K-theory and Related Topics

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Rough plan of the course.

(1) For the first two weeks, we will talk about Ajay Ramadoss's thesis.
(2) Two weeks from now, we will then talk about motives, Voevodsky's triangulated category and the bounded derived category of the category of motives and some K-theory.

1.1. Basics.

Recall the following. Consider a vector bundle V on a space X. The isomorphism class of the vector bundle V on X is noted by $[V]$. When we have a short exact sequence of vector bundles $0 \to V' \to V \to V'' \to 0$, we consider an abstract relation $[V] = [V'] + [V'']$. K-theory was defined like this:

$$K(X) := \frac{\text{the free abelian group on the isomorphisms classes of vector bundles}}{\text{the above relations}}$$

Example 1.1. (1) Let X be a topological space, or a C^∞-manifold. Then consider C^∞-vector bundles.

(2) Let X be a complex manifold and consider holomorphic vector bundles.

(3) Let X be a C^∞-manifold equipped with a foliation: $L \subset TX$ is an involutive subbundle such that, if l_1, l_2 are sections of L, then $[l_1, l_2]$ is a section of L again. Then, for all $p \in X$, there is a neighborhood U_p of p in X such that

$$\begin{tikzcd}
U_p \arrow[r, "\phi", "\simeq"'] & \mathbb{R}^n \arrow[d, "\pi"] & (x_1, \cdots, x_n) \arrow[d] \\
 & \mathbb{R}^k & (x_1, \cdots, x_k)
\end{tikzcd}$$

and for $q \in \mathbb{R}^k$, $T_x\left(\pi^{-1}q \cap U_p\right) \subset T_xX$ coincide with $L_x \subset T_xX$. Then, vector bundles V are equipped with a flat connection on the leaves: $T^*X = \Omega_X \to L^*$, usual connection : $\Gamma(X, V) \to \Gamma(X, \Omega_X \otimes V)$, the connection we need: $\Gamma(X, V) \to \Gamma(X, L^* \otimes V)$ together with Leibniz rule and flatness.

We remark that (2) is a special case of (3) if we allow $L \subset \mathbb{C} \otimes_{\mathbb{R}} TX$ and $[L, L] \subset L$.

Note also that, we have a sheaf of rings on X: $\mathcal{O}_X$ defined as

$$\begin{aligned}\Gamma(U_p, \mathcal{O}_X) &= \{f \circ \pi | f : \mathbb{R}^k \to \mathbb{C} \text{ is } C^\infty\} \\ &= \{f : U_p \to \mathbb{C} | vf = 0, \forall v \in \Gamma(U_p, L)\}\end{aligned}$$

(4) Much more generally, consider $(X, \mathcal{O}_X)$, a ringed space and locally free finitely generated sheaves of $\mathcal{O}_X$-modules on X. This case contains all the previous ones.

(5) Let X be a topological space and $g : X \to X$ is a homeomorphism. Consider

$$\{(V, \phi) | V \text{ vector bundles on } X, \phi : V \overset{\simeq}{\to} g^*V\}.$$

Here, to talk about exact sequences make sense. For example, consider $X = S^1$ and $\xi \in S^1$ which is not a root of 1. Let $g = T_\xi$ be the translation by ξ: $T_\xi(z) = \xi z$. Then we consider V with $V \overset{\simeq}{\to} T_\xi^* V$. It makes sense to talk about K-theory of this. They are the representations of a pro-algebraic group. It will come later.

There is a special example for (2). Let $X = \mathbb{R}^2/\mathbb{Z}^2$. For any $x \in X$, $T_xX = \mathbb{R}^2$. When α is an irrational number, consider $L_x = \mathbb{R} \cdot (1, \alpha)$.

(6) Let G be a group acting on X. We are interested in vector bundles with G-action, i.e. G acts on V. For $g \in G$, the diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "\pi"'] & V \arrow[d, "\pi"] \\ X \arrow[r, "g"] & X \end{tikzcd}$$

commutes and $g : \pi^{-1}X \to \pi^{-1}gX$ is a linear transformation.

For example, when X is a point, we have $\mathrm{Rep}(G)$.

Remark 1.2. $K(X)$ is known in case (1). For case (2), when X is a compact Riemann surface or a Stein manifold (Theorem of Grauert) we know them. For (6), when X is a point, we remarked that it is $\mathrm{Rep}(G)$.

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{O}_X$ is a sheaf of commutative rings. $K(X)$ is a commutative ring with a product structure defined by $[V] \cdot [W] = [V \otimes W]$. It is a special λ-ring: we have a λ-operation $V \mapsto \wedge^i V$ and so we can form $[V] \mapsto \sum_{i=0}^{\infty} t^i [\wedge^i V]$. It gives us a map

$$\lambda_t : K(X) \to K(X)[[t]]^{\times}.$$

One interesting fact is that $K(X) \otimes \mathbb{Q}$ is a graded ring.

1.2. **Some classical construction.**

Let X be a compact manifold. Consider $\mathbb{C}$-vector bundles. Recall the theorem of Atiyah-Hirzebruch:

$$K(X) \otimes \mathbb{Q} \simeq \bigoplus_{i=0}^{\infty} H^{2i}(X, \mathbb{Q})$$

Chern classes via connections and curvatures give this map. Let's recall the construction a little bit.

Given a C^∞-vector bundle V and a connection $D : V \to \Omega \otimes V$ in local coordinate, say, $X = \mathbb{R}^n \ni (x_1, \cdots, x_n)$. Giving a connection is same as giving $\frac{D}{\partial x_i} : \Gamma(X, V) \to \Gamma(X, V)$ with

$$\left[\frac{D}{\partial x_i}, \frac{D}{\partial x_j}\right](fv) = f\left[\frac{D}{\partial x_i}, \frac{D}{\partial x_j}\right]v$$

for all functions f and for all sections v of V, i.e. $\left[\frac{D}{\partial x_i}, \frac{D}{\partial x_j}\right]$ is a section of $\mathrm{End}(V)$.

The curvature $R = \sum_{i \le i < j \le n} dx_i dx_j \left[\frac{D}{\partial x_i}, \frac{D}{\partial x_j}\right]$ is a global section of $\Omega^2 \otimes \mathrm{End}(V)$. Hence,

$$\underbrace{R \wedge \cdots \wedge R}_{p\text{-times}} \in \Omega^{2p} \otimes \underbrace{\mathrm{End}(V) \otimes \mathrm{End}(V)}_{p\text{-times}} \overset{\Omega^{2p} \otimes l}{\to} \Omega^{2p}$$

where $l : M_n(\mathbb{C})^{\otimes p} \to \mathbb{C}$ satisfies $l(gh_1g^{-1} \otimes gh_2g^{-1} \otimes \cdots \otimes gh_pg^{-1}) = l(h_1 \otimes h_2 \otimes \cdots \otimes h_p)$ for all $h_i \in M_n(\mathbb{C})$ and for all $g \in GL_n(\mathbb{C})$. These give the characteristic classes.

1.3. **Ajay Ramadoss's thesis.**

Let X be a complex manifold and $\Omega = \Omega^1_X$ is the sheaf of Kähler differentials, i.e. the sheaf of sections of the holomorphic cotangent bundle.

Let V be a holomorphic bundle. Cover X by open subsets $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ so that $V|_{U_\alpha} \overset{\simeq}{\to} U_\alpha \times \mathbb{C}^n$ so that $V|_{U_\alpha}$ has a holomorphic connection D_α. On the overlaps $U_\alpha \cap U_\beta$, we have

$$V|_{U_\alpha \cap U_\beta} \overset{D_\beta - D_\alpha}{\longrightarrow} (\Omega \otimes V)|_{U_\alpha \cap U_\beta}$$

is in fact a holomorphic section $D_\beta - D_\alpha \in \Gamma(U_\alpha \cap U_\beta, \Omega \otimes \mathrm{End}(V))$ so that it gives rise to $\Theta(V)$ in $Z^1(\mathcal{U}, \Omega \otimes \mathrm{End}(V))$ so that a class in $H^1(X, \Omega \otimes \mathrm{End}(V))$. This class $\Theta(V)$ is usually called the *Atiyah class*.

Recall that we have a cup product $\cup$ so that if $u \in H^p(X, \mathcal{F})$, $v \in H^q(X, \mathcal{G})$, then $u \cup v \in H^{p+q}(X, \mathcal{F} \otimes \mathcal{G})$. Hence, we have $\underbrace{\Theta(V) \cup \cdots \cup \Theta(V)}_{k\text{-times}} \in H^k(X, \Omega^{\otimes k} \otimes_{\mathcal{O}} (\mathrm{End}(V))^{\otimes k}) \overset{(1_{\Omega^{\otimes k}})\otimes l}{\to} H^k(X, \Omega^{\otimes k})$. Therefore, $M_n(\mathbb{C})^{\otimes k} \xrightarrow{l} \mathbb{C}$, l are $GL_n(\mathbb{C})$-invariant.

Remark 1.3. Recall the theorem of Hermann Weyl. Let $W = \mathbb{C}^n$. Consider $W^* \otimes \cdots \otimes W^* \otimes W \otimes \cdots \otimes W$. Any l is a linear combination of (choose $\sigma \in S_k$) $w_1^* \otimes \cdots \otimes w_k^* \otimes w_1 \otimes \cdots \otimes w_k \mapsto w_{\sigma(1)}^*(w_1) w_{\sigma(2)}^*(w_2) \cdots w_{\sigma(k)}^*(w_k)$. For example, if $\sigma = (1, \cdots, n)$, the corresponding $l : M_k(\mathbb{C})^{\otimes k} \to \mathbb{C}$ will be given by $l(A_1 \otimes \cdots \otimes A_k) = \mathrm{tr}(A_1 \cdots A_k)$.

More generally, given $r_1 < r_2 < \cdots < r_p = k$, we have

$$l(A_1 \otimes \cdots \otimes A_k) = \mathrm{tr}(A_1 \cdots A_{r_1}) \mathrm{tr}(A_{r_1+1} \cdots A_{r_2}) \cdots \mathrm{tr}(A_{r_{p-1}+1} \cdots A_{r_p}).$$

This l is saturated by the action of S_k. They give all linear functions which are GL_n-invariant.

We now define $t_k(V)$. Consider $(\mathrm{End}(V))^{\otimes k} \xrightarrow{l} \mathbb{C}$ with $l(A_1 \otimes \cdots \otimes A_k) = \mathrm{tr}(A_1 \cdots A_k)$ and consider the image of $\Theta(V)^k \in H^k(X, \Omega^{\otimes k} \otimes (\mathrm{End}(V))^{\otimes k}) \overset{(1_{\Omega^{\otimes k}})\otimes l}{\to} H^k(X, \Omega^{\otimes k})$. It is defined to be $t_k(V)$.

Note that, in fact, one could have just taken

$$(\mathrm{End}(V))^{\otimes k} \xrightarrow{\widetilde{l}} \mathrm{End}(V), \widetilde{l}(A_1 \otimes \cdots A_k) = A_1 \cdots A_k$$

before taking the trace and consider the image $1_{\Omega^{\otimes k}} \otimes \widetilde{l}\left(\Theta(V)^k\right) \in H^k(X, \Omega^{\otimes k} \otimes \mathrm{End}(V))$ which is by definition $\widetilde{t}_k(V)$. Hence, obviously, $\mathrm{tr}\widetilde{t}_k(V) = t_k(V)$.

Remark 1.4. Note that when $k = 0$, we have

$$\widetilde{t}_0(V) = t_0(V) = \mathrm{rk}(V) \in H^0(X, \Omega^{\otimes 0}).$$

Consider the map

$$K(X) \otimes \mathbb{Q} \to \bigoplus_{k=0}^{\infty} H^k(X, \Omega^{\otimes k})$$

$$V \mapsto \sum_{k=0}^{\infty} t_k(V).$$

On $\bigoplus_k H^k(X, \Omega^{\otimes k})$, we want to put the structure of a commutative graded ring so that we will see that the map is a homomorphism of graded rings.

