \square^{n+1}$ (see (4.2.7)) under the closed immersion $\Delta_{\mathbb{G}_m}$.

Since the diagram

$$
\begin{array}{cc}
\mathbb{G}_m \times \square^{n+1} & \mathbb{G}_m \times \square^{n+2} \\
\downarrow{\Delta_{\mathbb{G}_m}} & \downarrow{\Delta_{\mathbb{G}_m}} \\
\mathbb{G}_m \times \mathbb{G}_m \times \square^{n+1} & \mathbb{G}_m \times \mathbb{G}_m \times \square^{n+2}
\end{array}
$$

commutes, it follows that $\delta^2(Z)$ is the image of $Z$ under the rational map $X \times \mathbb{G}_m \times \square^{n-1} \to X \times \mathbb{G}_m \times \square^{n+1}$.
Let $X$ be a smooth quasi-projective $k$-variety, let $(x,t_1,t_2,y_1,\ldots,y_{n-1},y,\lambda)$ be the coordinates of $X \times B_{2,n+3}$. Let $T_X$ be the closed subscheme of $X \times B_{2,n+3}$ defined by the equation $t_1y - 1 = \lambda(t_1t_2y - 1)$, and let $\bar{T}_X$ be its Zariski closure in $X \times \bar{B}_{2,n+3}$.

**Definition 4.3.1.** Let $Z \subset X \times B_{2,n+1}$ be an irreducible closed subvariety. Define $C_Z := T_X \cdot (Z \times \Box^2)$ on $X \times B_{2,n+3}$. Extend it to all cycles $Z$-linearly.

**Lemma 4.3.2.** Let $m_1,m_2 \geq 1$ be integers. Let $Z$ be a cycle on $X \times B_{2,n+1}$ satisfying the modulus $(m_1,m_2)$ condition. Then, $C_Z$ satisfies the modulus $(m_1,m_2)$ condition on $X \times B_{2,n+3}$.

**Proof.** We may assume that $Z$ is irreducible. Let $\overline{Z}$ and $\overline{C}_Z$ be the Zariski closures of $Z$ and $C_Z$ in $X \times \bar{B}_{2,n+1}$ and $X \times \bar{B}_{2,n+3}$, respectively. Then, we have the commutative diagram

$$
\begin{array}{c}
\overline{C}_Z \xrightarrow{\nu_C} \overline{C}_Z \xrightarrow{\iota_C} X \times \bar{B}_{2,n+3} \\
\downarrow{pr} \downarrow{pr} \downarrow{pr}
\end{array}
\begin{array}{c}
\overline{Z} \xrightarrow{\nu_Z} \overline{Z} \xrightarrow{\iota_Z} X \times \bar{B}_{2,n+1}
\end{array}
$$

where $pr : X \times \bar{B}_{2,n+3} \rightarrow X \times \bar{B}_{2,n+1}$ is the projection that ignores the last two coordinates $y,\lambda$, the maps $\nu_Z,\nu_C$ are the normalizations, and the dominant maps $\overline{pr},\overline{pr}_N$ are induced by $pr$. Since $Z$ satisfies the modulus $(m_1,m_2)$ condition, we have $\sum_{j=1}^{2}(m_1+1)[\nu_{C}^*\nu_{Z}^*\{t_j = 0\}] \leq \sum_{i=1}^{n-1}[\nu_{C}^*\nu_{Z}^*\{y_i = 1\}]$. Then, by pulling back via $\overline{pr}$ and using the commutativity of the diagram, we have $\sum_{j=1}^{2}(m_1+1)[\nu_{C}^*\nu_{Z}^*\{t_j = 0\}] \leq \sum_{i=1}^{n-1}[\nu_{C}^*\nu_{Z}^*\{y_i = 1\}]$. But $\overline{pr}^*\{t_j = 0\} = \{t_j = 0\}$ for $j = 1,2$ and $\overline{pr}^*\{y_i = 1\} = \{y_i = 1\}$ for $1 \leq i \leq n-1$. Hence,

$$
2 \sum_{j=1}^{2}(m_1+1)[\nu_{C}^*\nu_{Z}^*\{t_j = 0\}] \leq \sum_{i=1}^{n-1}[\nu_{C}^*\nu_{Z}^*\{y_i = 1\}] \leq \sum_{i=1}^{n+1}[\nu_{C}^*\nu_{Z}^*\{y_i = 1\}],
$$

which is the modulus $(m_1,m_2)$ condition for $C_Z$.\[ \square\]

**Corollary 4.3.3.** Let $m_1,m_2 \geq 1$. Let $Z$ be a cycle on $X \times B_{2,n+1}$ satisfying the modulus $(m_1,m_2)$ condition. Then, $\mu_*(C_Z)$ as in Definition 3.1.3 on $X \times B_{n+2}$ is well-defined, and it satisfies the modulus $\min\{m_1,m_2\}$ condition.

**Proof.** By Lemma 4.3.2, $C_Z$ satisfies the modulus $(m_1,m_2)$ condition so that $\mu_*(C_Z)$ satisfies the modulus $\min\{m_1,m_2\}$ condition by Proposition 3.1.6.\[ \square\]

**Definition 4.3.4.** $n = n_1 + n_2 - 1$. Let $Z_i \in T_{\mathbb{Z}}^q(X,n_i;m)$ for $i = 1,2$. Let $Z_1 \times \mu^* Z_2$ be the cycle $\sigma \cdot \mu_*(C_{Z_1 \times Z_2})$ on $X \times X \times B_{n+2}$, where $\sigma = (n+1,n,\cdots,1)$ is the square of the cyclic permutation on $\{1,2,\cdots,n+1\}$.

**Remark 4.3.5.** One can describe $Z_1 \times \mu^* Z_2$ as follows: consider the rational map

$$
\mu^* : X \times X \times \mathbb{G}_m \times \mathbb{G}_m \times \Box^{n-1} \times \Box \rightarrow X \times X \times \mathbb{G}_m \times \Box^{n+1}
$$

(4.3.1)

$$
\mu^*(x_1,x_2,t_1,t_2,y_1,\ldots,y_{n-1},y) = (x_1,x_2,t_1t_2,y_1,t_1y - \frac{1}{t_1t_2y - 1},y_1,\ldots,y_{n-1}).
$$

Then, set-theoretically $Z_1 \times \mu^* Z_2$ is equal to the Zariski closure of $\mu^*(Z_1 \times Z_2 \times \Box)$.\[ \square\]
Remark 4.3.6. The above product $\times_{\mu'}$ is motivated by the computation in [Park, 2007, Lemma 2.5].

For $t_1, t_2 \in \mathbb{G}_m$, let $C_{t_1, t_2} := \{t_1\} \times_{\mu'} \{t_2\} \subset \mathbb{G}_m \times \mathbb{D}^2$, which is the parameterized curve $(y \in k) \mapsto (t_1 t_2, y, (t_1 y - 1)/(t_1 t_2 y - 1)) \in \mathbb{G}_m \times \mathbb{D}^2$. This is an admissible 1-cycle and its boundary is given (loc.cit.) by $\partial C_{t_1, t_2} = -(t_1 t_2, 1/t_1) - (t_1 t_2, 1/t_2) + (t_1 t_2, t_1/t_2).

Let $n = n_1 + n_2 - 1$ and $q = q_1 + q_2 - 1$.

Proposition 4.3.7. $Z_1 \times_{\mu'} Z_2$ is an admissible cycle in $\mathbb{T}_qZ^{q+1}(X \times X, n + 2; m)$.

**Proof.** By Corollary 4.3.3, the cycle $Z_1 \times_{\mu'} Z_2$ satisfies the modulus $m$ condition.

We now prove that $Z' := Z_1 \times_{\mu'} Z_2$ intersects properly with all faces. It is easy to see from Definition 4.3.4 and (4.3.1) that when $\sigma_{n_1}$ is the permutation $(n_1, n_1 - 1, \ldots, 2, 1)$, we have

\[
\begin{aligned}
\partial^\infty_1(Z') &= \sigma_{n_1} \cdot (Z_1 \times_{\mu} \delta(Z_2)), \\
\partial^1_1(Z') &= 0,
\end{aligned}
\]

\[
\begin{aligned}
\partial^\infty_2(Z') &= \delta(Z_1 \times_{\mu} Z_2), \\
\partial^1_2(Z') &= \delta(Z_1) \times_{\mu} Z_2,
\end{aligned}
\]

(4.3.2)

Since $Z'_i$’s are admissible, this shows the proper intersection property of $Z'$, by Corollary 3.1.7, Proposition 4.1.4, and induction on the codimension of the faces.

By Proposition 4.3.7, we have the $\mathbb{Z}$-bilinear map

\[
\times_{\mu'} : \mathbb{T}_qZ^1(X, n_1; m) \otimes \mathbb{T}_qZ^2(X, n_2; m) \to \mathbb{T}_qZ^{q+1}(X \times X, n + 1; m).
\]

Proposition 4.3.8 (Leibniz rule). Let $X$ be a smooth quasi-projective variety, and let $\xi \in \mathbb{T}_qZ^1(X, n_1; m), \eta \in \mathbb{T}_qZ^2(X, n_2; m)$. Let $n = n_1 + n_2 - 1, q = q_1 + q_2 - 1$. Then, in the group $\mathbb{T}_qZ^{q+1}(X \times X, n + 1; m)$, we have

\[
\delta(\xi \times_{\mu'} \eta) - \delta(\xi \times_{\mu} \eta) - (-1)^{n_1 - 1}\xi \times_{\mu'} \eta \delta(\xi) - \partial(\xi) \times_{\mu'} \eta + \partial(\eta)
\]

(4.3.4)

where $\gamma$ is an admissible cycle as in Lemma 3.3.1, up to sign.

