A personal note on the Picard schemes

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1. DEFINITIONS AND BASIC PROPERTIES

Let X be a nonsingular projective variety over an algebraically closed field k of any characteristic.

Definition 1.1 (T-isomorphism)**.** *Let $T \in Sch_k$. Two line bundles L, L' on $X \times T$ are said to be* T-isomorphic *if there is a line bundle M on T such that $L \simeq L' \otimes p_2^* M$ where $p_2 : X \times T \to T$ is the projection.*

Definition 1.2 (The Picard functor)**.** *We define a contravariant functor:*

$$\mathbb{P}\mathrm{ic}_X : Sch_k \to Set$$

as follows:

1. *For $T \in Sch_k$, $\mathbb{P}\mathrm{ic}_X(T)$ is the set of all T-isomorphism classes of line bundles over $X \times T$.*
2. *For $f : T' \to T$, we define $\mathbb{P}\mathrm{ic}_X(f) : \mathbb{P}\mathrm{ic}_X(T) \to \mathbb{P}\mathrm{ic}_X(T')$ as $\mathbb{P}\mathrm{ic}_X(f)([L]) = [(\mathrm{id}_X \times f)^* L]$.*

Definition 1.3 ($\mathbb{P}\mathrm{ic}_X^0$)**.** *$\mathbb{P}\mathrm{ic}_X^0$ is a subfunctor of $\mathbb{P}\mathrm{ic}_X$ defined by*

$$\mathbb{P}\mathrm{ic}_X^0(T) = \{L \in \mathbb{P}\mathrm{ic}_X(T) | L_t := L|_{X \times \{t\}} \approx \mathcal{O}_X \text{ for all } t \in T\}$$

where $\approx$ is the algebraic equivalence.

Proposition 1.4. *$\mathbb{P}\mathrm{ic}_X$ and $\mathbb{P}\mathrm{ic}_X^0$ are representable and their representing pairs $(\underline{\mathrm{Pic}}(X), L)$, $(\underline{\mathrm{Pic}}^0(X), L')$ have the following properties:*

1. *$\underline{\mathrm{Pic}}(X)$ is a disjoint union of an infinite family of proper k-schemes.*
2. *L is a line bundle on $X \times \underline{\mathrm{Pic}}(X)$ and the k-rational points of $\underline{\mathrm{Pic}}(X)$ are in a natural bijective correspondence with the elements of the Picard group of X, $\mathrm{Pic}(X)$.*
3. *$\underline{\mathrm{Pic}}^0(X)$ is the connected component of the origin 0 in $\underline{\mathrm{Pic}}(X)$, where $0 \in \underline{\mathrm{Pic}}(X)$ is the k-rational point corresponding to the class of $\mathcal{O}_X$, and $\underline{\mathrm{Pic}}^0(X)$ is a proper scheme over k.*
4. *The sheaf L' is the restriction of L to $X \times \underline{\mathrm{Pic}}^0(X)$ and L' is uniquely determined up to a $\underline{\mathrm{Pic}}^0(X)$-isomorphism.*

Definition 1.5 (Poincaré invertible sheaf)**.** *Let $(\underline{\mathrm{Pic}}^0(X), L')$ be as above. Choose a closed point $x_0 \in X$. Let $M = L'|_{\{x_0\} \times \underline{\mathrm{Pic}}^0(X)}$. Replace L' by $L' \otimes p_2^* M^{-1}$. (They are $\underline{\mathrm{Pic}}^0(X)$-isomorphic, but not necessarily isomorphic.) This representation L' of the $\underline{\mathrm{Pic}}^0(X)$-isomorphism class of L' is called the* Poincaré invertible sheaf *or the* Poincaré line bundle *on $X \times \underline{\mathrm{Pic}}^0(X)$.*

Proposition 1.6. *The Poincaré line bundle is uniquely determined up to isomorphism, and it is completely characterized by the following properties:*

1. (a) *$L'|_{\{x_0\} \times \underline{\mathrm{Pic}}^0(X)} \simeq \mathcal{O}_{\underline{\mathrm{Pic}}^0(X)}$.*
   (b) *for every closed pont $a \in \underline{\mathrm{Pic}}^0(X)$, the invertible $\mathcal{O}_X$-module $L'_a := L'|_{X \times \{a\}}$ is algebraically equivalent to 0, and $a \mapsto L'_a$ defines a bijection between the set of closed points of $\underline{\mathrm{Pic}}^0(X)$ and $\mathrm{Pic}^0(X)$.*
2. *(Universal Mapping Property) Let T be an arbitrary k-scheme and L'' is an ivertible $\mathcal{O}_{X \times T}$-module such that*
   (a) *$L''_{\{x_0\} \times T} \simeq \mathcal{O}_T$ and*

(b) $L''_t = L''|_{X\times\{t\}}$ *is algebraically equivalent to* 0 *for all* $t \in T$. *Then, there is a unique morphism of* k*-schemes* $f : T \to \underline{\mathrm{Pic}}^0(X)$ *such that* $L'' \simeq (\mathrm{id}_X \times f)^*(L')$.