Lemma 1.5. *Let $0 \to V' \to V \to V'' \to 0$ be a short exact sequence of vector bundles. Then,*

$$t_k(V) = t_k(V') + t_k(V'').$$

Proof. There is an open cover $\{U_\alpha\}$ so that $V \underset{\simeq}{\overset{\phi}{\to}} U_\alpha \times \mathbb{C}^n$, $V' \simeq U_\alpha \times \mathbb{C}^m$ with $\mathbb{C}^m \hookrightarrow \mathbb{C}^n$, $(z_1, \cdots, z_m) \mapsto (z_1, \cdots, z_m, 0, \cdots, 0)$. Let D_α be the trivial connection given by this identification ϕ. Then, $D_\alpha(\Gamma(U_\alpha, V')) \subset \Gamma(U_\alpha, \Omega \otimes V')$ so that $D_\beta|_{U_\alpha \cap U_\beta} - D_\alpha|_{U_\alpha \cap U_\beta}$ therefore also has the same property. Hence, let $\mathrm{End}'(V) = \{u : V \to V | u(V') \subset V'\}$, then,

$D_\beta - D_\alpha \in \Gamma(U_\alpha \cap U_\beta | \Omega \otimes \mathrm{End}'(V))$ gives rise to a cocycle $\Theta'(V) \in H^1(X, \Omega \otimes \mathrm{End}'(V)) \mapsto \Theta(V) \in H^1(X, \Omega \otimes \mathrm{End}(V))$. On the other hand,

$$\begin{tikzcd}
H^k(X, \Omega^{\otimes k} \otimes (\mathrm{End}'(V))^{\otimes k}) \arrow[r, "\Theta'(V)^k \mapsto \Theta(V)^k"] \arrow[d] & H^k(X, \Omega^{\otimes k} \otimes (\mathrm{End}(V))^{\otimes k}) \arrow[d, "\Theta(V)^k \mapsto \widetilde{t}_k(V)"', "1_{\Omega^{\otimes k}} \otimes \widetilde{l}"] \\
H^k(X, \Omega^{\otimes k} \otimes \mathrm{End}'(V)) \arrow[r] & H^k(X, \Omega^{\otimes k} \otimes \mathrm{End}(V))
\end{tikzcd}$$

Notice that $\mathrm{End}'(V) \subset \mathrm{End}(V)$ is a sheaf of subalgebras so that the only matrices we are looking at are $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ and we know that $\mathrm{tr} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \mathrm{tr}(A) + \mathrm{tr}(D)$. There are two algebra homomorphisms $\mathrm{End}'(V) \xrightarrow{i} \mathrm{End}(V')$ and $\mathrm{End}'(V) \xrightarrow{j} \mathrm{End}(V'')$ defined by $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mapsto A$ and $\mapsto B$ respectively. And, we see that $1_{\Omega^{\otimes k}} \otimes j\Theta'(V) = \Theta(V'')$ and $1_{\Omega^{\otimes k}} \otimes i\Theta'(V) = \Theta(V')$. This finishes the proof. □

We look at the behaviour of t_k under $\otimes$.

Lemma 1.6.

$$\widetilde{t}_k(V \otimes W) = \sum_{r+s=k} \sum_{\sigma:(r,s)\text{-shuffle}} \epsilon(\sigma) \sigma \widetilde{t}_r(V) \cup \widetilde{t}_s(W)$$

Remark 1.7. (1) There is a map $\mathrm{End}(V) \otimes \mathrm{End}(W) \xrightarrow{\simeq} \mathrm{End}(V \otimes W)$ which is defined for $A : V \to V$ and $B : W \to W$ as a map $A \otimes B : V \otimes W \to V \otimes W$ sending $(A, B) \mapsto A \otimes V$ so that

$$\begin{cases} \widetilde{t}_s(V) \in H^r(X, \Omega^{\otimes r} \otimes \mathrm{End}(V)) \\ \widetilde{t}_r(W) \in H^s(X, \Omega^{\otimes s} \otimes \mathrm{End}(W)) \end{cases}$$

implies

$$\widetilde{t}_r(V) \cup \widetilde{t}_s(W) \in H^k(X, \Omega^{\otimes k} \otimes \mathrm{End}(V \otimes W)).$$

(2) Recall that $\mathrm{tr}(A \otimes B) = (\mathrm{tr} A)(\mathrm{tr} B)$.

Corollary 1.8.

$$t_k(V \otimes W) = \sum_{r+s=k} \sum_{\sigma:(r,s)\text{-shuffle}} \epsilon(\sigma) \sigma t_r(V) \cup t_s(W)$$

Suppose that $R \subset \{1, \cdots, k\}$, $|R| = r$. Let $S = R^c$ and $|S| = s$. Let $R = \{i_1, \cdots, i_r\}$, $i_1 < \cdots < i_r$ and $S = \{j_1, \cdots, j_s\}$, $j_1 < \cdots < j_s$. Consider $\sigma \in S_k$ which maps $1 \le x \le r$ to i_x and $r+1 \le y \le r+s = k$ to j_y. This is a shuffle.

In general, the right action of S_k on $W^{\otimes k}$ is given by

$$(w_1 \otimes \cdots \otimes w_k)\tau = w_{\tau(1)} \otimes \cdots \otimes w_{\tau(k)}.$$

Using it, for $\sigma \in S_k$ and $w \in W^{\otimes k}$, define $\sigma w = w\sigma^{-1}$, a left action.

2. September 30th, 2004

3. October 5th, 2004

3.1. Grothendieck's Descent theory.

Let's recall something about Grothendeick's descent (schemes). Given a morphism of schemes $f : Y \to X$ and a quasicoherent sheaf $\mathcal{F}$ on Y, our initial question is this: *Is there a quasicoherent sheaf $\mathcal{G}$ on X so that $f^*\mathcal{G} \simeq \mathcal{F}$?*

If this is the case, then we will have the following three properties:

D_1 Given any two morphisms $g_1, g_2 : Z \to Y$ such that $f \circ g_1 = f \circ g_2$, there is an isomorphism $c(g_1, g_2) : g_1^*\mathcal{F} \xrightarrow{\simeq} g_2^*\mathcal{F}$.

D_2 (funtoriality) Given any morphisms $h : W \to Z$ and $g_1, g_2 : Z \to Y$ such that $f \circ g_1 = f \circ g_2$, we have a commutative diagram

$$\begin{tikzcd}
h6 * g_1^*\mathcal{F} \arrow[r, "h^*c(g_1,g_2)"] \arrow[d, "\simeq"'] & h^*g_2^*\mathcal{F} \arrow[d, "\simeq"] \\
(g_1 \circ h)^*\mathcal{F} \arrow[r, "c(g_1\circ h, g_2\circ h)"] & (g_2 \circ h)^*\mathcal{F}
\end{tikzcd}$$

D_3 Given three morphisms $g_1, g_2, g_3 : Z \to Y$ such that $f \circ g_1 = f \circ g_2 = f \circ g_3$, we have a commutative diagram

$$\begin{tikzcd}
g_1^*\mathcal{F} \arrow[r, "c(g_1,g_2)"] \arrow[dr, "c(g_1,g_3)"'] & g_2^*\mathcal{F} \arrow[d, "c(g_2,g_3)"] \\
 & g_3^*\mathcal{F}
\end{tikzcd}$$

Note that the condition D_3 implies that $c(g, g) = \mathrm{id}_{g^*\mathcal{F}}$ and $c(a, b) = c(b, a)^{-1}$.

Definition 3.1 (descent data). *Let $f : Y \to X$ be a morphism of schemes and let $\mathcal{F}$ be a sheaf on Y. A descent data for $\mathcal{F}$ is the above three conditions D_1, D_2 and D_3.*

Theorem 3.2 (Grothendieck). *If $f : Y \to X$ is a flat surjective morphism, then, there is an equivalence of categories:*

$$\left\{ \begin{matrix} \textit{quasicoherent} \\ \textit{sheaves on } X \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{quasicoherent sheaves} \\ \textit{on } Y \textit{ with descent data} \end{matrix} \right\}$$

where the map correspondence is given by $\mathcal{G} \mapsto f^*\mathcal{G}$.

Construction. Given a quasicoherent sheaf $\mathcal{F}$ on Y with D_1, D_2 and D_3, Grothendieck constructs a quasicoherent sheaf $\overline{\mathcal{F}}$ on X as follows: first, we let

$$\Gamma(X, \overline{\mathcal{F}}) = \{s \in \Gamma(Y, \mathcal{F}) | \text{ for all } g_1, g_2 : Z \to Y, c(g_1, g_2)^* g_1^* s = g_2^* s\}.$$

We de the same for every open subset $U \hookrightarrow X$. Consider $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ and the descent data on $\mathcal{F}$ gives rise to a descent data on $\mathcal{F}|_{f^{-1}(U)}$ for the above morphism, so, define $\Gamma(U, \overline{\mathcal{F}})$ as above. It defines a presheaf, hence, sheafify it to obtain a sheaf. ☐

Example 3.3.

1. Two projections $p_1, p_2 : Y \times_X Y \to Y$ induce $c(p_1, p_2)^* : p_1^*\mathcal{F} \xrightarrow{\simeq} p_2^*\mathcal{F}$ obtained from D_1. Note that if we have $g_1, g_2 : Z \to Y$ with $f \circ g_1 = f \circ g_2$, then, we have $h := (g_1, g_2) : Z \to Y \times_X Y$ so that $p_1 \circ h = g_1, p_2 \circ h = g_2$ and we have $c(g_1, g_2) = (g_1^*, g_2^*)c(p_1, p_2) : g_1^*\mathcal{F} \to g_2^*\mathcal{F}$ by D_2.

   Hence, $D_1 + D_2$ is equivalent to giving $c(p_1, p_2)$. Note that D_3 is a statement about pull-baks of $\mathcal{F}$ on $Y \times_X Y \times_X Y$.
2. Note that $\overline{\mathcal{O}_Y} = \mathcal{O}_X$.

3.2. **Group schemes and principal bundles.**

Now, we work with schemes over a field k. Let H be an affine group scheme over k. (Autumatically the structure map $H \to \mathrm{Spec}(k)$ is flat. This is an important fact to note.)

Given a scheme Y over k, a right H-action on Y is defined by the following data:

1. $Y \times_k H \xrightarrow{A} Y$, A is the action.

(2) When $m : H \times H \to H$ is the group multiplication, we have a commutative diagram

$$\begin{tikzcd}
Y \times_k H \times_k H \arrow[r, "A \times \mathrm{id}_H"] \arrow[d, "\mathrm{id}_Y \times m"] & Y \times_k H \arrow[d, "A"] \\
Y \times_k H \arrow[r, "A"] & Y
\end{tikzcd}$$

Definition 3.4. *A* principal H-bundle *on X is a morphism $f : Y \to X$ where Y has a right H-action, X has the trivial action and f is H-equivariant such that the following conditions are satisfies:*

(1) *We have a commutative diagram of H-spaces over X:*

$$\begin{tikzcd}
Y \times_k H \arrow[r, "A"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

(2) *$f : Y \to X$ is a flat surjective morphism.*

(3) *the natural map $Y \times_k H \xrightarrow{u} Y \times_X Y$ given by $u = (p_1, A)$ is an isomorphism. (Note: $u(y,h) = (y, yh)$.)*

Let's examine D_1, D_2 and D_3 for $\mathcal{F}$ on Y. Fix a k-scheme Z and consider $\mathrm{Mor}_{\mathrm{Spec} k}(Z,T) = T(Z)$. The action map $A : Y \times H \to Y$ gives rise to a right action of the group $H(Z)$ on the set $Y(Z)$. (Consider $\mathrm{Mor}(Z, Y \times H) \to \mathrm{Mor}(Z,Y)$.) Then, (3) of the definition says that for all $g, g' : Z \to Y$ with $f \circ g = f \circ g'$, we have

$$\begin{tikzcd}
Z \arrow[rr, "(g{,}g')"] \arrow[dr, "(g{,}h)"'] & & Y \times_k Y \\
& Y \times_k H \arrow[ur, "\simeq"] &
\end{tikzcd}$$

so that there is a unique morphism $h : Z \to H$ such that $g' = gh$.

D_1 is now saying that for all $g : Z \to Y$ and for all $h : Z \to H$, we have $c(g,h) : g^*\mathcal{F} \overset{\simeq}{\to} (gh)^*\mathcal{F}$ and D_3 says that given any $g : Z \to Y$ and $h_1, h_2 : Z \to H$, we have a commutative diagram

$$\begin{tikzcd}
g^*\mathcal{F} \arrow[r, "c(g{,}h_1)"] \arrow[dr, "c(g{,}h_1h_2)"] & (gh_1)^*\mathcal{F} \arrow[d, "c(gh_1{,}h_2)"] \\
& (gh_1h_2)^*\mathcal{F}
\end{tikzcd} .$$

And, on $Y \times H$, $p_1^*\mathcal{F} \overset{c(p_1,p_2)}{\underset{\simeq}{\to}} A^*\mathcal{F}$. Hence, the data D_1, D_2, D_3 on $\mathcal{F}$ is in fact equivalent to an H-action on $\mathcal{F}$. Hence, the Grothendeick's theorem implies the following theorem:

Theorem 3.5. *Let $f : Y \to X$ be a principal H-bundle. Then, we have the following equivalence of categories:*

$$\left\{ \begin{matrix} \textit{quasicoherent} \\ \textit{sheaves on } X \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} H\textit{-quasicoherent} \\ \textit{sheaves on } Y \end{matrix} \right\}$$

Remark 3.6. (1) Note that on the other hand, we have a map

$$\mathrm{Rep}(H) \to \{H\text{-quasicoherent sheaves on } Y\}$$

.

(2) Note that $\mathcal{O}_Y$ is naturally an H-sheaf on Y.