**Proof.** It follows from (4.3.2) that

\[
\begin{aligned}
\delta(\xi \times_{\mu'} \eta) &= \sum_{i=1}^{n+1} (-1)^i (\partial^\infty_i - \partial^0_i)(\xi \times_{\mu'} \eta) \\
&= \delta(\xi \times_{\mu} \eta) - \{\sigma_{n_1} \cdot (\xi \times_{\mu} \delta(\eta)) + \delta(\xi) \times_{\mu} \eta\} + \sum_{i=3}^{n+1} (-1)^i (\partial^\infty_{i-2} - \partial^0_{i-2})(\xi) \times_{\mu} \eta \\
&\quad + \sum_{i=n_1+2}^{n+1} (-1)^i (\partial_{i-n_1-1} - \partial^0_{i-n_1-1})(\eta) \\
&= \delta(\xi \times_{\mu} \eta) - \{\sigma_{n_1} \cdot (\xi \times_{\mu} \delta(\eta)) + \delta(\xi) \times_{\mu} \eta\} + (\partial(\xi) \times_{\mu'} \eta + (-1)^{n_1 - 1}\xi \times_{\mu'} \partial(\eta)).
\end{aligned}
\]

This is equivalent to the identity

\[
\delta(\xi \times_{\mu} \eta) - \delta(\xi \times_{\mu} \eta - \sigma_{n_1} \cdot (\xi \times_{\mu} \delta(\eta)) = \delta(\xi \times_{\mu'} \eta) - \delta(\xi \times_{\mu'} \eta - (1)^{n_1 - 1}\xi \times_{\mu'} \partial(\eta).
\]

(4.3.5)

But, by Lemma 3.3.1, we have $\sigma_{n_1} \cdot (\xi \times_{\mu} \delta(\eta)) = (-1)^{n_1 - 1}\xi \times_{\mu} \delta(\eta) + \partial(\gamma)$ for an admissible cycle $\gamma$. Hence, we deduce (4.3.4) as desired.

We put together all pieces so far to conclude another main result of this paper.

Theorem 4.3.9. Let $X$ be a smooth projective variety over a perfect field $k$ of char $\neq 2$. Then, the additive higher Chow groups (TCH$(X; m), \wedge, \delta$) form a graded-commutative differential graded algebra, where $\delta$ is the graded derivation for the internal product $\wedge$. The derivation commutes with the pull-back and push-forward maps of additive higher Chow groups.

**Proof.** This follows from Theorem 3.5.1 and Proposition 4.3.8.

Remark 4.3.10. (cf. Remark 3.0.3) If the moving lemma [Krishna and Park, 2012a, Theorem 4.1] holds also for smooth quasi-projective varieties, then Theorem 4.3.9 extends to all smooth quasi-projective varieties as well.
5 Witt-complex structure

So far, we considered mostly additive higher Chow groups with a fixed modulus \( m \geq 1 \). In this section, we consider the projective system of additive higher Chow groups over all modulus \( m \geq 1 \), and we show that for smooth projective varieties, they form a restricted Witt-complex over \( k \). Recall:

**Definition 5.0.11 ([Rülling, 2007, Definition 1.14], [Hesselholt and Madsen, 2001, Definition 1.1.1])**. Let \( A \) be a commutative ring with unity. A restricted Witt-complex over \( A \) is defined to be a projective system of differential graded \( \mathbb{Z} \)-algebras

\[
((E_m)_{m \in \mathbb{N}}, R: E_{m+1} \to E_m),
\]

together with families of homomorphisms of graded rings

\[
(F_r: E_{rm+r-1} \to E_{m}^r)_{m,r \in \mathbb{N}},
\]

and homomorphisms of graded groups

\[
(V_r: E_m \to E_{rm+r-1})_{m,r \in \mathbb{N}},
\]

satisfying the following relations for all \( n, r \in \mathbb{N} \):

(i) \( R F_r = F_r R, R' V_r = V_r R, F_1 = V_1 = 1d, F_r F_s = F_{rs}, V_r V_s = V_{rs} \).

(ii) \( F_r V_r = r. \) When \( (r, s) = 1 \), then \( F_r V_s = V_s F_r \) on \( E_{rm+r-1} \).

(iii) \( V_r(F_r(x)y) = x V_r(y) \) for all \( x \in E_{rm+r-1} \) and \( y \in E_m \).

(iv) \( F_r dV_r = d \) (where \( d \) denotes the differential of the DGAs).

Furthermore, there is a homomorphism of projective systems of rings \( (\lambda: \mathbb{W}_m(A) \to E_m^0)_{m \in \mathbb{N}} \) that commutes with \( F_r \) and \( V_r \), and satisfies

\[
(\nu_r \circ \lambda(a)) = \lambda([a]^{-1})d\lambda([a]) \text{ for all } a \in A \text{ and } r \in \mathbb{N}.
\]

Here, \( \mathbb{W}_m(A) \) is the Witt ring of \( A \) associated to the truncation set \( \{1, 2, \cdots, m\} \), and \( [a] \) is the Teichmüller lift of \( a \).

The category of restricted Witt-complexes over \( A \) has an initial object given by the projective system \( (\mathbb{W}_m^\bullet(A)) \) of de Rham-Witt complexes of [Hesselholt and Madsen, 2001], and when \( A \) is any field \( k \) of characteristic \( \neq 2 \), it is proven in [Rülling, 2007] that this projective system is realized by the collection of additive higher Chow groups of zero cycles for \( X = \text{Spec} (k) \). Our goal is to study its higher dimensional analogue, Theorem 5.3.1: when \( X \) is smooth projective, the system of DGAs \( \{\text{TCH}(X; m)\}_{m \geq 1} \) has a structure of a restricted Witt-complex over \( k \).

5.1 Frobenius operators

For \( r \geq 1 \), consider the \( r \)-fold product \( \Phi_r: \mathbb{G}_m \to \mathbb{G}_m, a \mapsto a^r \). This naturally extends to \( \Phi_r: \mathbb{A}^1 \to \mathbb{A}^1 \). In both cases, the maps \( \Phi_r \) are finite and flat. For each equidimensional \( k \)-scheme \( X \), we have the induced finite flat morphisms \( X \times \mathbb{G}_m \to X \times \mathbb{G}_m \) and \( X \times B_n \to X \times B_n \), also denoted by \( \Phi_r \).

**Lemma 5.1.1.** Let \( Z \in \mathbb{N}^q(X; n; rm + r - 1) \) be an irreducible admissible cycle. Then, the push-forward \( \Phi_{rs}(Z) \) as an abstract algebraic cycle is admissible, and lies in \( \mathbb{N}^q(X; n; m) \).

**Proof.** The morphism \( \Phi_r: X \times \mathbb{G}_m \to X \times \mathbb{G}_m \) is finite. Hence, the cycle \( \Phi_{rs}(Z) \) is in \( z^q(X \times \mathbb{G}_m, n - 1) \), i.e., it is in good position with all faces.

To show that \( \Phi_{rs}(Z) \) satisfies the modulus \( m \) condition, it is enough to show that the image \( W := \Phi_r(Z) \) satisfies the modulus condition. (N.B. It is not a corollary of [Krishna and Levine, 2008, §3.3].) Let \( Z, W \) be the Zariski closures of \( Z \) and \( W \) in \( X \times \overline{B}_n \), and let \( \nu_Z \) and \( \nu_W \) be the normalizations of the closures. Then, we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}^N & \xrightarrow{\nu_Z} & Z \\
\downarrow \Phi^* & & \downarrow \overline{\Phi}\ \\
\overline{W}^N & \xrightarrow{\nu_W} & \overline{W}
\end{array}
\]

Then, \( D_m := \sum_{i=1}^{n-1} \{y_i = 1\} - (m + 1)t = 0 \). Then, \( \Phi^* D = \sum_{i=1}^{n-1} \{y_i = 1\} - r(m + 1)t = 0 \). By the given assumption, we have \( [\nu_Z^* \Phi^* D_{rm+r-1}] \geq 0 \), which is equivalent to \( ([\Phi^*])^* \nu_W^* t_W^* D_m \geq 0 \). By Lemma 2.4.1, we have \( [\nu_W^* t_W^* D_m] \geq 0 \), i.e., \( W \) satisfies the modulus \( m \) condition. This completes the proof. \( \blacksquare \)
Definition 5.1.2. For \( r \geq 1 \), let \( F_r = \phi_{r,*} \). We call it the \( r \)-th Frobenius on additive higher Chow cycles. One notes immediately that \( F_r \) commutes with the operations of intersecting with the faces so that \( F_r \) gives a morphism of complexes \( F_r : \text{Tz}^q(X, \bullet; rm + r - 1) \to \text{Tz}^q(X, \bullet; m) \) and a homomorphism \( F_r : \text{TCH}^q(X, n; rm + r - 1) \to \text{TCH}^q(X, n; m) \).