Proposition 1.7. *The Zariski tangent space* $T_{\underline{\mathrm{Pic}}^0(X),0}$ *of* $\underline{\mathrm{Pic}}^0(X)$ *at* 0 *is canonically isomorphic to the* k*-vector space* $H^1(X, \mathcal{O}_X)$.

Remark. For any $T \in Sch_k$ on $\mathbb{P}\mathrm{ic}^0(T)$ has a natural abelian group structure, which is given by the tensor product of invertible $\mathcal{O}_{X\times T}$-modules, so that in fact

$$\mathbb{P}\mathrm{ic}^0_X : Sch_k \to Ab \to Set.$$

Hence, $\underline{\mathrm{Pic}}^0(X)$ is in fact a commutative group scheme so that if $\underline{\mathrm{Pic}}^0(X)$ is reduced, it is an abelian variety.

Theorem 1.8 (Cartier)**.** *Every commutative group scheme is reduced if the characteristic of* k *is* 0.

Definition 1.9. *The group scheme* $\underline{\mathrm{Pic}}^0(X)$ *is called the* Picard scheme *of* X *and the commutative group* $\underline{\mathrm{Pic}}^0(X)_{\mathrm{red}}$ *is called the (classical)* Picard variety *of* X.

Remark. *Since* X *was a nonsingular projective variety over* k, $\underline{\mathrm{Pic}}^0(X)$ *is proper over* k *so that* $\underline{\mathrm{Pic}}^0(X)$ *is an abelian variety.*

Remark. $\mathbb{P}\mathrm{ic}^0_X$ *is representable even if* X *is not regular (but still projective), but in this case* $\underline{\mathrm{Pic}}^0(X)$ *may be nonproper over* k.

2. Review of abelian varieties

As usual, k is always algebraically closed.

Definition 2.1 (abelian variety)**.** *An* abelian variety X *is a complete algebraic vareity over* k *with a group law* $m : X \times X \to X$ *such that* m *and the inverse map are both morphisms of varieties.*

We have some properties of an abelian variety.

Proposition 2.2. *An abelian variety* X *of dimension* g *over* k *has the following properties:*

1. X *is commutative and divisible.*
2. *If* $n_X : X \to X$ *is multiplication by* $n > 0$, *then* $X_n = \ker n_X$ *is*
$$X_n = (\mathbb{Z}/n\mathbb{Z})^{2g} \quad \text{if char } k \not| n$$
$$X_{p^m} = (\mathbb{Z}/p^m\mathbb{Z})^i \quad \text{if } p = \text{ char } k, m > 0$$
where i *can take every value in the range* $0 \le i \le g = \dim X.i$ *is called the* p*-rank of* X.
3. *The cohomology groups*
$$H^q(X, \Omega^p) \simeq \Lambda^p H^0(X, \Omega^1) \otimes_k \Lambda^q H^1(X, \mathcal{O}_X)$$
and $\dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \Omega^1) = g$.
4. *The algebraic fundamental group* $\pi_1(X)$, *i.e., the inverse limit of finite groups of unramified Galois coverings, is isomorphic to*
$$\prod_l (\mathbb{Z}_l)^{2g} \text{ in char } 0$$
$$\prod_{l\neq p} (\mathbb{Z}_l)^{2g} \times \mathbb{Z}_p^i \text{ in char } p.$$

(5) *There is an exact sequence*

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}X \to NS(X) \to 0$$

where $\mathrm{Pic}^0(X)$ *is has a natural structure of an abelian variety and* $NS(X)$ *is a finitely generated free abelian group whose rank* ρ *is called the* base number *of* X*.*

(6) *Let* $\Omega_0 = T^*_{X,0}$*. Then,*

$$\Omega_0 \otimes_k \mathcal{O}_X \simeq \Omega^1_X.$$

(7) *If char* $p \not| n$*, then,* $n_X : X \to X$ *is surjective.*

To show the commutativity of an abelian variety, which *a priori* is not defined to be abelian, we need the following very useful lemma:

Lemma 2.3 (The Rigidity Lemma)**.** *Let* X *be a complete variety and* Y*,* Z *be any varieties, and let* $f : X \times Y \to Z$ *be a morphism such that for some* $y_0 \in Y$*,* $f(X \times \{y_0\})$ *is a single point* $z_0 \in Z$*. Then, there is a morphism* $g : Y \to Z$ *such that* f *factors through* g *and* $p_2 : X \times Y \to Y$*, i.e.* $f = g \circ p_2$*.*

Proof. Choose any point $x_0 \in X$, and define $g : Y \to Z$ by $g(y) = f(x_0, y)$. To prove that $f = g \circ p_2$, it is enough to show that they agree on an open subset of $X \times Y$.