(3) A representation V of H is naturally a H-sheaf on $\operatorname{Spec}(k)$. Hence, $\mathcal{O}_Y \otimes_k H$ is naturally a H-sheaf on $Y = Y \times_k \operatorname{Spec}(k)$.

(4) Recall the following from differential geometry. Let V be a left representation of H. Then, $\overline{V} := Y \times V/\sim \to X$ where $(yh, v) \sim (y, hv)$ is a vector bundle.

Corollary 3.7. *We get an exact functor commuting with* $\otimes$*:*

$$\{H\text{-}representations\} \xrightarrow{F} \left\{\begin{array}{c} locally\ free\ quasicoherent \\ sheaves\ on\ X \end{array}\right\}$$

defined by $F(V) = \overline{\mathcal{O}_Y \otimes_k V}$. *(Note that* $\bar{\ }$ *is the operation of Grothendieck.)*

Example 3.8. (1) $F(\text{trivial 1-dimensional representation}) = \mathcal{O}_X$.

(2) $F(\Gamma(H, \mathcal{O}_H)) = f_*\mathcal{O}_Y$. $(\rho(h)s = s \circ r_h.)$ Let's see why. (Sketch of proof of (2))

Let $\rho : H \to Aut(V)$ be a representation. Let $U \hookrightarrow X$ be an affine open subset. Then,

$$\Gamma(U, \overline{\mathcal{O}_Y \otimes_k V}) = \left\{ s \in \Gamma(f^{-1}(U), \mathcal{O}_Y \otimes_k V| \begin{array}{c} s : f^{-1}(U) \to V \text{ so that} \\ \rho(h)s(xh) = s(x) \forall x \in f^{-1}(U), \forall h \in H \end{array} \right\}.$$

Given $sin\Gamma(f^{-1}(U), \mathcal{O}_Y \times_k \Gamma(H, U)) = \Gamma(f^{-1}(U), \mathcal{O}) \otimes \Gamma(H, \mathcal{O})$, i.e. $\omega : f^{-1}(U) \times H \to \mathbb{A}^1$ with $\omega(xh, gh) = \omega(x, h)$ for all $x \in f^{-1}(U)$ and for all $g, h \in H$, the map $\omega \mapsto \omega|_{f^{-1}(U)}$ is a bijection from such ω's to $\Gamma(f^{-1}(U), \mathcal{O})$.

Let $X := Gr(q, V)$ be the Grassmannian and let $G = GL(V)$. Consider the subgroup $H := \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \subset G$ and let $Y = G$, $X = G/H$. Note that G is reductive.

Remark 3.9. In general, Chevalley proved that if H is a closed subgroup of an affine algebraic group G, then we have the existence of $X = G/H$ as a quasiprojective variety and $f : Y = G \to X = G/H$ is a flat surjective morphism.

Because the left G-action on G commutes with the right H-action, we have a functor $F : \{H\text{-representations}\} \to \{\text{ sheaves on } X\}$.

Theorem 3.10 (Bott)**.** *Let G be a reductive group with $char(k) = 0$. Then, when we regard k as the trivial representation, we have a natural isomorphism*

$$H^i(G/H, FV)^G \simeq \operatorname{Ext}^i_{H-rep}(k, V).$$

Proof. Put $T^iV = H^i(G/H, FV)^G$ for all $i \geq 0$. We check the following list of properties of Grothendieck's Tôhoku paper.

(a) $T^i : (H - rep) \to (k - vec)$ is a functor.

(b) Given short exact sequence $0 \to V' \xrightarrow{m} V \xrightarrow{n} V'' \to 0$ of H-representations, there is a long exact sequence

$$\cdots \to T^iV' \xrightarrow{T^i m} T^iV \xrightarrow{T^i n} T^iV'' \xrightarrow{\delta} T^{i+1}V' \to \cdots .$$

To prove it use two facts: from the given short exact sequence, obtain the long exact sequence of $H^i(-, \cdot)$. Now use the fact that if we have an exact sequence $W' \to W \to W''$ of G-representations, then, $W'^G \to W^G \to W''^G$ is exact.

(c) The data in (b) is functorial for short exact sequences.

(d) $T^0V = V^H$. $\Gamma(X, FV) = \{u : Y \to V | \rho(h)u(gh) = u(g) \forall g \in G, \forall h \in H\}$ and in $\Gamma(X, FV)^G$, $u(g_1g_2) = u(g_2)$ for all $g_1, g_2 \in G$. Thus $u(g) = u(e)$ for all $g \in G$ so that $\rho(h)u(h) = u(e)$ and $g = e$.

(e) (effaceability) For all $i > 0$, for all $\alpha \in T^i V$, there is a monomorphism $j : V \to W$ in $(H - rep)$ such that $(T^i j)\alpha = 0$.

In the language of Grothendieck's Tôhoku paper, $T^i = R^i T^0$ and $T^0 V = V^H = \mathrm{Hom}_{H-rep}(k, V)$.

We prove (e), which is the only non obvious one. Take $V = \Gamma(H, \mathcal{O})$. We know that $FV = f_* \mathcal{O}_G$. Observe that $G = Y$ is affine and $f : G \to G/H$ is also an affine morphism. Hence, when $\mathcal{F}$ is quasicoherent on Y, $H^i(Y, \mathcal{F}) = 0$ for all $i > 0$ since Y is affine, and thus, $R^i f_* \mathcal{F} = 0$ for all $i > 0$. Then, the Leray spectral sequence gives $E_2^{p,q} = H^p(X, R^q f_* \mathcal{F})$ and $E_2^{p,0} = H^p(X, f_* \mathcal{F})$. Take $p > 0$, then we deduce that $H^p(X, f_* \mathcal{O}_Y) = 0$ for all $p > 0$.

In fact, $V = \Gamma(H, \mathcal{O})$ is an injective object. Let W be a H-representation. Then, we have an isomorphism

$$\mathrm{Hom}_{H-rep}(W, \Gamma(H, \mathcal{O})) \overset{\cong}{\to} W^*$$

$$L \mapsto ev_{id} \circ L$$

where

$$W \xrightarrow{L} \Gamma(H, \mathcal{O}) \xrightarrow{ev_{id}} k.$$

Define $S : W^* \to \mathrm{Hom}_{H-rep}(W, \Gamma(H, \mathcal{O}))$ be the inverse. Let $u_1, u_2, \cdots \in W^*$ so that $\cap \ker u_i = 0$. We have $Su_i : W \to \Gamma(H, \mathcal{O})$ and thus, $W \to \bigoplus_i \Gamma(H, \mathcal{O})$ is one to one. It completes the proof of Bott's theorem. $\square$

Recall that $G = GL(V)$, $H = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, a parabolic subgroup. Let $U = \begin{pmatrix} \mathrm{id}_s & B \\ 0 & \mathrm{id}_q \end{pmatrix}$, the unipotent redicals of H where $q + s = n = \mathrm{rk}(V)$. By the Bott's theorem, $H^i(G/H, FV)^G = H^i(H - rep, V)$. The point is that, there is an exact sequence

$$1 \to U \to H \to H/U = GL_s \times GL_q \to 1.$$

We have Hochschild-Serre spectral sequence to compute $H^i(H - rep, V)$: $E_2^{p,q} = H^p(H/U - rep, H^q(U, V))$. We have $char(k) = 0$ so that $E_2^{p,q} = 0$ for all $p > 0$, so we end up having

$$H^q(U, V)^H = E_2^{0,q} \overset{\cong}{\leftarrow} H^p(H, V).$$

Suppose that V is a representation of H/U. Then, $H^q(U, V) \overset{\cong}{\leftarrow} H^q(U, k) \otimes_k V$ is an H/U-isomorphism. The question is this: What is $H^q(U, k)$? In fact, whenever U is commutative, $U = \mathrm{SpecSym} W$, then, $H^i(U, k) = \wedge^i W$.

4. October 7th, 2004

5. October 12th, 2004

5.1. **Forward for Motives.** From today, we will talk about motives.

Let X be a variety over $\mathbb{C}$ and $Y \hookrightarrow X$ is a subvariety. Let $H^i(X, Y)$ be the singular (relative) cohomology group with $\mathbb{Z}$-coefficient. Recall some elemenentary facts about it.

If $f : (X, Y) \to (X', Y')$ is a morphism, then it induces $H^i(f) : H^i(X', Y') \to H^i(X, Y)$. This is not an arbitrary map, for example, thanks to Deligne, these cohomology groups has Mixed Hodge Structures and $H^i(f)$ respects them.

In the following, we will see that there is an abelian category called ECM, the effective cohomological motives, and a faithful exact functor $F : ECM \to (Ab)$ so that $H^i(X, Y) \in \mathrm{Ob}(ECM)$ and $F(H^i(X, Y))$ is the usual singular cohomology group.

5.2. **First attempt.**

1. We want to think of the abelian category as modules over a ring E. In this setting, for all $Y \subset X$ defined over $\mathbb{C}$, for $i \geq 0$, $H^i(X,Y)$ will be a left E-module, i.e. there is a ring homomorphism $E \to \mathrm{End}_{\mathbb{Z}}(H^i(X,Y))$. Thus, we will have a ring homomorphism
$$E \to \prod_{(X,Y,i)} \mathrm{End}_{\mathbb{Z}} H^i(X,Y).$$
2. Meanwhile, we want $H^i(f)$ to be (left) E-module homomorphisms.
3. Also, when $X \supset Y \supset Z$ is a chain of closed subvarieties, we want to have a map $\delta : H^{i-1}(Y,Z) \to H^i(X,Y)$ which is an E-module homomorphism.

5.3. **Definition of E and ECM.** For what ring E can we achieve it? Let's take the following:

$$E = \left\{ a \in \prod_{(X,Y,i)} \mathrm{End}_{\mathbb{Z}} H^i(X,Y) | a \text{ satisfies the following } I \text{ and } II \right\}$$

(the components of a in this product will be denoted by $a(X,Y,i)$.)

(I) For all $f : (X,Y) \to (X',Y')$ where f is a morphism such that $f(Y) \subset Y'$, we have a commutative diagram:

$$\begin{tikzcd}
H^i(X',Y') \arrow[r, "H^i(f)"] \arrow[d, "a(X',Y',i)"'] & H^i(X,Y) \arrow[d, "a(X,Y,i)"] \\
H^i(X',Y') \arrow[r, "H^i(f)"] & H^i(X,Y)
\end{tikzcd}$$

(II) For all $X \subset Y \subset Z$ and for all i, we have a commutative diagram:

$$\begin{tikzcd}
H^{i-1}(Y,Z) \arrow[r, "\delta"] \arrow[d, "a(Y,Z,i-1)"'] & H^i(X,Y) \arrow[d, "a(X,Y,i)"] \\
H^{i-1}(Y,Z) \arrow[r, "\delta"] & H^i(X,Y)
\end{tikzcd}$$

Then, this E is a subring of $\prod_{(X,Y,i)} \mathrm{End}_{\mathbb{Z}} H^i(X,Y)$. We take ECM = the full subcategory of all left E-modules V that are finitely generated abelian groups such that there is a finite set S of (X,Y,i)'s with the following property: if $a(X,Y,i) = 0$ for all $(X,Y,i) \in S$, then, the action of a on V is 0. (In other words, V is a left module over $\mathrm{im}\Big(E \to \prod_{(X,Y,i)} \mathrm{End}_{\mathbb{Z}}(H^i(X,Y))\Big)$.)

Clearly this is an abelian category and we define the fibre functor $F : ECM \to (Ab)$ is the forgetful functor.

Remark 5.1. We easily see that $H^i(X,Y)$ are objects of ECM and the usual morphisms $H^i(f)$ induced from $f : (X,Y) \to (X',Y')$ and the usual coboundary morphisms are morphisms in the category ECM.

Remark 5.2.

1. One could define this category ECM also for singular cohomologies with coefficients in a commutative noetherian ring R. The definition works well, however, only if R is a field, for example, $R = \mathbb{Q}$. In fact, for $R = \mathbb{Z}$ (and with the base field $\mathbb{C}$) we will see that as rings $\mathbb{Z} \simeq E$.
2. We want to talk about it for other base fields other than $\mathbb{C}$ as well.

5.4. **More constructions.**

Definition 5.3 (diagram)**.** (Après Grothendieck's Tôhoku paper) *A* diagram D *is a category with compositions undefined, with the following properties:*

1. *The diagram D consists of an 4-tuple $(O(D), M(D), s, t)$, where $O(D)$, $M(D)$ are sets and $s, t : M(D) \to O(D)$ are functions.*
2. *For two objects p, q of $O(D)$, morphisms $M_D(p, q)$ from p to q in D is defined to be $M_D(p, q) = s^{-1}\{p\} \cap t^{-1}\{q\}$.*
3. *$m \in M_D(p, q)$ will be manytimes written as $m : p \to q$ in D and vice versa.*

Definition 5.4 (representation)**.** *Let D, D' be two diagrams. A* representation $F : D \to D'$ *consists of the following data:*

1. *A pair of functions $F : O(D) \to O(D')$ and $F : M(D) \to M(D')$ such that*
2. *for all $p, q \in O(D)$, we have $F(M_D(p, q)) \subset M_{D'}(F(p), F(q))$.*

A commutative noetherian ring R will be fixed in this section. Let $(R - mod)$ be the category of finitely generated left R-modules.