Lemma 5.1.3. Let \( f : X \to Y \) be a morphism of quasi-projective varieties over \( k \), where \( Y \) is smooth and projective. Then, the pull-backs \( f^* : \text{TCH}^q(Y, n; m) \to \text{TCH}^q(X, n; m) \) constructed in [Krishna and Park, 2012a, Theorem 7.1] are compatible with the Frobenius map \( F_r \) for each \( r \geq 1 \).

Proof. From the proof of [Krishna and Park, 2012a, Theorem 7.1], for some finite collection \( \mathcal{W} \) of locally closed subsets of \( Y \), we have chain maps \( \text{Tz}_W^q(Y, \bullet; m) \xrightarrow{pr_2^*} \text{Tz}_{[r]}^q(X \times Y, \bullet; m) \xrightarrow{gr_r^f} \text{Tz}^q(X, \bullet; m) \), whose composition gives \( f^* \). Here, \( \text{Tz}_W^q(Y, n; m) \) is defined in [Krishna and Park, 2012a, Definition 4.2]. Since all of \( F_r \), \( pr_2^* \), \( gr_r^f \) are equivariant with respect to taking faces, we need to check that the following diagram commutes, where \( m_r := rm + r - 1 \):

\[
\begin{array}{ccc}
\text{Tz}_W^q(Y, n; m_r) & \xrightarrow{pr_2^*} & \text{Tz}_{[r]}^q(X \times Y, n; m_r) \\
& \downarrow F_r & \downarrow F_r \\
\text{Tz}_W^q(Y, n; m) & \xrightarrow{pr_2^*} & \text{Tz}_{[r]}^q(X \times Y, n; m) \\
& \downarrow F_r & \\
& \text{Tz}^q(X, n; m_r) & \xrightarrow{gr_r^f} \text{Tz}^q(X, n; m). \\
\end{array}
\]

But the equality \( F_r \circ pr_2^* = pr_2^* \circ F_r \) just holds by definition, as long as the flat-pull-back \( pr_2^* \) and finite push-forward \( F_r \) are well-defined, which is the case. (One can also deduce it from [Fulton, 1998, Proposition 1.7] regarding the cycles as Chow cycles.) Similarly, we have the equality \( F_r \circ gr_r^f = gr_r^f \circ F_r \) since \( gr_r^f \) is the well-defined Gysin pull-back, by [Krishna and Park, 2012a, Corollary 7.2]. Taking the homology groups and using the moving lemma [Krishna and Park, 2012a, Theorem 4.1], we obtain the desired compatibility \( F_r \circ f^* = f^* \circ F_r \).

Lemma 5.1.4. Let \( X \) and \( Y \) be equidimensional \( k \)-schemes. Then, the external product \( \times_{\mu} \) is compatible with the Frobenius \( F_r \) for each \( r \geq 1 \). More precisely, the following diagram commutes, where \( m_r := rm + r - 1 \):

\[
\begin{array}{ccc}
\text{TCH}(X; m_r) \otimes \text{TCH}(Y; m_r) & \xrightarrow{\times_{\mu}^r} & \text{TCH}(X \times Y; m_r) \\
& \downarrow F_r \otimes F_r & \downarrow F_r \\
\text{TCH}(X; m) \otimes \text{TCH}(Y; m) & \xrightarrow{\times_{\mu}^r} & \text{TCH}(X \times Y; m). \\
\end{array}
\]

Proof. This immediately follows from Theorem 3.5.1 together with the fact that \( (ab)^r = a^r \cdot b^r \) for \( a, b \in \mathbb{G}_m \).

Corollary 5.1.5. Let \( X \) be a smooth projective variety over \( k \). Then, the internal product \( \wedge \) is compatible with the Frobenius map \( F_r \) for each \( r \geq 1 \). In particular, \( F_r \) is a graded ring homomorphism with respect to the internal product \( \wedge \).

Proof. The statement is equivalent to the commutativity of the outer rectangle of the following diagram, where \( m_r := rm + r - 1 \):

\[
\begin{array}{ccc}
\text{TCH}(X; m_r) \otimes \text{TCH}(X; m_r) & \xrightarrow{\times_{\mu}^r} & \text{TCH}(X \times X; m_r) \\
& \downarrow F_r \otimes F_r & \downarrow F_r \\
\text{TCH}(X; m) \otimes \text{TCH}(X; m) & \xrightarrow{\times_{\mu}^r} & \text{TCH}(X \times X; m) \\
& \downarrow F_r & \downarrow F_r \\
& \text{TCH}(X; m) & \xrightarrow{\Delta^r} \text{TCH}(X; m). \\
\end{array}
\]

But, by Lemma 5.1.3 the right square is commutative and by Lemma 5.1.4 the left square is commutative. Hence, the outer rectangle is commutative, as desired.
5.2 Verschiebung operators

We now define the Verschiebung operators. Let $X$ be an equidimensional $k$-scheme, and let $r \geq 1$ and $\phi_r$ be as in Section 5.1.

**Lemma 5.2.1.** Let $Z \in Tz^q(X, n; m)$ be an irreducible admissible cycle. Then, the pull-back $\phi_r^*(Z)$ as an abstract algebraic cycle is admissible, thus lies in $Tz^q(X, n; rm + r - 1)$.

**Proof.** The morphism $\phi_r: X \times \mathbb{G}_m \rightarrow X \times \mathbb{G}_m$ is flat. Hence, the cycle $\phi_r^*(Z)$ is in $z^q(X \times \mathbb{G}_m, n-1)$, i.e., it is in good position with all faces.

It remains to show that $\phi_r^*(Z)$ satisfies the modulus $(rm + r - 1)$ condition. (N.B. It is not a corollary of [Krishna and Levine, 2008, §3.4] Let $W$ be an irreducible component of $\phi_r^*(Z)$. It is enough to prove the modulus condition for $W$. Let $\overline{W}$ and $\overline{Z}$ be the Zariski closures of $W$ and $Z$ in $X \times \overline{\mathbb{G}}_n$, and let $\nu_W$ and $\nu_Z$ be the normalizations of the closures. Then, we have a commutative diagram

$$
\begin{array}{ccc}
\overline{W}^N & \xrightarrow{\nu_W} & \overline{W} \\
\phi_r^N & \xrightarrow{\phi_r} & \overline{Z} \\
\overline{Z}^N & \xrightarrow{\nu_Z} & \overline{Z} \\
\end{array}
$$

Here, let $D_m := \sum_{i=1}^{n-1} \{ y_i = 1 \} - (m+1)t \{ t = 0 \}$. By the modulus $m$ condition satisfied by $Z$, we have $[\nu_W^* \nu_Z^* D_m] \geq 0$. Hence, $[(\phi_r^N)^* \nu_Z^* D_m] \geq 0$. This is equivalent to $[\nu_W^* \nu_Z^* D_m] \geq 0$. But, $\phi_r^* D_m = D_{rm+r-1}$ as seen in the proof of Lemma 5.1.1, so that $W$ satisfies the modulus $(rm + r - 1)$ condition, as desired. ■

**Definition 5.2.2.** For $r \geq 1$, let $V_r = \phi_r^*$. We call it the $r$-th Verschiebung on additive higher Chow cycles. One notes immediately that $V_r$ commutes with intersections with the faces so that $V_r$ gives a morphism of complexes $V_r: Tz^q(X, \bullet, m) \rightarrow Tz^q(X, \bullet, rm + r - 1)$ and a homomorphism $V_r: TCH^q(X, n; m) \rightarrow TCH^q(X, n; rm + r - 1)$.

**Lemma 5.2.3.** Let $f: X \rightarrow Y$ be a morphism of quasi-projective varieties over $k$, where $Y$ is smooth projective. Then, the pull-backs $f^*: TCH^q(Y, n; m) \rightarrow TCH^q(X, n; m)$ constructed in [Krishna and Park, 2012a, Theorem 7.1] are compatible with the Verschiebung map $V_r$ for each $r \geq 1$.

**Proof.** As in the proof of Lemma 5.1.3, by [Krishna and Park, 2012a, Theorem 7.1], for some finite collection $W$ of locally closed subsets of $Y$, we have that $Tz^q_W(Y, \bullet, m) \xrightarrow{pr_2^*} Tz^q_{\{ f \}}(X \times Y, \bullet, m)$ whose composition gives $f^*$. Since all of $V_r, pr_2^*, gr_f^*$ are equivariant with respect to taking faces, it is enough to check that the following diagram commutes, where $m_r := rm + r - 1$:

$$
\begin{array}{ccc}
Tz^q_W(Y, n; m) & \xrightarrow{pr_2^*} & Tz^q_{\{ f \}}(X \times Y, n; m) \\
\downarrow V_r & & \downarrow V_r \\
Tz^q_Y(X, n; m_r) & \xrightarrow{pr_2^*} & Tz^q_Y(X, n; m_r) \\
\end{array}
$$

But, the equalities $V_r \circ pr_2^* = pr_2^* \circ V_r$ and $V_r \circ gr_f^* = gr_f^* \circ V_r$ just hold by definition as long as all pull-backs in the expressions make sense, which is the case, because $pr_2$ is flat. $\phi_r^* D_m$ is the Gysin pull-back of [Krishna and Park, 2012a, Corollary 7.2]. Taking the homology groups and using the moving lemma [Krishna and Park, 2012a, Theorem 4.1], we obtain the desired compatibility $V_r \circ f^* = f^* \circ V_r$. ■

The following observations about the Frobenius and the Verschiebung maps in the positive characteristic will be useful in proving some of the axioms of the Witt-complex structure on the additive higher Chow groups.