Let U be an affine open neighborhood of $z_0 \in Z$, and let $F = Z - U$, $G = p_2(f^{-1}(F))$. Since X is complete, p_2 is a closed map, so that G is closed in Y. Since $f(X \times \{y_0\}) = \{z_0\}$ and $z_0 \notin F$, we must have $y_0 \notin G$. Hence $V = Y - G$ is a nonempty open set in Y. Now, for $y \in V$, $X \times \{y\} \xrightarrow{f} U$ and $X \times \{y\}$ is complete and U is affine. Hence it is constant.

(Recall: X is a complete k-variety, then

$$\mathrm{Hom}_{Sch_k}(X, \mathrm{Spec}A) = \mathrm{Hom}_{k-alg}(A, \Gamma(X, \mathcal{O}_X)) = \mathrm{Hom}_{k-alg}(A, k) = \mathrm{Hom}_{Sch_k}(\mathrm{Spec}k, \mathrm{Spec}A)$$

which is the collection of k-rational points of $\mathrm{Spec}A$. Hence all morphisms are constant.)

Hence, in particular, for $(x, y) \in X \times V$,

$$f(x, y) = f(x_0, y) = g(y) = g(p_2(x, y)).$$

□

Above lemma has the following very interesting and useful corollaries.

Corollary 2.4. *Let* X*,* Y *be abelian varieties and* $f : X \to Y$ *be any morphism. Then, there is a homomorphism* $h : X \to Y$ *of abelian varieties with* $a \in Y$ *such that* $f(x) = h(x) + a$*, i.e. any morphism between abelian varieties is a translation of a homomorphism.*

Proof. Replace f by $f - f(0)$, so that we may asume that $f(0) = 0$, and then we have to show that f is a homomorphism. Consider $\phi : X \times X \to Y$ defined by $\phi(x, y) = f(x + y) - f(x) - f(y)$. Then, $\phi(x, 0) = f(x) - f(x) = 0$ so that by the Rigidity lemma, there is $g : X \to Y$ such that $g(y) = \phi(x, y) = f(x + y) - f(x) - f(y)$. Note that the LHS depends only of y, so that we can put $x = 0$ in RHS which gives $g(y) = f(y) - f(y) = 0$ for all y, i.e. $g \equiv 0$, i.e. $f(x + y) \equiv f(x) + f(y)$. Hence f is a homomorphism. □

Corollary 2.5. X *is an abelian group.*

Proof. The morphism $\phi : X \to X$, $x \mapsto x^{-1}$ is a homomorphism by above corollary. Hence, $(xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1}$ which is $xy = yx$. □

Corollary 2.6. *Let* X *be an abelian variety with base point* 0*. Let* $\mathcal{C}_0$ *be the category of based complete varieties over* k*, with morphisms are morphisms of varieties preserving basepoints. Let* $S \in \mathcal{C}_0$*. Note that* $\mathrm{Hom}_{\mathcal{C}_0}(S, X)$ *is an abelian group. Then, the contravariant functor*

$$\mathcal{C}_0 \to Ab, S \mapsto \mathrm{Hom}_{\mathcal{C}_0}(S, X)$$

is linear*, in the sense that*

$$\mathrm{Hom}_{\mathcal{C}_0}(S,X)\times \mathrm{Hom}_{\mathcal{C}_0}(T,X)\to \mathrm{Hom}_{\mathcal{C}_0}(S\times T,X),\quad (f,g)\mapsto h,\quad h(s,t)=f(s)+g(t)$$

is a bijection.

Proof. Injectivity is obvious. For surjectivity, let $h\in \mathrm{Hom}_{\mathcal{C}_0}(S\times T,X)$. Let $f(s)=h(s,t_0)$, $g(t)=h(s_0,t)$, $k(s,t)=h(s,t)-f(s)-g(t)$. Then, $k(S\times t_0)=0$, so that by the Rigidity lemma, $k\equiv 0$. ☐

3. The Picard variety and the Albanese variety

Let's recall some definitions.