Definition 5.5. (1) *A category $\mathcal{A}$ is an* R-additive category *if $\mathcal{A}$ is an additive category in the usual sense with the following additional property:*
   (a) *for all $A, B \in \mathrm{Ob}(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}}(A, B) \in \mathrm{Ob}(R - mod)$.*
   (b) *for all $A, B, C \in \mathrm{Ob}(\mathcal{A})$, the composition map*

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \times \mathrm{Hom}_{\mathcal{A}}(B, C) \to \mathrm{Hom}_{\mathcal{A}}(A, C)$$

given by $(f, g) \mapsto g \circ f$ is R-linear in each variable.

(2) *A category $\mathcal{A}$ is* R-abelian category *if it is an R-additive category and in addition,* ker, coker *of morphisms exist and whenever $f : A \to B$ is a morphism with* $\ker f = \mathrm{coker} f = 0$, *then f is an isomorphism.*

Given a representation $T : D \to (R - mod)$, we shall construct an R-Abelian category $C(T)$ such that we have the following commutative diagram of representations:

$$\begin{tikzcd} D \arrow[r, "\widetilde{T}"] \arrow[dr, "T"'] & C(T) \arrow[d, "ff_T"] \\ & (R - mod) \end{tikzcd}$$

where $\widetilde{T}$ is a representation, ff_T is an R-additive faithful exact functor.

Definition 5.6. (1) *A diagram D is* finite *if $O(D)$ and $M(D)$ are finite sets.*

(2) *Let F be a finite subdiagramof D. We may restrict T to F and define* $\mathrm{End}(T|F)$ *to be*

$$\mathrm{End}(T|F) = \left\{ a \in \prod_{p \in O(F)} \mathrm{End}_R(Tp) \middle| \forall m : p \to \text{ in } F, \begin{array}{ccc} Tp & \xrightarrow{Tm} & Tq \\ {\scriptstyle a(p)}\big\downarrow & & \big\downarrow{\scriptstyle a(q)} \\ Tp & \xrightarrow{Tm} & Tq \end{array} \text{ commutes.} \right\}$$

Remark 5.7. (1) $\mathrm{End}(T|F)$ is a R-subalgebra which is finitely generated as an R-module.

(2) For all finite subdiagrams F, G with $F \subset G \subset D$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{End}(T|G) \arrow[r] \arrow[d] & \prod_{p \in O(G)} \mathrm{End}_R(Tp) \arrow[d] \\
\mathrm{End}(T|F) \arrow[r] & \prod_{p \in O(F)} \mathrm{End}_R(Tp)
\end{tikzcd}$$

Definition 5.8. *The category $C(T)$ is defined to be*

$$C(T) = \varinjlim_{F \subset D \text{ finite subdiagram}} \mathrm{End}_R(T|F) - mod.$$

More precisely,

$$\mathrm{Ob}(C(T)) = \left\{ (V, F, e) \middle| \begin{array}{c} V \in \mathrm{Ob}(R-mod) \\ F \subset D : \textit{finite subdiagram} \\ e : \mathrm{End}(T|F) \to \mathrm{End}_R(V) \textit{ is an } R\textit{-algebra homomorphism.} \end{array} \right\}$$

$\mathrm{Mor}_{C(T)}((V_1, F_1, e_1), (V_2, F_2, e_2)) =$

$\{f : V_1 \to V_2 | f$ *is a* $\mathrm{End}(T|F_3)$*-module homomorphism for some finite subdiagram* $F_3 \supset F_1 \cup F_2\}$ *with* $ff_T : C(T) \to R - mod$ *is defined to be* $ff_T(V, F, e) = V$.

It remains to define $\widetilde{T} : D \to C(T)$. *For all* $p \in O(T)$, *define* $F_p \subset D$ *by* $M(F_p) = \phi$, $O(F_p) = \{p\}$, $\mathrm{End}(T|F_p) = \mathrm{End}_R(Tp)$. *We then define*

$$\widetilde{T}p = \left(Tp, Fp, \mathrm{id}_{\mathrm{End}(T|F_p)}\right).$$

For all $m : p \to q$ *in* $M(D)$, *define* F_m *by* $O(F_m) = \{p, q\}$, $M(F_m) = m$ *and let* $\widetilde{T}m : \widetilde{T}p \to \widetilde{T}q$ *be given by " $f = Tm$ " in the above definition.*

This definition has a certain universal property:

Theorem 5.9. *Suppose that we are given a diagram*

$$\begin{tikzcd}
D \arrow[r, "F"] \arrow[dr, "T"'] & \mathcal{A} \arrow[d, "G"] \\
 & R - mod
\end{tikzcd}$$

where $\mathcal{A}$ is an R-abelian category and G is R-additive faithful exact functor. Then, there is a unique factorization

$$\begin{tikzcd}
D \arrow[r, "\widetilde{T}"] \arrow[dr, "F"'] & C(T) \arrow[r, "ff_T"] \arrow[d] & R - mod \\
 & \mathcal{A} \arrow[ur, "G"'] &
\end{tikzcd}$$

with both diagrams commute such that $C(T) \to \mathcal{A}$ is R-additive. (This is automatically faithful and exact.)

Remark 5.10. If R is artinian, (in particular a field) then this construction of $C(T)$ coincides with the full subcategory of finitely generated continuous left $\mathrm{End}(T)$-modules.

Let k be a field and let $\sigma : k \to \mathbb{C}$ be an embedding. All schemes (over k) considered are assumed to be reduced and of finite type.

Definition 5.11 (good pair)**.** *A good pair (X, Y) consists of the following information: X is an affine variety, $Y \subset X$ is closed in the sense of Zariski and $X - Y$ is nonempty smooth of pure dimension n such that* $\dim Y < n$ *so that the q-th singular homology of (X^σ, Y^σ) is 0 for all $q \neq n$.*

In this case, $H_n(X^\sigma, Y^\sigma)$ is a finitely generated free abelian group thanks to some generalizations of the theorem of Andreoth-Frankel.

Equivalently, $H^q(X^\sigma, Y^\sigma; \mathbb{Z}/n\mathbb{Z}) = 0$ for all $q \neq n$ and $n \geq 1$. When $\overline{k}$ is the algebraic closure of k, then accoding to Grothendieck, $H^q(X^\sigma, Y^\sigma; \mathbb{Z}/n\mathbb{Z}) \simeq H^q_{et}(X_{\overline{k}}, Y_{\overline{k}}; \mathbb{Z}/n\mathbb{Z})$ so that the definition of *good pair*, in fact, doesn't depend on the choice of a particular embedding $\sigma : k \hookrightarrow \mathbb{C}$.

Definition 5.12. *The diagram of good paris $D_g(k)$ is defined as follows:*

(1) *For objects,*

$$\mathrm{Ob}(D_g(k)) = \coprod_{N \geq 0} \left\{ (X,Y) | (X,Y) \ : \textit{ good pair}, X \overset{closed}{\hookrightarrow} \mathbb{A}^N_k \right\}$$

(2) *For morphisms* $\mathrm{Mor}(D_g(k))$, *we have two types:*
   - (I) *if (X, Y) and (X', Y') are good pairs of the same dimension, then all $m : (X,Y) \to (X',Y')$ in* $\mathrm{Mor}(D_g(k))$ *are morphisms of varieties $f : (X,Y) \to (X',Y')$.*
   - (II) *Assume that $\mathbb{A}^N_k \supset X \supset Y \supset Z$ and* $\dim X = \dim Y + 1$, *and (X, Y), (Y, Z) are both good pairs. Then, we have a unique morphism $\partial(X,Y,Z) : (X,Y) \to (Y,Z)$ i.e.* $\mathrm{Mor}(D_g(k))\left((X,Y),(Y,Z)\right)$ *is a singleton.*

Fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$.

Definition 5.13 (ECM, EHM and etc.). *We have $H^\sigma_* : D_g(k) \to \mathbb{Z} - mod$ given by $H^\sigma_*(X,Y) = H_n(X^\sigma, y^\sigma)$, the unique nonvanishing singular homology with $\mathbb{Z}$-coefficient, $n = \dim X$. Similarly, we have a functor $H^*_\sigma : D_g(k)^{op} \to \mathbb{Z} - mod$ given by $H^*_\sigma(X,Y) = H^n(X^\sigma, Y^\sigma)$, the unique nonvanishing singular cohomology with $\mathbb{Z}$-coefficient.*

We define ECM to be the category $C(T)$ with $T = H^*_\sigma : D_g(k)^{op} \to \mathbb{Z} - mod$ and EHM to be the category $C(T)$ with $T = H^\sigma_* : D_g(k) \to \mathbb{Z} - mod$.

Remark 5.14.
(1) We commonly write $ff_T : ECM \to \mathbb{Z} - mod$ and $ff_T : EHM \to \mathbb{Z} - mod$ by $b(\sigma)$ to mean the Betti-realization.
(2) By the above construction, for all good pairs (X, Y) with $n = \dim X$, we now have $\widetilde{T}(X,Y) \in \mathrm{Ob}(ECM)$ such that $b(\sigma)\widetilde{T}(X,Y) = H^n(X^\sigma, Y^\sigma)$.

6. October 14th, 2004

Our objective was this: to construct an affine monoid scheme over $\mathrm{Spec}(\mathbb{Z})$ defined by $MotMon(\sigma)$ so that the category of left representations is ECM and the right representations is EHM.

Definition 6.1. *We say that $T : D \to R - mod$ is* special *when R is a Dedekind domain or a field and when for all $p \in O(D)$, Tp is a torsion free module (i.e. projective).*

In this case, we have $\mathrm{End}(T|F) \subset \prod_{p \in O(F)} \mathrm{End}_R(Tp)$ is torsion free, hence a projective R-module, and we take $\mathrm{End}^*(T|F) = \mathrm{Hom}_R(\mathrm{End}(T|F), R)$, the collection of all R-module homomorphisms. Multiplication on $\mathrm{End}(T|F)$ turns $\mathrm{End}^*(T|F)$ into an R-coalgebra (coassociative with counit). Finally, $\mathrm{End}^*(T) = \varinjlim_{\substack{F \subset D \\ \text{finite subdiagram}}} \mathrm{End}^*(T|F)$ is once again a coalgebra (coassociative with counit).

Remark 6.2. $C(T)$ is equivalent to the category of $\mathrm{End}^*(T)$-comodules that are finitely generated as R-modules. We can prove it as follows. Suppose that $(V, F, \rho) \in \mathrm{Ob}(C(T))$,

$V \in \mathrm{Ob}(R-mod)$, $\rho : \mathrm{End}(T|F) \to \mathrm{End}_R(V)$. Data $\mathrm{End}(T|F) \otimes_R V \to V$ is equivalent to having $V \to \mathrm{End}^*(T|F) \otimes_R V$ because $\mathrm{End}(T|F)$ is finitely generated projective. Thus, this gives $V \to \mathrm{End}^*(T|F) \otimes_R V$. This give $C(T) \to \mathrm{End}^*(T) - comod$. Observe that if V is a finitely presented R-module, then,

$$\varinjlim_{\lambda} \mathrm{Hom}(V, W_\lambda) \xrightarrow{\simeq} \mathrm{Hom}(V, \varinjlim_{\lambda} W_\lambda) :$$

given a finitely generated R-module V that is a comodule over $\mathrm{End}^*(T)$, we have

$$V \xrightarrow{A} \mathrm{End}^*(T) \otimes_R^V = \left(\varinjlim_{F} \mathrm{End}^*(T|F) \right) \otimes_R V \xleftarrow{\simeq} \varinjlim_{F} \mathrm{End}^*(T|F) \otimes_R V.$$

A yields therefore a map $A_F : V \to \mathrm{End}^*(T|F) \otimes_R V$ for some F by the above observation (surjectivity part).