**Lemma 5.2.4.** Assume that the ground field $k$ is of characteristic $p > 0$ and let $Z \subseteq X \times \mathbb{G}_m \times \square^{n-1}$ be an integral closed subvariety representing an element $[Z] \in Tz^q(X, n; m)$, where $X$ is any smooth quasi-projective variety over $k$. Let $Z'$ denote the scheme theoretic inverse image of $Z$ under the map $\phi_p$. Then, $Z'$ is irreducible and one of the following holds:

(i) $Z'$ is reduced. In this case, $[\kappa(Z') : \kappa(Z)] = [\kappa(\delta(Z')) : \kappa(\phi_p \circ \delta(Z'))] = p$ and $\phi_p^*[Z] = [Z']$.
(ii) $Z'$ is non-reduced. In this case, $[\kappa(Z'_\text{red}) : \kappa(Z)] = [\kappa(\delta(Z'_\text{red})) : \kappa(\phi_p \circ \delta(Z'_\text{red}))] = 1$ and $\phi_p^*[Z] = p[Z'_\text{red}]$. 


In both cases, one has $|\kappa(\phi_p \circ \delta(Z'_\text{red})) : \kappa(\chi_{n+1,p} \circ \phi_p \circ \delta(Z'_\text{red}))| = 1.$

**Proof.** Since $k$ is of characteristic $p > 0$, it follows easily from the definition of $\phi_p$ that $Z'$ is irreducible, and it is either reduced, or defined by the $p$-th power of the ideal defining $Z'_\text{red}$ in $X \times \mathbb{G}_m \times \square^{n-1}$. One has $\phi_p^*(\kappa(Z)) = [Z']$ in the first case and $\phi_p^*(\kappa(Z)) = p[Z'_\text{red}]$ in the second. Since $F_p \circ V_p = p$ (see Claim (ii) of the proof of Theorem 5.3.1, Part 1), we must have $|\kappa(Z') : \kappa(Z)| = p$ in the first case and $|\kappa(Z'_\text{red}) : \kappa(Z)| = 1$ in the second.

To check the degrees of the other function field extensions, we use the commutative squares of finite maps (cf. (5.5.1))

$$
\begin{array}{ccc}
Z'_\text{red} & \xrightarrow{\delta} & Z'_\text{red} \\
\phi_p & \downarrow & \phi_p \\
Z & \xrightarrow{\delta} & Z.
\end{array}
$$

Here, the horizontal maps in the right square are induced by the projection map $X \times \mathbb{G}_m \times \square^{n-1} \rightarrow X \times \mathbb{G}_m \times \square^{n-1}$ which ignores the first box. The assertions about the degrees of the other field extensions follow easily from the commutativity of the above squares and the fact that the horizontal maps in the two squares are isomorphisms.

Given a finite map $f : X \rightarrow Y$ of smooth quasi-projective varieties over a field $k$ and a cycle $\gamma = \sum_{i=1}^d m_i V_i \in \mathbb{T}Z^q(X, m)$, let $f(\gamma) \in \mathbb{T}Z^q(Y, m)$ be the cycle $f(\gamma) := \sum_{i=1}^d m_i f(V_i)$.

**Lemma 5.2.5.** Let $X$ be a smooth quasi-projective variety over a field $k$ of characteristic $p > 0$. Let $\gamma = \sum_{i=1}^d m_i V_i \in \mathbb{T}Z^q(X, n; m)$ be a cycle such that $\chi_{n,p}(\gamma) = 0$. Then $\gamma = 0$.

**Proof.** This is an immediate consequence of the fact that $\chi_{n,p} : V \rightarrow \chi_{n,p}(V)$ is bijective for each $i$ and that $\mathbb{T}Z^q(X, m)$ is a free abelian group on integral admissible cycles on $X \times \mathbb{G}_m \times \square^{n-1}$.

**Lemma 5.2.6.** Let $X$ be a smooth quasi-projective variety over a field $k$ of characteristic $p > 0$. Given a cycle $\gamma = \sum_{i=1}^d m_i V_i \in \mathbb{T}Z^q(X, n; m)$, let $\xi = \sum_{i=1}^d m_i \phi_p(V_i)$. Assume that $\partial(\xi) = 0$. Then, $\partial_{n+1,1}(\phi_p(\delta(\gamma))) = 0 = \partial(\phi_p(\delta(\gamma)))$.

**Proof.** We have shown in the proof of Proposition 4.1.4 that $\{\delta(V_i) \cap \{y_i = 0, \infty\} = \emptyset$. Since the restriction to a face of $\square^n$ commutes with $\phi_p$, we see that $\{\phi_p(\delta(V_i)) \cap \{y_i = 0, \infty\} = \emptyset$. Since $\chi_{n+1,p}(\phi_p(V)) = \chi_{n+1,p}(V)$ for $\epsilon = 0, \infty$, we also get $\chi_{n+1,p}(\phi_p(V_i)) \cap \{y_i = 0, \infty\} = \emptyset$. We conclude for each $i \geq 1$ that $\partial_{n+1,1}(\phi_p(\delta(V_i))) = 0 = \partial_{n+1,1}(\chi_{n+1,p}(\phi_p(V_i)))$. In particular, we get

$$
\partial_{n+1,1}(\phi_p(\delta(\gamma))) = 0 = \partial_{n+1,1}(\chi_{n+1,p}(\phi_p(\delta(\gamma))).
$$

On the other hand, we know that

$$
0 = -\delta \partial(\xi) = \partial(\delta(\xi)) = \partial(\delta(V_i)) = \partial(V_i) + \chi_{n+1,p}(\phi_p(\delta(\gamma))),
$$

where the second and the fourth equalities follow from Proposition 4.1.4 and the diagram (5.5.1) below, respectively. We conclude from (5.2.2) that $\sum_{i=2}^n \chi_{n+1,p}(\phi_p(\delta(\gamma))) = 0$. As $\chi_{n+1,p}(\phi_p(\delta(\gamma))) = 0$, we conclude for each $i \geq 1$ that $\partial_{n+1,1}(\phi_p(\delta(V_i))) = 0 = \partial_{n+1,1}(\chi_{n+1,p}(\phi_p(V_i)))$. In particular, we get

$$
\partial_{n+1,1}(\phi_p(\delta(\gamma))) = 0 = \partial_{n+1,1}(\chi_{n+1,p}(\phi_p(V_i))).
$$

5.3 Witt-complex structure

From [Rülling, 2007], recall that for each $m \geq 1$, we have an isomorphism $\lambda : \mathcal{W}_m(k) \xrightarrow{\sim} \mathcal{TCH}^q(X, m)$. Furthermore, the structure map $f : X \rightarrow \text{Spec}(k)$ induces the pull-back $f^* : \mathcal{TCH}^q(k, 1; m) \rightarrow \mathcal{TCH}^q(X, 1; m)$. We let $\lambda_X := f^* \circ \lambda$. On the other hand, the natural injections $\mathbb{T}Z^q(X, \bullet; m + 1) \rightarrow \mathbb{T}Z^q(X, \bullet; m)$ induce the natural maps $R : \mathcal{TCH}^q(X, n; m + 1) \rightarrow \mathcal{TCH}^q(X, n; m)$.

The final objective of the paper is to show the following:
**Theorem 5.3.1.** Let $X$ be a smooth projective variety over a perfect field $k$ of characteristic $\neq 2$. Let $E_m := \text{TCH}(X; m) = \bigoplus_{q,n \geq 1} \text{TCH}^q(X; n; m)$ with its DGA-structure considered as in Theorem 4.3.9. Then, the projective system $(E_m, R)_{m \in \mathbb{N}}$ together with the maps $F_r, V_r, \lambda_X, \delta$ is a restricted Witt-complex over $k$ in the sense of Definition 5.0.11.

**Proof.** (of Theorem 5.3.1; Part 1) By Theorem 4.3.9, each $E_m$ is a DGA, and by Corollary 5.1.5 each $F_r: E_{rm+r-1} \rightarrow E_m$ is a graded ring homomorphism. Furthermore, by Lemmas 5.1.3 and 5.2.3 applied to $f$, we have $\lambda_X F_r = F_r \lambda_X$ and $\lambda_X V_r = V_r \lambda_X$. Hence, it is enough to check (i) - (v) in Definition 5.0.11 for $A = k$. Here, we shrink the shorthand product notation $\wedge$. Most of the arguments below are similar to those in the proof of [Rülling, 2007, Proposition 3.17], except (iv), that we check separately.

Claim: (i) $RF_r = F_r R^r$, $R^r V_r = V_r R$, $F_1 = 1 = Id$, $F_r F_s = F_s F_r$, $V_r V_s = V_s V_r$.

The identity $RF_r = F_r R^r$ (where $R = R \circ \cdots \circ R$) is immediate from the definitions. The same for the identity $R^r V_r = V_r R$. The identity $F_1 = 1 = Id$ holds since $\phi_1 = \phi_1^* = Id$. That $F_r F_s = F_s F_r$ and $V_r V_s = V_s V_r$ holds because $\phi_r \phi_s = \phi_s \phi_r = \phi_{rs}$, i.e., $(a^s)^r = (a^r)^s = a^{rs}$ for $a \in \mathbb{G}_m$.