Definition 3.1 (Albanese variety)**.** *Let X be a nonsingular projective variety and let $x_0\in X$ be a fixed closed point. A pair (A,α), A: an abelian variety, and $\alpha : X\to A$ a morphism such that $\alpha(x_0)=0$ is called the Albanese variety of X if, for every morphism $f : X\to B$ such that B is an abelian variety and $f(x_0)=0$, there is a unique homomorphism of abelian varieties $g : A\to b$ such that $g\circ\alpha=f$*

Remark. One coroallary of the previous section tells us that a morphism between abelian varieties is always a homomorphism plus a constant, so that above definition can be in fact made without resorting to a base point $x_0\in X$.

Remark. Also, above definition is stated in terms of a universal property, so that all Albanese varieties of X must be isomorphic, and so, we can just say that $(Alb(X),\alpha)$ is the Albanese variety of X, once it exists.

We'll show the existence of the Albanese variety and its construction will show the relationship with the Picard scheme.

Definition 3.2 (dual abelian variety)**.** *Let B be an abelian variety. Mumford proved that then $\underline{\mathrm{Pic}}^0(B)$ is always reduced, hence an abelian variety. We write it as $\hat{B}=\underline{\mathrm{Pic}}^0(B)$ and call it the* dual abelian variety *of B.*

Note that $\dim\hat{B}=\dim T_{B,0}=\dim H^1(B,\mathcal{O}_B)=\dim B$.

Let X be an arbitrary nonsingular projective variety. Let $P(X)$ be the Picard variety of X, i.e. $P(X)=\underline{\mathrm{Pic}}^0(X)_{\mathrm{red}}$.

Lemma 3.3. *There is a canonical morphism*

$$u_X : X\to \widehat{P(X)}=\underline{\mathrm{Pic}}^0(P(X)).$$

Proof. Let $x_0\in X$. Let L' be the Poincaré line bundle on $X\times\underline{\mathrm{Pic}}^0(X)$ and L'' be the Poincaré line bundle on $P(X)\times\widehat{P(X)}$. Let L'_{red} be the restriction of L' on $X\times P(X)$.

Let $T=X$, and let $v : P(X)\times X\to X\times P(X)$ be defined by $(a,b)\mapsto(b,a)$, which is an isomorphism. Consider $v^*L'_{\mathrm{red}}$.

Choose $a_0\in P(X)$, which corresponds to $\mathcal{O}_X$, in the sense of Prop (1.6) (1) (b), then, by our choise, $L'_{\mathrm{red}}|_{X\times\{a_0\}}\simeq\mathcal{O}_X$, so that

(a) $v^*L'_{\mathrm{red}}|_{\{a_0\}\times X}\simeq L'_{\mathrm{red}}|_{X\times\{a_0\}}\simeq\mathcal{O}_X$ and
(b) $v^*L'_{\mathrm{red}}|_{P(X)\times\{x\}}\simeq L'_{\mathrm{red}}|_{\{x\}\times P(X)}\simeq\mathcal{O}_{P(X)}$ since L' was the Poincaré line bundle, and in particular, it is algebraically equivalent to $\mathcal{O}_X$.

Hence by the UMP of $\underline{\mathrm{Pic}}^0(P(X))$, there is a unique morphism

$$u_X : X\to\underline{\mathrm{Pic}}^0(P(X))=\widehat{P(X)}$$

such that $v^*(L'_{\mathrm{red}}) \simeq (\mathrm{id}_{P(X)} \times u_X)^*(L'')$, which proves the lemma.

□

Remark (Mumford)**.** *If X is an abelian variety, then the canonical morphism $u_X : X \to \widehat{P(X)} = \widehat{\widehat{X}}$ is an isomorphism.*

Let $Alb(X) = \widehat{P(X)}$ and let $\alpha = u_X$. We show that it indeed is the Albanese variety.

Theorem 3.4. *If X is a nonsingular projective variety, then the Albanese variety $(Alb(X), \alpha)$ exists and it is unique up to isomorphism, and $Alb(X)$ is the dual of the Picard variety $\underline{\mathrm{Pic}}^0(X)_{\mathrm{red}}$ of X.*

Proof. By construction, $u_X(x_0) = 0$. Let $f : X \to B$ be a morphism with B: an abelian variety and $f(x_0) = 0$. Then, $P(f) : P(B) \to P(X)$ and $\widehat{P(f)} : \widehat{P(X)} \to \widehat{P(X)}$ which is

$$\widehat{(P(f)} : Alb(X) \to Alb(B) = \widehat{\widehat{B}} = B.$$

□

Remark. When X is a nonsingular projective curve, then, the Picard scheme $\underline{\mathrm{Pic}}^0(X)$ is automatically reduced, so that it is the Picard variety. Furthermore, $\underline{\mathrm{Pic}}^0(X) \simeq \widehat{\underline{\mathrm{Pic}}^0(X)}$ so that $\underline{\mathrm{Pic}}^0(X) \simeq Alb(X)$, which is also called the Jacobian of X.