Let $\mu_F : \mathrm{End}^*(T|F) \to \mathrm{End}^*(T|F) \otimes_R \mathrm{End}^*(T|F)$ be the comultiplication. For V to be a comodule, we need the commutativity of

$$\begin{tikzcd}
V \arrow[r, "A_F"] \arrow[d, "A_F"'] & \mathrm{End}^*(T|F) \otimes_R V \arrow[d, "\mathrm{id}_{\mathrm{End}^*(T|F)} \otimes A_F"] \\
\mathrm{End}^*(T|F) \otimes_R^V \arrow[r, "\mu_F \otimes \mathrm{id}_V"] & \mathrm{End}^*(T|F) \otimes_R \mathrm{End}^*(T|F) \otimes_R V
\end{tikzcd}$$

Let's see this. Let $\xi : V \to \mathrm{End}^*(T|F) \otimes_R \mathrm{End}^*(T|F) \otimes_R V$ be the difference of two compositions. We know that $j \circ \xi = 0$, where $j : \mathrm{End}^*(T|F) \otimes_R \mathrm{End}^*(T|F) \otimes_R V \to \mathrm{End}^*(T) \otimes_R \mathrm{End}^*(T) \otimes_R V$, because V is $\mathrm{End}^*(T)$-comodule. Therefore, the injectivity part of the observation implies that there is a finite subdiagram G containing F such that the above diagram commutes with F replaced by G. One checks that V as a $\mathrm{End}^*(T|G)$-module is counitary, i.e. V is a module over $\mathrm{End}(T|G)$.

Remark 6.3. Given $T : D \to R-mod$ and a commutative R-algebra S which is noetherian, consider $B : R-mod \to S-mod$ given by $M \mapsto S \otimes_R M$. Then, for all finite subdiagram F of D, we have $S \otimes_R \mathrm{End}(T|F) \xrightarrow{\simeq} \mathrm{End}(B \circ T|F)$.

Recall the following. Let $m : p \to q$ be in F and we have

$$\begin{tikzcd}
Tp \arrow[r, "a(p)"] \arrow[d, "Tm"'] & Tp \arrow[d, "Tm"] \\
Tq \arrow[r, "a(q)"] & Tq
\end{tikzcd}$$

where $p = s(m), q = t(m)$. For all $m \in M(F)$, $a \mapsto Tm \circ a(p) - a(q) \circ Tm$,

$$w(m) : \prod_{p \in O(F)} \mathrm{End}_R(Tp) \to \mathrm{Hom}_R(Tsm, Ttm).$$

In other words, the exactness of the sequence

$$0 \to \mathrm{End}(T|F) \to \prod_{p \in O(F)} \mathrm{End}_R(Tp) \to \prod_{m \in M(F)} \mathrm{Hom}(Tsm, Ttm)$$

is a definition of $\mathrm{End}(T|F)$. By taking $S \otimes_R -$, we have a left exact sequence. To complete the proof, we need to say that

$$S \otimes_R \mathrm{Hom}(M, N) \xrightarrow{\simeq} \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$$

for all finitely presented R-modules M.

Remark 6.4. Assume that $T : D \to R - mod$ is special and S is an R-flat Dedekind domain. Then,

(i) there is a natural isomorphism

$$S \otimes_R \operatorname{End}^*(T) \simeq \operatorname{End}^*(B \circ T),$$

(ii) the category of $S \otimes_R \operatorname{End}^*(T)$ comodules is precisely $C(B \circ T)$.

6.1. **Product structure.** We shall show that ECM, EHM are tensor categories. So, now,

$$D = D_g(k) = \coprod_{N \geq 0} \{(X, Y) \text{ is a good pair.} | X \hookrightarrow \mathbb{A}_k^N\}$$

with morphisms $\operatorname{Mor}_D((X, Y), (X', Y'))$ is the usual morphism of pair of varieties if $\dim X = \dim X'$. If $X \supset Y \supset Z$ and if (X, Y) and (Y, Z) are good pairs and $\dim X = \dim Y + 1$, then, $\operatorname{Mor}_D((X, Y), (Y, Z))$ is the singleton $\partial(X, Y, Z)$.

If (X_i, Y_i) is a good pair with $\dim X_i n_i$, $i = 1, 2$, then, $(X_1, Y_1) \times (X_2, Y_2) := (X_1 \times X_2, Y_1 \times X_2 \cup X_1 \times Y_2)$ is a good pair of dimension $n_1 + n_2$ and we have the Eilenberg-Zilber isomorphism

$$EZ : H^{n_1+n_2}((X_1, Y_1)^\sigma \times (X_2, Y_2)^\sigma) \overset{\simeq}{\to} H^{n_1}(X_1^\sigma, Y_1^\sigma) \otimes_{\mathbb{Z}} H^{n_2}(X_2^\sigma, Y_2^\sigma)$$

.

Given $(N_i, X_i, Y_i) \in \operatorname{Ob}(D_g(k))$ with $X_i \hookrightarrow \mathbb{A}_k^{N_i}$, we have $(N_1 + N_2, (X_1 \times X_2, Y_1 \times X_2 \cup X_1 \times Y_2)) \in \operatorname{Ob}(D_g(k))$ so that there is a product structure: $\operatorname{Ob}(D_g(k)) \times \operatorname{Ob}(D_g(k)) \to O(D_g(k))$.

Note that given an object $p \in \operatorname{Ob}(D_g(k))$ and $m : q \to r$, we get a morphism $1_p \times m : p \times q \to p \times r$ defined in an obvious manner:

type I A morphism of pair of varieties $f : (X_2, Y_2) \to (X_3, Y_3)$ yields $1_{X_1} \times f : (X_1, Y_1) \times (X_2, Y_2) \to (X_1, Y_1) \times (X_3, Y_3)$.

type II Supposed that we are given $A \supset B \supset C$ such that $\partial(A, B, C)$ is defined. Then, for any good pair (X, Y), $1_{(X,Y)} \times \partial(A, B, C) : (X, Y) \times (A, B) \to (X, Y) \times (B, C)$ is not defined: this corresponds to the excision map in the following diagram:

$$\begin{tikzcd}
H_*(X \times A, Y \times A \cup X \times B) \arrow[r] & H_{*-1}(Y \times A \cup X \times B, Y \times A) \\
 & H_{*-1}(X \times B, (X \times B) \cap (Y \times A)) \arrow[u, "\text{excision}"']
\end{tikzcd}$$

It is to be clarified later.

6.2. **Alternative description of** $\operatorname{End}(T|F)$**.** We had $\operatorname{End}(T|F) \subset \prod_{p \in O(F)} \operatorname{End}(Tp) \subset \operatorname{End}(\bigoplus_{p \in O(F)} Tp)$. Put $TF = \bigoplus_{p \in O(F)} Tp$ and let $i(p) : Tp \to TF$ and $\pi(p) : TF \to Tp$ be the natural inclusion and projection. Let $e(p) = i(p) \circ \pi(p)$ for all $p \in O(F)$. For all $m : p \to q$ in $M(F)$, we have

$$\begin{tikzcd}
TF \arrow[r, "\pi(p)"] \arrow[drr, "w(m)\text{:composition}"] & Tp \arrow[r, "Tm"] & Tq \arrow[d, "i(q)"] \\
 & & TF
\end{tikzcd}$$

Claim.

$$\operatorname{End}(T|F) = \{h \in \operatorname{End}_R(TF) | h \textit{ commutes with } e(p) \textit{ and } w(m) \forall p \in O(F), \forall m \in M(F)\}$$

Proof. $he(p) = e(p)h$ for all $p \in O(F)$ implies that $h \in \prod_{p\in O(F)} \mathrm{End}(Tp)$ and $hw(m) = w(m)h$ now implies that $h \in \mathrm{End}(T|F)$. ☐

Definition 6.5 (product of diagram)**.** *Given diagrams D_1 and D_2, the product diagram $D = D_1 \times D_2$ is defined as follows:*

(1) *its objects* $\mathrm{Ob}(D) = \mathrm{Ob}(D_1) \times \mathrm{Ob}(D_2)$.
(2) *morphisms are defined as follows:*

$$\mathrm{Mor}_D\left((p_1,p_2),(q_1,q_2)\right) = \begin{cases} \phi & \text{if } p_1 \neq q_1 \text{ and } p_2 \neq q_2 \\ \mathrm{Mor}_{D_2}(p_2,q_2) & \text{if } p_1 = q_1 \text{ and } p_2 \neq q_2 \\ \mathrm{Mor}_{D_1}(p_1,q_1) & \text{if } p_2 = q_2 \text{ and } p_1 \neq q_1 \\ \mathrm{Mor}_{D_1}(p_1,q_1) \cup \mathrm{Mor}_{D_2}(p_2,q_2) & \text{if } p_1 = q_1 \text{ and } p_2 = q_2 \end{cases}.$$

Definition 6.6 (product of T_i)**.** *Let $I_i : D_i \to R-mod$ be given for $i = 1,2$. We define $T = T_1 \times T_2 : D_1 \times D_2 \to R-mod$ as follows: $T(p_1,p_2) = Tp_1 \otimes Tp_2$ for all $p_1 \in \mathrm{Ob}(F_1)$ and $p_2 \in \mathrm{Ob}(F_2)$.*

Given case 2 or 4, $\mathrm{Mor}_D((p_1,p_2),(q_1,q_2)) = \mathrm{Mor}_{D_2}(p_2,q_2)$, so, given, $m_2 : p_2 \to q_2$, $\mathrm{id}_{Tp} \otimes Tm_2 = T_1p \otimes T_2p_2 \to T_1p \otimes T_2q_2$. We do this similarly for case 3.

Lemma 6.7. *Assume that $T_i : D_i \to R-mod$ are special, where $i = 1,2$. Let $F_i \subset D_i$ be finite subdiagrams. Then,*

$$\mathrm{End}(T_1|F_1) \otimes \mathrm{End}(T_2|F_2) \xrightarrow{\simeq} \mathrm{End}(T_1 \times T_2|F_1 \times F_2).$$

$T_iF_i = \bigoplus_{p\in\mathrm{Ob}(F_i)} T_ip$, $i = 1,2$. We are looking at $\mathrm{End}(T_i|F) \hookrightarrow \mathrm{End}(T_iF_i)$ which is the commutant of $S_i = \{e(p_i), w(m_i) | \forall p \in \mathrm{Ob}(F_i), \forall m_i \mathrm{Ob}(M(F_i))\} \subset \mathrm{End}(T_iF_i)$. $T = T_1 \times T_2$, $F = F_1 \times F_2$, $TF = \bigoplus_{\substack{p_1 \in \mathrm{Ob}(F_1) \\ p_2 \in \mathrm{Ob}(F_2)}} \cdots = T_1F_1 \otimes T_2F_2$.

Claim. *The R-subalgebra of* $\mathrm{End}(T_1F_1) \otimes_R \mathrm{End}(T_2F_2)$ *generated by* $S_1 \otimes \mathrm{id}_{T_2F_2}$ *and* $\mathrm{id}_{T_1F_1} \otimes S_2$ *also equals the R-subalgebra generated by $S = \{e(p), w(m) | \forall p \in \mathrm{Ob}(F), m \in \mathrm{Mor}(F)\}$.*

Proof. $S = \{e(p_1) \otimes e(p_2), e(p_1) \otimes w(m_2), w(m_1) \otimes e(p_2) | \forall p_i \in \mathrm{Ob}(F_i), \forall m_i \in \mathrm{Mor}(F_i), i = 1,2\}$ so that one inclusion ($\supset$) is clear. ($\subset$) is just easy: note that $\sum_{p\in\mathrm{Ob}(F_1)} e(p) = \mathrm{id}_{T_1F_1}$ and for a fixed m_2, $\sum_{p_1\in\mathrm{Ob}(F_1)} e(p_1) \otimes w(m_2) = \mathrm{id} \otimes w(m_2)$, the same remark applies. ☐

7. October 19th, 2004

8. October 21st, 2004

8.1. From affine varieties to chain complexes in EHM.

Let's recall some definitions from the previous classes. Let $\sigma : k \hookrightarrow \mathbb{C}$ be an embedding. Let X be a variety over k and X^σ be the resulting variety over $\mathbb{C}$.

(a) We had $H_i((X^\sigma)_{an}, (Y^\sigma)_{an}) = H_i^\sigma(X,Y)$.
(b) $b(\sigma) : EHM \to \mathbb{Z}-mod$ is the forgetful functor. Let (X,Y) be a good pair over k. We define $H_*^\sigma(X,Y) := H_{\dim X}^\sigma(X,Y)$. In general we had $D \xrightarrow{\widetilde{T}} C(T) \xrightarrow{ff_T} R-mod$. In particular, when $D = D_g(k)$ and $EHM = C(T)$, we have

$$D_g(k) \xrightarrow{H_*} EHM \xrightarrow{b(\sigma)} \mathbb{Z}-mod,$$

thus we have $H_*(X,Y) \in \mathrm{Ob}(EHM)$ and $b(\sigma)H_*(X,Y) = H_{\dim X}^\sigma(X,Y)$.

Definition 8.1. (1) *Let X be an affine variety over k. An* admissible filtration *is a sequence $X_0 \subset X_1 \subset \cdots \subset X$ of closed subvarieties so that* $\dim X_i \leq i$.