Claim: (ii) $F_r V_r = r$. When $(r, s) = 1$, then $F_r V_s = V_s F_r$ on $E_{rm+r-1}$.

That $F_r V_r = r$ follows from the general identity on Chow cycles $(g, g^s) \cdot Id$ for any finite and flat morphism $g$. When $(r, s) = 1$, by applying [Fulton, 1998, Proposition 1.7] to the following Cartesian square, we deduce that $F_r V_s = V_s F_r$.

\[ \begin{array}{ccc}
\mathbb{G}_m & \xrightarrow{\phi_s} & \mathbb{G}_m \\
\phi_r & \downarrow & \phi_r \\
\mathbb{G}_m & \xrightarrow{\phi_s} & \mathbb{G}_m
\end{array} \]

Claim: (iii) $V_r(F_r(x)y) = x V_r(y)$ for all $x \in E_{rm+r-1}$ and $y \in E_m$.

By applying again [Fulton, 1998, Proposition 1.7] to the following Cartesian square,

\[ \begin{array}{ccc}
\mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{\text{Id} \times \phi_r} & \mathbb{G}_m \times \mathbb{G}_m \\
\downarrow \mu & & \downarrow \mu \circ (\phi_r \times \text{Id}) \\
\mathbb{G}_m & \xrightarrow{\phi_r} & \mathbb{G}_m
\end{array} \]

we deduce $\mu((\text{Id} \times \phi_r)^s) = \phi_r^s((\mu \circ (\phi_r \times \text{Id}))_s)$, which is (iii).

Claim: (v) $F_r \delta \lambda_X([a]) = \lambda_X([a]^{-1}) \delta \lambda_X([a])$ for all $a \in k$ and $r \in \mathbb{N}$.

By [Rülling, 2007, Proposition 3.17], we have $F_r \delta \lambda([a]) = \lambda([a]^{-1}) \delta \lambda([a])$. Hence by pulling back via $f$, we obtain the desired relation.

Thus, we checked everything except (iv) in Definition 5.0.11. This remaining piece will be checked in Section 5.5 after several calculations in Section 5.4.

---

### 5.4 Some more cycle computations

Let $X$ be a smooth quasi-projective $k$-variety. Let $n \geq 2$ and $i \geq 1$. Given an irreducible closed subvariety $Z \subset X \times B_n$, define $L_{n,i}(Z)$ to be the cycle $\sigma \cdot \gamma^i_Z$ coming from the cycle defined in Definition 3.2.3 and (3.2.4), where $\sigma = (n, n-1, \cdots, 1)^2$ is the square of a cyclic permutation on $\{1, 2, \cdots, n\}$. We extend this definition $\mathbb{Z}$-linearly so that $L_{n,i}(Z)$ is defined for all cycles $Z$.

**Remark 5.4.1.** One can also describe this cycle as follows: consider the rational map $l_{n,i}: X \times \mathbb{G}_m \times \Box^{n-1} \times \Box \rightarrow X \times \mathbb{G}_m \times \Box^n$

\[ l_{n,i}: (x, t, y_1, \cdots, y_{n-1}) \times y \mapsto (x, t, y, \frac{y-y_1^n}{y_1^{i-1}}, y_2, \cdots, y_{n-1}). \]  

(5.4.1)

Given an irreducible closed subvariety $Z \subset X \times \mathbb{G}_m \times \Box^{n-1}$, the cycle $L_{n,i}(Z)$ is the Zariski closure of $l_{n,i}(Z \times \Box)$ in $X \times \mathbb{G}_m \times \Box^n$.

**Lemma 5.4.2.** Let $Z \in \mathbb{T}^q(X; n; m)$. Then, $L_{n,i}(Z) \in \mathbb{T}^q(X, n + 1; m)$.

**Proof.** By definition, the cycle $L_{n,i}(Z)$ is a permutation of the admissible (by Lemma 3.2.5) cycle $\gamma^i_Z$. Hence, $L_{n,i}(Z)$ is also admissible.

\[ \square \]
Theorem 5.3.1. Let $Z \in T \mathbb{Q}^{q}(X, n; m)$ be a cycle such that $\partial Z = \partial_{1}(Z)$. Then,

$$\partial (L_{n,i}(Z)) = \left\{ \begin{array}{ll}
Z + Z\{i-1\} - Z\{i\}, & \text{if } n = 2, \\
Z + Z\{i-1\} - Z\{i\} - L_{n-1,i}(\partial(Z)), & \text{if } n > 2,
\end{array} \right.$$  \hspace{1cm} (5.4.2)

where $Z\{i\}$ is as defined in Section 3.2.1.

Proof. We see (cf. (5.4.1)) that $\partial Z = \partial_{1}(Z) = Z$ and $\partial_{2}(Z) = Z\{i-1\}$. Hence, $\partial (L_{n,i}(Z)) = Z\{i\}$. So, when $n > 2$, we have $\partial (L_{n,i}(Z)) = Z + Z\{i-1\} - Z\{i\}$.

When $n = 2$, for $j \geq 3$ and $\epsilon \in \{0, \infty\}$, we have $\delta j(\partial Z) = L_{n-1,i}(\partial(Z))$. Hence

$$\sum_{j=3}^{n} (-1)^{j} (\partial_{j}^{\infty} - \partial_{j}(Z)) = - \sum_{j=2}^{n-1} (-1)^{j} L_{n-1,i}(\partial_{j}^{\infty}(Z) - \partial_{j}(Z)) = - \sum_{j=1}^{n-1} (-1)^{j} L_{n-1,i}(\partial_{j}^{\infty}(Z) - \partial_{j}(Z)),$$

which is just $- L_{n-1,i}(\partial(Z))$. Here, $\dagger$ follows from the given assumption that $\partial Z = \partial_{1}(Z)$. Hence, $\partial (L_{n,i}(Z)) = Z + Z\{i-1\} - Z\{i\} - L_{n-1,i}(\partial(Z))$, as desired.

Proposition 5.4.4. Let $Z \in T \mathbb{Q}^{q}(X, n; m)$ be a cycle such that $\partial Z = \partial_{1}(Z)$ and $\partial(Z) = 0$. Then, there is a cycle $L_{n,r}(Z) \in T \mathbb{Q}^{q}(X, n, 1; m)$ such that $\partial (L_{n,r}(Z)) = rZ - Z\{r\}$.

Proof. Since $Z\{1\} = Z$, we can assume $r \geq 2$. We set $L_{n,r}(Z) := \sum_{i=2}^{r} L_{n,i}(Z)$. By Lemma 5.4.2, this cycle is admissible. On the other hand, by Lemma 5.4.3 we have

$$\partial (L_{n,r}(Z)) = \sum_{i=2}^{r} \partial (L_{n,i}(Z)) = \sum_{i=2}^{r} (Z + Z\{i-1\} - Z\{i\}) = (r-1)Z + Z\{1\} - Z\{r\}.$$

Since $Z\{1\} = Z$, we have $\partial (L_{n,r}(Z)) = rZ - Z\{r\}$, as desired.

Corollary 5.4.5. Let $W \in T \mathbb{Q}^{q}(X, n; m)$ be a cycle such that $\partial(W) = 0$. Then, we have $\partial(L_{n+1,r}(F_{r}\delta(W))) = r(F_{r}\delta(W)) - F_{r}\delta(W)\{r\}$. In particular, $r(F_{r}\delta(W)) = F_{r}\delta(W)\{r\}$ in $T \mathbb{Q}^{r+1}(X, n+m)$.

Proof. We saw from the first part of Proposition 4.1.4 (and from the Claim 1 in its proof) that $\delta(W)$ is an admissible cycle such that $\partial_{1}(\delta(W)) = \partial_{1}(\delta(W)) = 0$. However, obviously $F_{r}\delta_{1} = \delta_{1}F_{r}$ for all $i$ and $\epsilon \in \{0, \infty\}$ so that $\partial_{1}(F_{r}\delta(W)) = \partial_{1}(F_{r}\delta(W)) = 0$. Similarly, we have $\partial(F_{r}\delta(W)) = F_{r}\delta(W)\{r\} = F_{r}(-\delta(W)) = 0$, where $\dagger$ is by the second part of Proposition 4.1.4. Hence, by Proposition 5.4.4 applied to $F_{r}\delta(W)$, we are done.

5.5 Witt-complex structure (continued)

We now complete the proof of Theorem 5.3.1 using the calculations in Section 5.4.

Proof. (Theorem 5.3.1; Part 2) It only remains to prove the property (iv) in Definition 5.0.11 of a restricted Witt-complex. We first prove the

Claim: $rF_{r}\delta = \delta F_{r}$.