(2) *An admissible filtration* X_n *is called a* good filtration *if it satisfies the following two properties:*
   (a) $\bigcup_n X_n = X$
   (b) *Let* Z_k *be the union of irreducible components of* X_k *of dimension* k*. If* $Z_k \neq \phi$*, then,* $(Z_k, Z_k \cap X_{k-1})$ *is a good pair.*

Lemma 8.2. *If* $X.$ *is a good filtration, then for all* $k \geq 0$*,* $(Z_k \cap X_{k-1}, Z_k \cap X_{k-2})$ *is also a good filtration.*

Proof. That $(Z_k, Z_k \cap X_{k-1})$ is a good pair implies that $Z_k \cap X_{k-1}$ has pure dimension $k-1$ and thus it is in Z_{k-1} by the definition of Z_{k-1}. And, recall the following fact: if (P, Q) is a good pair of dimension e, and if P' is the union of irreducible components (??? of what???), then, $(P', P' \cap Q)$ is a good pair. Apply this to $(P, Q) = (Z_{k-1}, Z_{k-1} \cap X_{k-2})$ and $P' = Z_k \cap X_{k-1}$. So, here, $P' \cap Q = Z_k \cap X_{k-1} \cap Z_{k-1} \cap X_{k-2} = Z_k \cap X_{k-2}$. $\square$

We have triples $(Z_k, Z_k \cap X_{k-1}, Z_k \cap X_{k-2}) \hookrightarrow (X_k, X_{k-1}, X_{k-2})$, so, writing down the boundary maps in singular homology, we have a commutative diagram

$$\begin{tikzcd}
H_k^\sigma(Z_k, Z_k \cap X_{k-1}) \arrow[r, "\partial'"] \arrow[dd, "\simeq"] & H_{k-1}^\sigma(Z_k \cap X_{k-1}, Z_k \cap X_{k-2}) \arrow[d, "i"] \\
 & H_{k-1}^\sigma(Z_{k-1}, Z_{k-1} \cap X_{k-2}) \arrow[d, "\simeq"] \\
H_k^\sigma(X_k, X_{k-1}) \arrow[r, "\sigma"] & H_{k-1}^\sigma(X_{k-1}, X_{k-2})
\end{tikzcd}$$

We have

$$\begin{tikzcd}
p = (Z_k, Z_k \cap X_{k-1}) \arrow[r, "\partial(p,q)"] & q = (Z_k \cap X_{k-1}, Z_k \cap X_{k-2}) \arrow[d, "i''"] \\
 & (Z_{k-1}, Z_{k-1} \cap X_{k-2})
\end{tikzcd}$$

where $\partial(p, q)$ is a morphism of type II and i'' is a morphism of type I in $D_g(k)$. Thus we have a morphism in EHM

$$H_*(i'') \circ H_*(m) : H_*(Z_k, Z_k \cap X_{k-1}) \to H_*(Z_{k-1}, Z_{k-1} \cap X_{k-2})$$

and $b(\sigma)H_*(i'') \circ H_*(m) = i \circ \partial'$. This gives a chain complex $C.(X, \{X_n\}_n)$ in EHM:

$$\cdots \to H_*(Z_k, Z_k \cap X_{k-1}) \to H_*(Z_{k-1}, Z_{k-1} \cap X_{k-2}) \to \cdots$$

so that when we apply $b(\sigma)$ to the complex, $b(\sigma)C.(X, \{X_n\}_n)$ is the chain complex

$$\cdots \to H_k^\sigma(X_k, X_{k-1}) \to H_{k-1}^\sigma(X_{k-1}, X_{k-2}) \to \cdots .$$

In particular, we can define the following: $H_i(X) := H_i(C.(X, \{X_n\}_n)) \in \mathrm{Ob}(EHM)$. We then see that $b(\sigma)H_i(X) = H_i^\sigma(X)$.

Corollary 8.3. *(of the Lemma) Any admissible filtration* $\{X_n\}$ *on an affine variety* X *is contained in a good filtration.*

(Use a decreasing induction on n.)

Remark 8.4. Let $f : Y \to X$ be a morphism of affine varieties. Choose a good filtration $\{Y_n\}_n$ on Y. Then, let X'_n be the Zariski closure of the images of $f(Y_n)$. This forms an admissible filtration, therefore, there is a good filtration $\{X_n\}$ such that $X'_n \subset X_n$.

This induces a chain map in EHM: $C.(f) : C.(Y, \{Y_n\}_n) \to C.(X, \{X_n\}_n)$.

8.2. **The category** Ind($\mathcal{A}$) **by Deligne.** Basic reference for this digression is the following paper:

Pierre Deligne, *Le group fondemental de droite projective moins trois points* in Galois groups of $\mathbb{Q}$, p.79-297, MSRI Publ. 16, Springer, NY, 1989

Given a category $\mathcal{A}$, we can construct a new category Ind($\mathcal{A}$) as follows: its objects are $\{(A, X_a) | a \in A\}$ where A is a directed set and $\{a \mapsto X_a\}$ is a directed system of objects in $\mathcal{A}$. An object (A, X_a) will be denoted by $\varinjlim_{a\in A} X_a$. Its morphisms are

$$\mathrm{Mor}_{\mathrm{Ind}(\mathcal{A})}\left(\varinjlim_{a\in A} X_a, \varinjlim_{b\in B} Y_b\right) = \varprojlim_{a\in A} \mathrm{Mor}_{\mathrm{Ind}(\mathcal{A})}\left(X_a, \varinjlim_{b\in B} Y_b\right) = \varprojlim_{a\in A}\varinjlim_{b\in B} \mathrm{Mor}_{\mathcal{A}}(X_a, Y_b)$$

where $X_a S$ is defined as $(\{a\}, X_a)$.

Example 8.5.
Recall that $\mathbb{Z} - mod$ is the category of finitely generated abelian groups. We have $\mathrm{Ind}(\mathbb{Z} - mod) = (Ab)$.
But, $\mathrm{Ind}(Ab) \neq (Ab)$.

Let $\mathcal{A}$ be an R-abelian category such that every object satisfies the ascending chain condition. Then, $\mathcal{A}$ is a full subcategory of the abelian category Ind($\mathcal{A}$) under $a \mapsto (\{a\}, a)$.

Let $\mathcal{D}^b_{\mathcal{A}}(\mathrm{Ind}(\mathcal{A}))$ be the derived category of bounded chain complexes $C.$ in Ind($\mathcal{A}$) such that $H_i(C.) \in \mathrm{Ob}(\mathcal{A})$ for all i. Then, Deligne shows that

$$\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b_{\mathcal{A}}(\mathrm{Ind}(\mathcal{A}))$$

is an equivalence of triangulated categories.

8.3. **Chain complexes.** Let $Ch'(\mathrm{Ind}(EHM))$ be the category of bounded chain complexes in $\mathrm{Ind}(EHM)$ such that $H_i(C.) \in \mathrm{Ob}(EHM)$ for all i.

Theorem 8.6. *There is a functor* $C. : (AffVar/k) \to Ch'(\mathrm{Ind}(EHM))$ *so that for all* $i \geq 0$, $b(\sigma)H_i(C.(X))$ *is naturally equivalent to* $H^{\sigma}_i(X)$.

Here, we use this: the Betti realization $b(\sigma) : EHM \to \mathbb{Z}{-}mod$ *induces* $b(\sigma) : \mathrm{Ind}(EHM) \to \mathrm{Ind}(\mathbb{Z} - mod) = (Ab)$.

Definition 8.7. *Let X be an affine variety over k. Note that if $\{X'_n\}_n$ and $\{X''_n\}_n$ are good filtrations, then, $\{X'_n \cup X''_n\}_n$ is admissible so that it is contained in a good filtration $\{X'''_n\}_n$. Thus, the collection of good filtrations is a directed set and it makes sense to define, for all $r \geq 0$,*

$$C_r(X) = \varinjlim_{\substack{F\ :\ good \\ filtrations}} C_r(X, F).$$

Lemma 8.8. *Let X be an affine variety over k. Then there is a natural map*

$$\bigoplus_{\substack{W \subset X\ closed \\ irreducible \\ \dim W = r}} C_r(W) \to C.(X)$$

which is an isomorphism.

Proof. Fix W. Define $p_W : C_r(X) \to C_r(W)$. We have to define $H_r(X_r, X_{r-1}) \to C_r(W)$ for all $X_{r-1} \subset X_r \subset X$ closed and (X_r, X_{r-1}) is a good pair of dimension r, in a compatible manner. This is done as follows:

(a) If $W \not\subset X_r$, then, the desired map is defined to be 0.
(b) Assume that $W \subset X_r$. Let W' be the union of remaining irreducible components of X_r. We have seen that $(W, W \cap X_{r-1})$ and $(W', W' \cap X_{r-1})$ are good paris and $H_*(W, W \cap X_{r-1}) \oplus H_*(W', W' \cap X_{r-1}) \to H_*(X_r, X_{r-1})$ is an isomorphism in EHM. The desired map is defined to be the projection to the first factor.

1. Check compatibility to show that p_W is well-defined.
2. $C_r(W) \to C_r(X) \overset{p_W}{\to} C_r(W)$ is the identity map. (Obvious form the definition.)
3. Given $\alpha \in C_r(X)$, $\{W | p_W(\alpha) \neq 0\}$ is finite. (Clear, because there are only finitely many irreducible components.)
4. For all $\alpha \in C_rX)$, $\sum_W i_W p_W \alpha = \alpha$.

□

Theorem 8.9 (continued)**.** *We have two functors*

$$\begin{tikzcd}
(AffVar/k) \times (AffVar/k) \arrow[r] \arrow[d, "C.\otimes C."] & AffVar/k \arrow[r, "C."] & Ch'(\mathrm{Ind}(EHM)) \\
CH'(\mathrm{Ind}(EHM)) & &
\end{tikzcd}$$

These are not the same functors, but, there is a natural transformation

$$N : C. \otimes C. \to C. \circ (product)$$

such that for (X, Y), $N(X, Y) : C.(X) \otimes C.(Y) \to C.(X \times Y)$ *is a quasi-isomorphism, i.e. induces an isomorphism on homology level.*

Let $X_r \subset X$, $Y_s \subset Y$ be closed subvarieties such that (X_r, X_{r-1}) and (Y_s, Y_{s-1}) are good pairs of dimensions r and s respectively. Then, $H_r(X_r, X_{r-1}) \otimes H_s(Y_s, Y_{s-1}) \overset{\simeq}{\to} H_{r+s}((X_r, X_{r-1} \times (Y_s, Y_{s-1}))$ and $(X_r, X_{r-1}) \times (Y_s, Y_{s-1})$ is a good pair on $X \times Y$.

Proposition 8.10. *UNFINISHED YET*

9. October 28th, 2004

10. November 2nd, 2004

Some Applications of étale cohomology is what we want to do this week. As always, an embedding $\sigma k \hookrightarrow \mathbb{C}$ is fixed. Recall that we had $D_g(k)$ = the diagram of good pairs. And we had $D_g(k)^{op} \overset{H^*_\sigma}{\to} \mathbb{Z} - mod), C(H^*_\sigma) := ECM$.

R is a commutative noetherian ring. Fix a good pair (X, Y), we can consider $H^*_\sigma(R)(X, Y) = H^{\dim X}_\sigma(X, Y; R)$. This gives us $D_g(k)@ > H^*_\sigma(R) >> R-mod$. If we take the corresponding functor $C(H^*_\sigma(R))) := ECM(R)$. This is nice only when R is flat or $\mathbb{Z}$.

For a finite subdiagram $F \subset D_g(k)$, recall that we had $0 \to \mathrm{End}(H^*_\sigma|_F) \hookrightarrow \mathrm{End}(\oplus_{p \in D(F)} H^*_\sigma p) \to Q \to 0$ and Q is torsion free. This sequence is exact. Hence, by tensoring with R, we obtain $0 \to \mathrm{End}(H^*_\sigma|_F) \otimes R \hookrightarrow \mathrm{End}(\oplus_{p \in D(F)} H^*_\sigma(p)) \otimes R \to Q \otimes R \to 0$ is exact. $H^*_\sigma(p)$ is free abelian, so that, $\mathrm{End}(\oplus_{p \in D(F)} H^*_\sigma p) \otimes R \overset{\simeq}{\to} \mathrm{End}(\oplus_{p \in D(F)} H^*_\sigma(R) p)$.

What we are getting at is the containments: $R \otimes_{\mathbb{Z}} \mathrm{End}(H^*_\sigma|_F) \hookrightarrow \mathrm{End}(H^*_\sigma(R)|_F) \hookrightarrow \prod_{p \in D(F)} \mathrm{End}_R H^*_\sigma(R)(p)$. So, the point is that this induces (by restriction), for instant $R = \mathbb{Z}/n\mathbb{Z}$, a faithful exact functor $ECM(R) \to ECM$.

Theorem 10.1. *When $n \in \mathbb{N}$, $ECM(\mathbb{Z}/n\mathbb{Z})$ is in fact equivalent to to the category of continuous $Gal(\overline{k}/k)$-modules which are finite and annihilated by n.*

This theorem is an application of étale cohomology and the comparison theorem.