Let $\xi \in T \mathbb{Q}^{q}(X, n; m)$ be a cycle class. We choose its cycle representative (also denoted by $\xi$) in $T \mathbb{Q}^{q}(X, n; m)$ such that $\partial(\xi) = 0$ and let $m_{r} := rm + r - 1$. Since the square of finite maps

$$X \times \mathbb{G}_{m} \times \square^{n-1} \overset{\delta}{\longrightarrow} X \times \mathbb{G}_{m} \times \square^{n} \hspace{1cm} (5.5.1)$$

is commutative, it follows from Corollary 5.4.5 that

$$rF_{r}\delta(\xi) = F_{r}\delta(\xi)\{r\} = (\chi_{n+1,r})_{*} \circ (\phi_{r})_{*} \circ \delta(\xi) = (\chi_{n+1,r} \circ \phi_{r})_{*} \circ F_{r}(\delta)(\xi) = \delta \circ (\phi_{r})_{*}(\xi) = \delta F_{r}(\xi), \hspace{1cm} (5.5.2)$$

which proves the Claim.

We now prove (iv) in the following three steps. Let $\xi \in T \mathbb{Q}^{q}(X, n; m)$, with $\partial(\xi) = 0$, be a representative of a given cycle class in $T \mathbb{Q}^{q}(X, n; m)$ as above.

Step 1: We first assume that either char($k$) = 0, or, char($k$) = $p > 0$ and $(r, p) = 1$. 

In such cases, it follows from the **Claim** above that \( rF_\delta V_r = \delta F_r V_r = r\delta \), where the second equality follows from the property (ii) (already proven in Section 5.3). We conclude that \( r(\delta - F_r V_r) = 0 \). On the other hand, we have shown in Corollary 3.5.4 that \( TCH^p(X, n; m) \) is a \( \mathcal{W}_m(k) \)-module. Since \( r \in \mathcal{W}_m(k) \) is the Teichmüller lift of \( r \in k \) and since the Teichmüller lift is multiplicative, it follows from our assumption that \( r \in \mathcal{W}_m(k) \) is invertible. We conclude that \( \delta - F_r V_r = 0 \).

**Step 2:** Now we consider the case when \( \text{char}(k) = p > 0 \) and \( r = p \).

In this case, we can write \( \xi = \sum_{i=1}^{d_1} m_i V_i + \sum_{j=1}^{d_2} n_j W_j \) in such a way that \( \phi^\circ_p(V_i) \) is non-reduced for each \( i \) and \( \phi^\circ_p(W_j) \) is reduced for each \( j \). Set \( V'_i := (\phi^\circ_p(V_i))_{\text{red}}, W' := (\phi^\circ_p(W_j))_{\text{red}} = \phi^\circ_p(W_j) \), \( \alpha := \sum_{i=1}^{d_1} m_i V'_i \), \( \beta := \sum_{j=1}^{d_2} n_j W'_j \), and \( \gamma = \alpha + \beta \).

As \( \partial(\xi) = 0 \), it follows from Lemma 5.2.6 that

\[
\partial_{n+1}^\infty(\phi_\delta(\gamma)) = 0 = \partial(\phi_\delta(\gamma)). \tag{5.5.3}
\]

By Lemma 5.2.4, we have \( V_\delta(\xi) = p\alpha + \beta \). Thus,

\[
F_p \delta V_\delta(\xi) = F_p \delta(p\alpha + \beta) = p F_p \delta(\alpha) + F_p \delta(\beta) = p \phi_\delta(\alpha) + p \phi_\delta(\beta) = p \phi_\delta(\gamma), \tag{5.5.4}
\]

where \( \gamma \) holds by Lemma 5.2.4. On the other hand, by (5.5.3) and Proposition 5.4.4, the right-hand side is equal to

\[
\partial(\tilde{L}_{n+1,p}(\phi_\delta(\gamma))) \equiv \partial(\tilde{L}_{n+1,p}(\phi_\delta(\gamma))) + \partial(\tilde{L}_{n+1,p}(\phi_\delta(\gamma))) \equiv \partial(\tilde{L}_{n+1,p}(\phi_\delta(\gamma))) + \partial(\tilde{L}_{n+1,p}(\phi_\delta(\gamma))) + \delta(\xi),
\]

where \( \delta \) holds by Lemma 5.2.4 and \( \gamma \) holds by the diagram (5.5.1). Combining this with (5.5.4), we conclude that \( F_p \delta V_\delta(\xi) - \partial(\xi) = \partial(\tilde{L}_{n+1,p}(\phi_\delta(\gamma))) \). In particular, \( F_p \delta V_\delta(\xi) \equiv \partial(\xi) \) in \( TCH^p(X, n; m) \).

**Step 3:** We finally consider the following remaining case: suppose \( \text{char}(k) = p > 0 \) and \( r \geq 1 \) is any integer divisible by \( p \). Write \( r = p^s \cdot r' \) for some \( s \geq 1 \) and \( r' \geq 1 \) with \( (r', p) = 1 \). By (i) (already proven in Section 5.3), we have \( F_r = (F_p)^s F_{r'} \) and \( V_r = V_{r'}(V_p)^s \). Thus, \( F_r \delta V_r = (F_p)^s F_{r'} \delta V_{r'}(V_p)^s = (F_p)^s \delta(V_p)^s \), where the last equality was already proven (in **Step 1**, because \( (r', p) = 1 \)). But, we also saw that \( F_p \delta V_p = \delta(\delta(V_p)^s) \) (by **Step 2**). Hence, \( (F_p)^s \delta(V_p)^s = (F_p)^s - 1 \delta(V_p)^s - 1 = (F_p)^s - 1 \delta(V_p)^s - 1 \), and inductively this is equal to \( \delta \). Thus, \( F_r \delta V_r = \delta \) in this case, too. This verifies the property (iv) in Definition 5.0.11, and it completes the proof of Theorem 5.3.1.

**Corollary 5.5.1.** Let \( g : Y \to X \) be a morphism of smooth projective varieties over \( k \). Then, the induced morphisms \( \{ g^* : TCH(Y;m) \to TCH(X;m) \}_{m \geq 1} \) form a morphism of restricted Witt complexes over \( k \).

**Proof.** All the operations \( F_r, V_r, R, \wedge, \delta, \lambda_X \) are functorial with respect to pull-backs of morphisms between smooth projective varieties. Thus, the corollary follows from Theorem 5.3.1.

6 **Appendix : the normalization theorem**

The purpose of this section is to prove the normalization theorem (Theorem 6.1.2). Roughly speaking, this allows us to replace the additive higher Chow complex by a smaller “concentrated” subcomplex consisting of cycles whose all but one face are trivial. The idea is to use a result of [Levine, 2009, Section 1], which shows that extended cubical objects have such normalization theorems.

In Section 6.1, we define the notion of normalized additive cycle complex. In Section 6.2, we recall the idea of extended cubical objects from [Levine, 2009, Section 1]. In Section 6.3, we prove for a smooth quasi-projective variety that the additive higher Chow complex is an extended cubical object. Here, as said, we use the modulus condition \( M_{\text{sum}} \).

6.1 **Normalized additive cycle complex**

**Definition 6.1.1.** Let \( X \) be an equidimensional \( k \)-scheme. For \( n, m \geq 1 \), let \( T^{\delta}_{\mathcal{N}}(X, n; m) \) be the subgroup of \( T^\delta(X, n; m) \) of cycles \( \alpha \) such that \( \partial_i^\delta(\alpha) = 0 \) for \( 1 \leq i \leq n - 1 \) and \( \partial_n^\infty(\alpha) = 0 \) for \( 2 \leq i \leq n - 1 \). Using the cubical structure, we see that for \( \alpha \in T^\delta_{\mathcal{N}}(X, n; m) \), one has \( \partial_i^\infty \circ \partial_n^\infty(\alpha) = 0 \). Writing \( \partial_i^\infty \) as \( \partial_i^N \), we get a subcomplex \( (T^\delta_{\mathcal{N}}(X, \bullet; m), \partial_i^N) \) of \( (T^\delta(X, \bullet; m), \partial) \). We call it the normalized additive cycle complex. The normalized additive higher Chow group \( TCH^N_{\mathcal{N}}(X, n; m) \) of \( X \) is by definition its homology \( \mathcal{H}_n(T^\delta_{\mathcal{N}}(X, \bullet; m)). \)

By [Bloch, online note, Theorem 4.4.2], the normalized higher Chow complex is quasi-isomorphic to the full higher Chow complex. We prove its additive version:

**Theorem 6.1.2.** Let \( X \) be a smooth quasi-projective variety. Then, the natural inclusion \( T^\delta_{\mathcal{N}}(X, \bullet; m) \to T^\delta(X, \bullet; m) \) of complexes is a quasi-isomorphism.
6.2 Extended cubical objects

Recall from [Krishna and Levine, 2008, Section 1] that the category **Cube** is the subcategory of the category **Set** of sets with objects \( n = \{0, 1\}^n \), \( n = 0, 1, 2, \cdots \), and morphisms generated by

(i) Inclusions: \( \iota_{n,i,e} : n - 1 \rightarrow n, i = 1, \cdots, n, \epsilon = 0, 1; \)
\[
\iota_{n,i,e}(y_1, \cdots, y_{n-1}) = (y_1, \cdots, y_{i-1}, \epsilon, y_i, \cdots, y_{n-1}).
\]

(ii) Projections: \( p_{n,i} : n \rightarrow n - 1, i = 1, \cdots, n. \)

(iii) Permutations of factors: \( (y_1, \cdots, y_n) \mapsto (y_{\sigma(1)}, \cdots, y_{\sigma(n)}) \) for \( \sigma \in S_n. \)

(iv) Involutions: \( \tau_{n,i} : y_1, \cdots, y_n \mapsto (y_1, \cdots, y_{i-1}, 1 - y_i, y_{i+1}, \cdots, y_n). \)

A cubical object in a category \( C \) is a functor \( **Cube**^{op} \rightarrow C \). A cubical object \( A \) in a pseudo-abelian category \( C \) gives a complex \( (A_n, d) \), where \( A_n = A(n)/A(n)_{\text{degn}}, A(n)_{\text{degn}} = \sum p_{n,i}^* A(n-1) \), and the differential \( d_n = \sum_{i=1}^{n} (-1)^i (\iota_{n,i,0}^* - \iota_{n,i,1}^*) \).