Among the various properties, we state things we are going to use only: Sheaves on étale topology.

Let $Sh(X_{et}; R)$ be the category of sheaves on X_{et} of R-modules. Among the properties this thing has, we want the following.

(a) X/k, $\tau : k \to L$ embedding, $\mathcal{F} \mapsto \mathcal{F}^\tau$ gives an exact functor $Sh(X_{et}; R) \to Sh(X^\tau_{et}; R)$.

(b) When $X/\mathbb{C}$, $\mathcal{F} \mapsto \mathcal{F}_{an}$ is an exact functor from the sheaves $Sh(X_{et}; R) \to Sh(X_{an}; R)$. This induces $H^i(X_{et}, \mathcal{F}) \to H^i(X_{an}, \mathcal{F})$. This loses lots of information unless $\mathcal{F}$ is a torsion.

The Artin-Grothendieck comparison theorem says that the above arrow is an isomorphism for torsion sheaves $\mathcal{F}$. Furthermore if we look at the proof, we never have to go to a larger integer n which annihilates the torsion sheaves.

(c) if $\tau ; k_1 \to k_2$ is a homomorphism of algebraically closed fields, and we have a variety X/k_1 and we have a sheaf $\mathcal{F} \in \mathrm{Ob}(Sh(X_{et}; R)$ with $nR = 0$. Then, the natural map

$$H^i(X_{et}, \mathcal{F}) \to H^i(X^\tau, \mathcal{F}^\tau)$$

is an isomorphism for all $i \geq 0$.

(d) Suppose X/k is a variety. Let $\overline{X} = X \times_k \overline{k}$. Suppose that $nR = 0$ and $\mathcal{F} \in Sh(X_{et}, R)$. Then, $\mathcal{F} \mapsto H^i(\overline{X}, \overline{\mathcal{F}})$ is the derived functor from $Sh(X_{et}, R) \to (All\mathrm{R}{-}moduesl)$ of the functor $\mathcal{F} \mapsto \Gamma(\overline{X}, \overline{\mathcal{F}})$ where $\tau : k \to \overline{k}, \overline{\mathcal{F}} = \mathcal{F}^\tau$.

One way to see this is to consider $X \xrightarrow{p} \mathrm{Spec}(k)$ and we want to coonsider the sheaves on Spec$k$. Then the stalks at Spec$\overline{k}$ is the ones we are looking at, i.e. $H^i(\overline{X}, \overline{\mathcal{F}})$ is the stalk at Spec$\overline{k} \to$ Spec$k$) of $R^i p_* \mathcal{F}$.

Any of the above properties will be just used without proofs.

Quick sketch of what we are going to do from now on. Given $\mathcal{F} \in \mathrm{Ob}(Sh(X_{et}; R))$, where R is any commutative noetherian ring, X/k, $\sigma : k \to \mathbb{C}$. $\mathcal{F}$ is constructible, i.e. there is a stratification, etc.

Choose a good filtration $\mathcal{F} = F^0\mathcal{F} \supset F^1\mathcal{F} \supset \cdots F^{n+1}\mathcal{F} = 0$, (Assume further that X is affine of dimension n.) so that $\mathrm{Supp}(F^k\mathcal{F}/F^{k+1}\mathcal{F}) = Z_k$ closed of pure dimension k contained in X. We have $Y_k \subset Z_k$ closed and the sheaf $F^k\mathcal{F}/F^{k+1}\mathcal{F}|Y_k = 0$ and $|_{Z_k - Y_k}$ is locally constant. Let $Z_k = L1 \cup L - 2 \cup \cdots \cup L_m$, L_i irreducible components. Then, $L_j, Y_k \cap L_j)$ is a super pair for all j.

(Recall : super pair. X, affine of dimension n, $Y \subset X$ is closed not equal to X. $X - Y$ is affine smooth. For all $\mathcal{F}$ on X^σ_{an}, $\mathcal{F}|Y^\sigma_{an} = 0$, $\mathcal{F}|(X - Y)^\sigma_{an}$ is locally constant, and $H^i(X^\sigma_{an}, \mathcal{F}) = 0$ for all $q \neq \dim X$.)

In particular, $H^q(X^\sigma_{an}, F^k\mathcal{F}/F^{k+1}\mathcal{F}^\sigma_{an}) = 0$ for all $q \neq k$. This defines a complex $H^\cdot(X^\sigma_{an}, \left(F^\cdot\mathcal{F}/F^{\cdot+1}\mathcal{F}\right)^\sigma_{an})$ of R-moduesl which computes $H^k(X^\sigma_{an}, \mathcal{F}^\sigma_{an})$.

So, we require to endow $H^k(X^\sigma_{an}, (F^k\mathcal{F}/F^{k+1}\mathcal{F})^\sigma_{an})$ the structure of an object of $ECM(R)$.

All this will show that if we take X an affine variety over k, we will have functors $H^i(X, \cdot) : Sh(X_{et}; R) \to ECM(R)$, $\forall i \geq 0$. This is what Grothendieck calls the cohomological δ-functor. If I take the betti realization $b(\sigma) \circ H^i(X, \mathcal{F})$, then, it is the same thing as $H^i(X^\sigma_{an}, \mathcal{F}^\sigma_{an})$.

Now, if $R = \mathbb{Z}/n\mathbb{Z}$, $\mathcal{F}$; constructible sheave on X_{et} with R-coefficient, $k \xrightarrow{\sigma} \mathbb{C}$, $k \to \overline{k}$, $\overline{k} \xrightarrow{k} \mathbb{C}$. (triangle)

Then, $H^i(\overline{X}_{et}, \overline{\mathcal{F}}) \simeq H^i(X^\sigma_{an}, \mathcal{F}^\sigma_{an})$. For all $q > 0$, there is an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of constructible sheaes in $Sh(X_{et}; R)$,

$H^{q-1}(\overline{X}_{et}, \mathcal{H}) \to^{surj} H^q(\overline{X}_{et}, \overline{\mathcal{F}} \to^0 H^q(\overline{X}_{et}, \mathcal{G})$

Therefore, by induction, for all $\mathcal{F}$, q, there exists a constructible sheaf $\mathcal{F}'$ on X and an epimorphism from $H^0(\overline{X}_{et}, \overline{\mathcal{F}'}) \to H^q(\overline{X}_{et}, \overline{calF})$. Remember that the comparison theorem gives an isomorphism from the above ones with $H^\cdot(X^\sigma_{an}, \mathcal{F}(or\mathcal{F}')^\sigma_{an})$.

Conclude from this that any object of $ECM(R)$ is a subquotient of a finite direct sum of $H^q(X - bar_et, \mathcal{F} - bar)$ so that $\Gamma((X_ibar)_{et}, \mathcal{F}_i)$. These are Galois modules. The ECM-construction using only H^0 is precisely finitely generated R-modules equipped with continuous Galois actions of $Gal(\overline{k}/k)$. This is the sketch of the idea.

This theorem has a nice consequences.

We go to some categorical fuss.

Avoiding natural transformations.

Definition 10.2. *Let $F\mathcal{A} \to \mathcal{B}$ be a functor of categories. F is irredundant if*

(i) *$f : P \to Q$ is an isomorphism in $\mathcal{A}$ and $FP = FQ$ and $Ff = \mathrm{id}_{FP}$. Then, $P = Q$ and $f = \mathrm{id}_P$.*

(ii) *Given an object $P \in \mathrm{Ob}(\mathcal{A})$, and an isomorphism $g : FP \to Q'$ in $\mathcal{B}$, there exists an isomorphism $f : P \to Q$ in $\mathcal{A}$ so that $g = Ff$. (The pair (Q, f) is unique by property (i).)*

Exercise 10.3. *Show that given any functor $F : \mathcal{A} \to \mathcal{B}$, there is a factoring of F, $G : \mathcal{A} \to \mathcal{C}$, $H : \mathcal{C} \to \mathcal{B}$ so that G is an equivalence and G is irredundant.*

Example 10.4. (1) Recall that MHS consists of the following data: $(L, \{W_n : n \in \mathbb{Z}\}, \{F^p : p \in \mathbb{Z}\})$, L is a finitely generated abelian group, W_n inc. F^n dec. filtrations, etc.

Let $T(L, \cdot, , \cdots) = L$. Then, T is irredudant.

(2) Galrep (Galois representations). Its objects are $\{(L, \rho) | L \in (\mathbb{Z}{-}mod), \rho : Gal(\overline{k}/k) \to Aut(\breve{\mathbb{Z}} \otimes L) isacontinuoushomomorphism\}$. Then, the functor $(L, \rho) \mapsto L$ is an irredundant functor.

(3) Let D be a diagram, $T : D \to (R - mod)$ be a representation. We had

$$\begin{tikzcd}
D \arrow[r, "\widetilde{T}"] \arrow[d, "T"] & C(T) \arrow[d, "ff_T"] \\
{} & (R - mod)
\end{tikzcd}$$

Here ff_T is not irredundant. But, we can turn it into an irredundant one by following procedure:

Objects of $C(T)$ are $\{(V, F, \rho) | V \in \mathrm{Ob}(R - mod), F \subset D, \text{finite}, \rho : \mathrm{End}(T|F) \to \mathrm{End}_R M : is \text{R}{-}\text{alge. homo}\}$. Define $C'(T)$ as equivalence calsses of $\mathrm{Ob}(C(T))$ via $(V_1, F_1, \rho_1) \sim (V_2, F - 2, \rho_2)$ if (a) $V_1 = V_2$ and, (b) there is a finite subdiagram

$F_3 \supset F_i$, $i = 1, 2$ and the diagram

$$\begin{tikzcd}
 & End(T|F_1) \arrow[dr, "\rho_1"] & \\
End(T|F_3) \arrow[ur] \arrow[dr] & & End_R(V_1) \\
 & End(T|F_2) \arrow[ur, "\rho_2"] &
\end{tikzcd}$$

commutes.

Then, we have

$$\begin{tikzcd}
D \arrow[r, "\tilde{T}"] \arrow[dr, "\tilde{T'}"] & C(T) \arrow[r, "ff_T"] \arrow[d] & (R-mod) \\
 & C'(T) \arrow[ur, "ff'_T"] &
\end{tikzcd}$$

where $ff_T(V, F, \rho) = V$.

In our context, $D = D_g(k)^{op}$, $T = H^*_\sigma$, and the diagram

$$\begin{tikzcd}
D_g(k)^{op} \arrow[r, "H^*"] \arrow[dr, "H^*"] & ECM(R) \arrow[r, "b(\sigma)"] \arrow[d, "equiv"] & R-mod \\
 & ECM'(R) \arrow[ur, "b(\sigma)"] &
\end{tikzcd}$$

We are working with $ECM'(R)$ until the theorem is proved.Thus $b(\sigma) : ECM'(R) \to R-mod$ is irredundant.

Definition 10.5. *We define the category* $D_s(k;R)$*.* $\mathrm{Ob}(D_s(k;R))$ *consists of* $(X, Y, \mathcal{F})$*,* (X, Y) *is a super pair and* $\mathcal{F} \in Sh(X_{et}; R)$*,* $\mathcal{F}|Y_{et} = 0$ *and* $\mathcal{F}|_{(X-Y)_{et}}$ *is locally constant and all stalks of* $\mathcal{F}$ *are finitely generated* R*-modules.*

type I morphisms: assume that $\dim X = \dim X'$*. Then,* $\mathrm{Mor}_{D_s(k;R)}((X, Y, \mathcal{F}), (X', Y', \mathcal{F}')) = \{(f, h) : f : X \to X' ismorphismofvarietiesoverkandh : f^*\mathcal{F}' \to \mathcal{F} isahomomorphismofsheavesinSh(X_{et}; R)$

type II morphisms: $(X, Y, \mathcal{F}) \to (X', Y', \mathcal{F}')$ *is empty unless* X' *is an irreducible component of* Y *and* $\mathcal{F}' = j_!\mathcal{F}|_{X'-Y'}$ *where* $j : X' - Y' \hookrightarrow X'$*. Here there is a unique morphism.*

11. November 4th, 2004

12. November 9th, 2004

The last step is to look at the coboundary operator. Situation is the following: we have a short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

in $Sh(X_{et}; R)$. Suppose that $X \supset Y \supset Z \supset W$ are all closed and suppose that (X, Y), (Z, W) are super pairs of dimension n and dimension $n - 1$, respectively.

The hypothesis (A) is this: $\mathcal{F}' \in \mathcal{C}(X, Y)$, and $\mathrm{Supp}\mathcal{F}'' \subset Z$ and $\mathcal{F}''|_Z \in \mathcal{C}(Z, W)$.