Let \( **ECube** \) denote the smallest symmetric monoidal subcategory of \( **Set** \) containing \( **Cube** \) with the same objects, and containing the morphism
\[
\mu : \mathbb{2} \rightarrow 1; \mu(y_1, y_2) = y_1 y_2.
\]

(6.2.1)

An extended cubical object in a category \( C \) is a functor \( **ECube^{op}** \rightarrow C \). An extended cubical object has the following interesting property:

**Lemma 6.2.1** ([Levine, 2009, Lemma 1.6]). Let \( A \) be an extended cubical object in a pseudo-abelian category \( C \) and let \( NA \subset A \) be the subobject defined by
\[
NA(n) = \bigcap_{i=2}^{n} \ker(\iota_{n,i,0}^*) \cap \bigcap_{i=1}^{n} \ker(\iota_{n,i,1}^*), \text{ with } d_n^N := \iota_{n,1,0}^*.
\]

Then the natural map \( (NA_\bullet, d^N) \rightarrow (A_\bullet, d) \) is a homotopy equivalence.

\( \square \)

6.3 Homotopy variety and the normalization theorem

Let \( X \) be a smooth quasi-projective variety. The technical objective of Section 6.3 is to prove Proposition 6.3.1, that together with Lemma 6.2.1, essentially implies Theorem 6.1.2.

Let \( q_1 : \square^2 \rightarrow \square \) the morphism
\[
q_1(y_1, y_2) = \begin{cases} y_1 + y_2 - y_1 y_2, & \text{if } y_1, y_2 \neq \infty, \\ \infty & \text{otherwise}. \end{cases}
\]

(6.3.1)

The fact that \( q_1 \) is indeed a morphism of varieties easily follows from the observation that this transforms to the morphism \( q_1, \psi : \mathbb{A}^2 \rightarrow \mathbb{A}^1; q_1, \psi(y_1, y_2) = y_1 y_2 \) under the identification \( \psi : \square \xrightarrow{\sim} \mathbb{A}^1 \) via \( y \mapsto 1/(1 - y) \). We shall write the morphism \( q_1 \) of (6.3.1) simply as \( q_1(y_1, y_2) = y_1 + y_2 - y_1 y_2 \).

**Proposition 6.3.1.** Let \( X \) be a smooth quasi-projective variety. Let \( q_n : X \times B_{n+2} \rightarrow X \times B_{n+1} \) be \((x, t, y_1, \cdots, y_{n+1}) \mapsto (x, t, y_1, \cdots, y_n, y_n + y_{n+1} - y_n y_{n+1})\). Then, for \( Z \in \mathbb{T}^2(X, n + 1; m) \), we have \( q_n^*(Z) \in \mathbb{T}^2(X, n; m) \).

To prove Proposition 6.3.1, one should, for instance, check that the pull-back \( q_n^* \) respects the modulus conditions so that we should consider the Zariski closures of the cycles \( Z \) and \( q_n^*(Z) \) in the spaces \( X \times \overline{B}_{n+1} \) and \( X \times \overline{B}_{n+2} \). Its proof is not really trivial because the product \( q_1 : \square^2 \rightarrow \square \) does not extend to \( q_1 : (\mathbb{P}^1)^2 \rightarrow \mathbb{P}^1 \) so that one cannot find a morphism \( q_n \) on \( X \times \overline{B}_{n+2} \) that restricts to \( q_n \). A detour is to use the “homotopy varieties” \( \overline{W}_n^X \) and \( \overline{W}_n^X \) below originating from [Levine, 1994, §4 p.30]. The equations (6.3.7) and (6.3.8) show how they are related to Proposition 6.3.1.

Since it is more convenient to work with \( \{1, 0\} \subset \mathbb{P}^1 \) as its faces via the isomorphism \( \psi \), we use the cube \( \square_\psi = (\mathbb{A}^1, \{1, 0\}) \) instead of \( \square = (\mathbb{P}^1 \setminus \{1\}, \{0, \infty\}) \), and \( \mathbb{P}_\psi^1 \) instead of \( \mathbb{P}^1 \). One takes the boundary \( \partial_\psi \psi = \sum_i (-1)^i (\partial_i^0 - \partial_i^1) \) instead of \( \partial = \sum_i (-1)^i (\partial_i^0 - \partial_i^1) \). The map \( q_\psi \) can be written as \( q_\psi(x, t, y_1, \cdots, y_{n+1}) = (x, t, y_1, \cdots, y_n, y_{n+1}) \). For the modulus condition \( M_{\text{sum}} \) in Definition 2.1.1, we replace the Cartier divisors \( F_{n,i} = \{y_i = 1\} \) and \( F_n = \sum_i F_{n,i} \) by \( F_{n,i} = \{y_i = \infty\} \) and \( F_n = \sum_i F_{n,i}^\infty \), respectively. The group of additive cycles based on \( \square_\psi \) is denoted by \( \mathbb{T}^2_\psi(X, n; m) \).
For \( n \geq 1 \), let \( i_n : W^X_n \hookrightarrow X \times G_m \times \Box^{n+1}_\psi \times \mathbb{P}^1 \) be the subscheme defined by

\[
t_0y_ny_{n+1} = t_1,
\]

(6.3.2)

where \((y_1, \ldots, y_{n+1}) \in \Box^{n+1}_\psi\) and \((t_0, t_1) \in \mathbb{P}^1\) are the coordinates. We let \( y := t_1/t_0 \).

Let \( \overline{i}_n : \overline{W}^X_n \hookrightarrow X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n+1} \times \mathbb{P}^1 \) be the Zariski closure of \( W^X_n \). Identify \( G_m \) with \( \mathbb{A}^1\setminus\{0\} \). In terms of the homogeneous coordinates \((u_{1,0}; u_{1,1}), \ldots, (u_{n+1,0}; u_{n+1,1})\) of \((\mathbb{P}^1)^{n+1}\), where \( y_i = u_i^1/u_0^1\), its defining equation is

\[
t_0u_{n,0}u_{n+1,1} = u_{n,0}u_{n+1,0}t_1.
\]

(6.3.3)

- Consider the morphism \( X \times G_m \times \Box^{n+1}_\psi \times \mathbb{P}^1 \to X \times G_m \times \Box^{n-1}_\psi \),

\[
(x, t, y_1, \ldots, y_{n+1}, (t_0; t_1)) \mapsto (x, t, y_1, \ldots, y_{n-1}, y_n y_{n+1}),
\]

and let \( \pi_n : W^X_n \to X \times G \times \Box^n \) be its restriction on \( \overline{W}^X_n \). To extend \( \pi_n \) on \( W^X_n \) to a morphism \( \overline{\pi}_n \) on \( \overline{W}^X_n \), we use the map \( \overline{\theta}_n \) below : \(\overline{\theta}_n : X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n+1} \times \mathbb{P}^1 \to X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1} \times \mathbb{P}^1 \) be the projection that discards the coordinates \((u_{n,0}; u_{n,1}), (u_{n+1,0}; u_{n+1,1})\).

Let \( p_n : X \times G_m \times \Box^{n+1}_\psi \times \mathbb{P}^1 \to X \times G_m \times \Box^{n+1}_\psi \) be the natural projection.

**Lemma 6.3.2.** Let \( n \geq 1 \). Then the following statements hold:

i) \( W^X_n \cap \{t_0 = 0\} = \emptyset \), thus \( W^X_n \subset X \times G_m \times \Box^{n+1}_\psi \times \mathbb{P}^1 \).

ii) \( \theta_n|_{W^X_n} = \pi_n \). Let \( \overline{\pi}_n : \overline{W}^X_n \to X \times \mathbb{A}^1 \times (\mathbb{P}^1)^n \) be the restriction of \( \overline{\theta}_n \).

iii) The varieties \( W^X_n \) and \( \overline{W}^X_n \) are smooth.

iv) Both \( \pi_n \) and \( \overline{\pi}_n \) are flat surjective morphisms of relative dimension 1.

**Proof.** (i) is immediate from the defining equation (6.3.2) of \( W^X_n \), and (ii) holds by definition. Now, write the restriction of \( p_n \) on \( W^X_n \) as

\[
\rho_n := p_n|_{W^X_n} : W^X_n \hookrightarrow X \times G_m \times \Box^{n+1}_\psi \times \mathbb{P}^1 \to X \times G_m \times \Box^{n+1}_\psi
\]

where the first inclusion is given by the equation \( y = y_n y_{n+1} \). Since \( X \) is smooth, using the Jacobian criterion we see that \( W^X_n \) is smooth, and the above composite map is an isomorphism. Under this isomorphism, the map \( \pi_n \) can be identified with the projection \((x, t, y_1, \ldots, y_n, y) \mapsto (x, t, y_1, \ldots, y_{n-1}, y)\), as we see from the equation (6.3.2) of \( W^X_n \). This shows that \( \pi_n \) is smooth and surjective map of relative dimension one. To prove that \( \overline{W}^X_n \) is smooth, we can check it locally on each open set where either of \( u_{n,i}, u_{n+1,i} \), \( t_i \) is nonzero for \( i = 0, 1 \). In any such open set, \( \overline{W}^X_n \) has the equation of the form (6.3.2) that defines \( W^X_n \) and hence smooth. Similarly, using these local coordinates, we see \( \overline{\pi}_n \) is of relative dimension one. Moreover, as \( \overline{\theta}_n \) is projective and \( \pi_n \) is surjective, we see that \( \overline{\pi}_n \) is projective and surjective. In particular, it is flat (cf. [Hartshorne, 1977, Exercise III-10.9]). Thus we proved (iii) and (iv).

**Lemma 6.3.3.** Let \( n \geq 1 \) and let \( Z \subset X \times G_m \times \Box^n_\psi \) be a closed subvariety which satisfies the modulus condition. Then \( Z' := (i_n)_* (\pi_n^* (Z)) \) also satisfies the modulus condition.

**Proof.** Let \( Z \) and \( Z' \) denote the Zariski closures of \( Z \) and \( Z' \) in \( X \times \mathbb{A}^1 \times (\mathbb{P}^1)^n \) and \( X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n+1} \times \mathbb{P}^1 \), respectively. Let \( Z^N \) and \( Z'^N \) denote the normalizations of \( Z \) and \( Z' \), respectively. Using Lemma 6.3.2 and the projectivity of the map \( \overline{\theta}_n \), we see that \( \overline{\theta}_n (Z') = Z \). Since \( \overline{W}^X_n \) is smooth, we get the commutative diagram:

\[
\begin{array}{ccc}
Z^N & \xrightarrow{\nu_2} & \overline{W}^X_n \\
\downarrow f & & \downarrow \overline{\pi}_n \\
Z & \xrightarrow{\nu_1} & X \times \mathbb{A}^1 \times (\mathbb{P}^1)^n \times \mathbb{P}^1
\end{array}
\]

(6.3.4)
Here is the map of normal $k$-schemes induced by the surjective map $Z' \to Z$. From the equation of $W_n^X$ in (6.3.3) and $\theta_n(F_{n+1}^\infty) = F_{n+3,n+2}^\infty = \{y = \infty\}$, we have

$$
\pi_n^*F_{n+1}^\infty = \pi_n^*\{y = \infty\} = \pi_n^*\{t_0 = 0\} \leq \pi_n^*\{t_0 = 0\} + \{u_{n+1} = 0\} = \{y_n = \infty\} + \{y_{n+1} = \infty\} = \pi_n^*F_{n+3,n+1}^\infty + \pi_n^*F_{n+3,n+1}^\infty.
$$

(6.3.5)

First consider the case $n \geq 2$. Since $g: Z' \to W_n^X$ is a map of normal $k$-schemes, and since $\theta_n(F_{n+1}^\infty) = F_{n+3,i}^\infty$ for $1 \leq i < n$, we have

$$
\nu_n^{\infty}(F_{n+1}^\infty) = \sum_{i=1}^{n-1} \nu_n^{\infty}(F_{n+3,i}^\infty) + g^*\pi_n^*(F_{n+1}^\infty) \leq \nu_n^*(F_{n+1}^\infty)
$$

(6.3.6)

where the inequality $\dagger$ follows from (6.3.5). If $n = 1$, we simply ignore the sums $\sum_{i=1}^{n-1}$ in the above. Now the modulus condition for $Z$ means $\nu_n^{\infty}((m+1)F_{n+1,0}) \leq \nu_n^*(F_{n+1}^\infty)$, thus, $f^*\nu_n^{\infty}((m+1)F_{n+1,0}) \leq f^*\nu_n^*(F_{n+1}^\infty)$. This in turn shows that $\nu_n^{\infty}(m+1)F_{n+1,0} \leq \nu_n^*(F_{n+1}^\infty)\otimes g^*\pi_n^*(F_{n+1}^\infty)$. Since $\theta_n(F_{n+1,0}) = F_{n+3,0}$, we conclude from (6.3.6) that $\nu_n^*(m+1)F_{n+1,0} \leq \nu_n^*(F_{n+3}^n)$. This is the modulus condition for $Z'$.

- For any closed subvariety $Z \subset X \times G_m \times \square^\infty_\psi$ with $n \geq 1$, set

$$
W_n^X(Z) := p_{n,*} \circ i_{n,*} \circ \pi_n^*(Z).
$$

(6.3.7)

Since $p_n$ is projective, $W_n^X(Z)$ is a closed subvariety of $X \times G_m \times \square^\infty_\psi$.

**Lemma 6.3.4** (cf. Levine, 1994, Lemma 4.1). Let $n \geq 1$. For $Z$ as above, if $Z$ intersects properly with all faces of $X \times G_m \times \square^\infty_\psi$, then $W_n^X(Z)$ intersects properly with all faces of $X \times G_m \times \square^\infty_\psi$.

**Proof.** From the defining equation of $W_n^X$, note that our $W_n^X$ is equal to the variety $\tau^*\bar{\tau}_n^*\rho_n^*W_n^X \times G_m$, where $\tau_i$ is the involution that sends $y_i$ to $1 - y_i$, and $\tau$ is one that sends $-\infty$ to $y$, where $y = t_1/t_0$, and $W_n^X \times G_m$ follows the notations of [Levine, 1994, Lemma 4.1]. These involutions send the faces of $\square^\infty_\psi$ to the faces of $\square^\infty_\psi$. Hence, Lemma 6.3.4 is just a special case of [Levine, 1994, Lemma 4.1].

**Proof.** (of Proposition 6.3.1) Via the isomorphism $\psi, y \mapsto 1/(1 - y)$, we work with $\square_\psi$. Thus, we prove that if $Z_\psi$ is admissible, then so is $q_{n,\psi}^*(Z_\psi)$. The boundary map is $\partial_\psi = \sum_i(-1)^i(\partial_i^\infty + \partial_i^1)$. Consider the commutative diagram, where $\theta_n$ is the restriction of $\bar{\theta}_n$:

$$
\begin{array}{ccc}
W_n^X & \xrightarrow{i_n} & X \times G_m \times \square^\infty_\psi \times \mathbb{P}^1 \\
\downarrow p_n & & \downarrow \theta_n \\
X \times G_m \times \square^\infty_\psi & \xrightarrow{q_{n,\psi}} & X \times G_m \times \square^\infty_\psi \times \square_\psi
\end{array}
$$

where $q_{n,\psi} \circ \rho_n = \pi_n$ by definition. Recall that we saw in the proof of Lemma 6.3.2 that $\rho_n$ is an isomorphism and $\pi_n$ is flat and surjective. Hence, $q_{n,\psi}$ is also flat and surjective. Furthermore, since $\rho_n$ is an isomorphism, we deduce

$$
q_{n,\psi}^*(Z_\psi) = \rho_{n,*} \circ q_{n,\psi}^*(Z_\psi) = p_{n,*} \circ \pi_n^*(Z_\psi) = p_{n,*} \circ (\pi_n^*(Z_\psi)) = W_n^X(Z_\psi).
$$

(6.3.8)

Thus we are reduced to prove $W_n^X(Z_\psi) \in T_{2q}(X, n + 2; m)$. By Lemmas 6.3.3 and 6.3.4 we know $i_{n,*} \circ \pi_n^*(Z_\psi) \in T_{2q}(X \times \mathbb{P}^1, n + 2; m)$. Since the map $p_n$ is projective, by [Krishna and Park, 2012a, Proposition 5.2] and (6.3.7), we conclude $W_n^X(Z_\psi) \in T_{2q}(X, n + 2; m)$. This shows that $q_{n,\psi}^*(Z) \in T_{2q}(X, n + 2; m)$ via $\psi$ again.

**Corollary 6.3.5.** Let $X$ be a smooth quasi-projective variety. Then, the cubical abelian group $(n \mapsto T_{2q}(X, n; m))$ is an extended cubical abelian group.
Proof. As in the proof of Proposition 6.3.1, we again identify □ with $\mathbb{A}^1$ via the isomorphism $\psi : y \mapsto 1/(1-y)$ and work with $\square_{\psi} = (\mathbb{A}^1, 1, 0)$ instead of $\square = (\mathbb{P}^1 \setminus \{1\}, 0, \infty)$. Since $(n \mapsto \mathbb{T}_n^\alpha(X, n; m))$ is already a cubical abelian group, it only remains to prove that the pull-back via the map

$$q_1, \psi : X \times \mathbb{G}_m \times \square_{\psi}^2 \to X \times \mathbb{G}_m \times \square_{\psi}; (x, t, y_1, y_2) \mapsto (x, t, y_1y_2)$$

preserves the admissibility of given additive cycles. This is just a special case of Proposition 6.3.1. This shows that $(n \mapsto \mathbb{T}_n^\alpha(X, n; m))$ is an extended cubical abelian group.

Proof. (of Theorem 6.1.2) By Corollary 6.3.5, the functor $(n \mapsto \mathbb{T}_n^\alpha(X, n; m))$ is an extended cubical abelian group. The theorem then follows by Lemma 6.2.1.

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