Consider

$$\begin{tikzcd}
H^{n-1}_\sigma(X, \mathcal{F}'') \arrow[r, "\delta"] \arrow[d, "\simeq"] & H^n_\sigma(X, \mathcal{F}') \\
H^{n-1}_\sigma(Z, \mathcal{F}''|_Z) \arrow[ur, "\delta_1"] &
\end{tikzcd}$$

Under additional hypothesis, we'll show that there is $h : H^{n-1}(Z, \mathcal{F}''|Z) \to H^n(X, \mathcal{F}')$ and h is in $ECM(R)$ so that $b(\sigma)h = \delta_1$.

Suppose that we have a commutative diagram of sheaves in $Sh(X_{et}; R)$ with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0
\end{tikzcd}$$

Suppose further that

(i) $\mathcal{F}' \to \mathcal{G}'$ is a monomorphism.
(ii) $\mathcal{G}', \mathcal{G}''$ satisfy the above hypothesis (A).
(1) the result holds for the bottom exact sequence.

Then, we claim that it holds for the first exact sequence.

We want to check that δ_1 is $\mathrm{End}(H^*_\sigma(R)|F)$-homomorphism for some $F \subset D_g(k)$.

$$\begin{tikzcd}
H^{n-1}_\sigma(Z, \mathcal{F}''|_Z) \arrow[r, "\delta_1"] \arrow[d, "\alpha"] & H^n_\sigma(X, \mathcal{F}') \arrow[d, "\beta"] \\
H^{n-1}_\sigma(Z, \mathcal{G}''|_Z) \arrow[r, "\delta_2"] & H^n_\sigma(X, \mathcal{G}'')
\end{tikzcd}$$

β is a monomorphism because $\mathcal{A} \mapsto H^n_\sigma(X, \mathcal{A})$ is exact in $\mathcal{C}(X, Y)$.

α, β are $\mathrm{End}H^*_\sigma(R)|_F$-module homomorphism for suitable $F \subset D_g(k)$. By the assumption (iii), δ_2 is also a $\mathrm{End}H^*_\sigma(R)|_F$-module homomorphism. Hence, δ_1 is also a $\mathrm{End}H^*_\sigma(R)|_F$-module homomorphism.

let $U = X - Y \subset V = (X - Y) \cup (Z - W) \hookrightarrow X$, U, V are open, and $u : U \to X, v : V \to X$ are inclusions. For any sheaf $\mathcal{F}$ on X, we have a natural map $\mathcal{F} \to u_*u^*\mathcal{F}$. In our situation, $\mathcal{F}|_{X-V} = 0$ and $\mathcal{F} \to v_!v^*u_*v^*\mathcal{F} \hookrightarrow u_*u^*\mathcal{F}$, so, if $u^*\mathcal{F} \hookrightarrow \mathcal{H}$, put $\mathcal{G} = v_!v^*|_{u^*\mathcal{H}}$, $\mathcal{G}' = u_!\mathcal{H}$, $\mathcal{G}'' = \mathcal{G}/\mathcal{G}'$. This gives a commutative diagram as above.

The choice of $\mathcal{H}$ Let $\pi : X' \to X$ be a finite surjective morphism and X' is a normal irreducible variety. Let $U' = \pi^{-1}(U)$ and

$$\begin{tikzcd}
U' \arrow[r, "u'"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\
U \arrow[r, "u"] & X
\end{tikzcd}$$

Assume that $\pi'^*u^*\mathcal{F} = \mathcal{M}_{U'}$, $\mathcal{M} \in \mathrm{Ob}(R - mod)$. Put $\mathcal{H} = \pi'_*\pi'^*\mathcal{F}$.

Now, we check (i).

$\mathcal{F}' \to \mathcal{G}'$ is monomorphism, i.e. $\mathcal{F}'|U \to \mathcal{G}'|U$ is mono, but, $\mathcal{P} \to \pi'_*(\pi')^*\mathcal{P}$ is mono for any $\mathcal{P}$ on U.

Check (ii). $\mathcal{G}'|Y = 0$. $\mathcal{G}'|X - Y$ is locally constant because direct image of locally constant sheaves under finite etale morphism is locally constant.

$\mathcal{G}''|W = 0$ is clear from the definition. We need the following thing here: $\mathcal{G}''|Z - W$ is locally constant.

Let's find out what $u_*\mathcal{H}$ is. Our $\mathcal{H} = \pi'_*M_{U'}$. So, $u_*\mathcal{H} = u_*\pi'_*M_{U'} = \pi_*u'_*M_{U'}$. Since X' is normal, (which implies that it is analytically irreducible) we see that $u'_*M_{U'} = M_{X'}$. Thus, $u_*\mathcal{H} = \pi_*M_{X'}$. We see that, therefore, $\mathcal{G}''|_{Z-W} = (\pi_*M_{X'})\,|_{Z-W}$ where $\pi^{-1}Z = Z'$, $\pi^{-1}W = W'$ and $\pi'' : Z' - W' \to Z - W$.

Thus $\mathcal{G}''|_{Z-W} = \pi''_*(M_{Z'-W'})$.

There are two possible hypotheses:

(B1) $(Z'-W')_{red} \overset{\pi''_{red}}{\to} Z-W$ is a finite étale morphism. So, it follows that $\mathcal{G}''|_{Z-W}$ is locally constant.

Check (iii). $\delta_2 : H^{n-1}_\sigma(Z, \mathcal{G}'') \to H^n_\sigma(X, \mathcal{G}')$ is $\mathrm{End}H^*_\sigma(R)|F$-module homomorphism. But, $H^{n-1}_\sigma(Z, \mathcal{G}'') = H^{n-1}(Z', W'; M) \to H^n(Z', Y'; M)$. Since (X, Y) and (Z, W) are super pairs, (X', Y') and (Z', W') are good pairs. hence, δ_2 with $M = R$ is by definition of a morphism of type II, $\mathrm{End}H^*_\sigma(R)|F)$-module homomorphism.

That was the last step.

Recall the definition of the good filtration. Let X be affine. We review a little bit of general topology.

Definition 12.1. *$Q \subset P$ is a closed subset of a topological space P, $q \in Q$. We say that (P, Q) is tame at q if there is a pointed topological space (V, v_0), a neighborhood U of q in Q and a homeomorphism $\phi : U \times V \to W$ where W is open in P such that $\phi(q, v_0) = u$ for all $u \in U$, i.e. locally the space is expressed as a product.*

Note: $X \subset Y$, Y is a Zariski closed subset in X/k. It is true that there is a Zariski dense open subset $Y' \subset Y'$ so that the pair $(X^\sigma_{an}, Y^\sigma_{an})$ is tame at points of $Y'(\mathbb{C})$. (Proved by Whiney stratification. c.f. "Stratified Morse theory" by Goresky and MacPherson)

The hypothesis B2 is the following: $X \supset Y \supset Z \subset W$. Let $(X^\sigma_{an}, Y^\sigma_{an})$ is tame at every point of $Z(\mathbb{C}) - W(\mathbb{C})$. Then, in fact B2 implies that B1.

Definition 12.2. *$X_{-1} = \phi \subset X_0 \subset X_1 \subset \cdots \subset X$ is a* good filtration *if*

(a) *for every irreducible component Z of X_q with $Z \not\subset X_{q-1}$, $(Z, Z \cap X_{q-1})$ is a super pair of $\dim q$.*

(Of course this immediately implies that $\dim X_q \leq q$ by induction on q. Also, $X_n = X$ if $n = \dim X$.)

(b) *For all Z in (a) above, the pair $(Z, Z \cap X_{q-1})^\sigma_{an}$ is tame at every point of $Z \cap (X_{q-1} - X_{q-2})$.*

(c) *a filtration is good for $\mathcal{F}$ if $\mathcal{F}|_{X_q - X_{q-1}}$ is locally constant. for all q, where $\mathcal{F} \in Sh(X_{et}; R)$ is constructible.*

Definition 12.3. *Let X be an affine variety. We define $C^\cdot(X < \mathcal{F})$ for a good filtration $\mathcal{X}$. for $\mathcal{F}$. Let $i : A \hookrightarrow X$ is a closed embedding, let $\mathcal{F}_A = i_* i^* \mathcal{F}$ so that we have a surjection $\mathcal{F} \to \mathcal{F}_A$. Define $\mathcal{F} = F^0\mathcal{F} \supset F^1\mathcal{F} \supset \cdots \supset F^{n+1}\mathcal{F} = 0$ ($n = \dim X$) by $F^{q+1}\mathcal{F} = \ker(\mathcal{F} \to \mathcal{F}_{X_q})$. (Then, $\left(F^q\mathcal{F}/F^{q+1}\mathcal{F}\right)_x = 0$ for all $x \notin X_q - X_{q-1}$ and $\mathcal{F}|_{X_q - X_{q-1}} \simeq F^q\mathcal{F}/F^{q+1}\mathcal{F}|_{X_q - X_{q-1}}$.*

For a moment, fix q. I want to define $C^q(X, \mathcal{F}) \in \mathrm{Ob}ECM'(R)$. Let $A_1, A_2, \cdots, A_m$ be the irreducible components of X_q with $A_i \not\subset X_{q-1}$. Let $C^q(X, \mathcal{F}) = \bigoplus_{i=1}^m H^q(A_i, (F^q\mathcal{F}/F^{q+1}\mathcal{F})|_{A_i})$.

The last step will show that $C^q(X, \mathcal{F}) \xrightarrow{\delta^q} C^{q+1}(X, \mathcal{F})$ is defined and $b(\sigma)(\delta^{q+1} \circ \delta^q) = 0$ implies that $\delta^{q+1}\delta^q = 0$. hence this is a complex and $C^\cdot(X, \mathcal{F}) \in Ch(ECM'(R))$.

Now, we want to define a cohomology.

Definition 12.4. *We will define $H^q(X, \mathcal{F})$. We have*

(i) $H^q(C^\cdot(X, \mathcal{F})) \in \mathrm{Ob}(ECM'(R))$.

(ii) *there is a canonical isomorphism $b(\sigma)H^q(C^\cdot(X\mathcal{F})) \simeq H^q_\sigma(X, \mathcal{F})$.*

Here, $ECM'(R) \to (R-mod)$ is irredundant. Therefore there is a unique object denoted by $H^q(X, \mathcal{F})$ and $\phi' : H^q(C^\cdot(X, \mathcal{F})) \to H^q(X, \mathcal{F})$ so that $b(\sigma)\phi' = \phi$.

We want to show that this has all the good properties.

Existence of good filtration Given X affine of dimension n and given a sequence $P_0 \subset P_1 subset \cdots$ of closed subsets in X with $\dim P_j \leq j$ for all j and a given constructible sheaf $\mathcal{F} \in Sh(X_{et}; R))$, there is a good filtration $X_q \subset \cdots$ so that $P_q \subset X_q$ for all q.

Proof. By decreasing induction on q, $X_n = X$ if $\dim X = n$. Suppose that $P_q \subset X_q$ is defined. $X - q = A_1 \cup \cdots \cup A_m \cup T$, A_i are irreducible of dimension q and T is closed of dimension $T \leq q - 1$. Put $A_1^1 = A_1 \cap (A_2 \cup \cdots \cup A_m \cup T \cup P_{q-1})$. Let $A_1^1 \subset A_1^2 \subset A_1$, A_1^2 is closed such that $\mathcal{F}|_{A_1 - A_1^2}$ is locally constant. Let Z_j $j = 1, \cdots, e$ be the irreducible components of X_{q+1} of dimension $q + 1$ so that $Z \supset A_1$. Assume that $(Z, Z \cap X^q)$ is tame at every point of $A_1 - A_1^3$. So we got $A_1^2 \subset A_1^3 \subset A_1$. This can be done by the Whitney stratification lemma.

Finally, the Basic Lemma gives $A_1^3 \subset A_1^4 \subset A_1$ so that (A_1, A_1^4) is a super pair.

Let $A_1' = A_1^4$ and similarly define $A_j' \subset A_j$ for $j = 1, \cdots, m$. Then, define $X_{q-1} = T \cup P_{q-1} \cup (\bigcup_{i=1}^m A_i')$.

This shows the existence of good filtration.

□

Lemma 12.5. *Let* $0\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *be a short exact sequence of constructible sheaves on* X_{et}. *let* $X.$ *be good for* $\mathcal{F}' \oplus \mathcal{F}'' \oplus \mathcal{F}$. *Then,* $0 \to C^\cdot(X, \mathcal{F}') \to C^\cdot(X, \mathcal{F}) \to C^\cdot(X, \mathcal{F}'') \to 0$ *is a short exact sequence in* $Ch(ECM'(R))$. *(It uses the old lemma that* $\mathcal{F} \mapsto H^*(X, \mathcal{F})$ *is exact for* $\mathcal{F} \in \mathcal{C}(X, Y)$ *when* (X, Y) *is a super pair.)*

Proof. Let X be afffine. There is a cohomological δ-functor $\{constructiblesheavesof\mathrm{R}-modulesonX_{et}\} \to ECM'(R)$ so that $b(\sigma)H^q(X, \mathcal{F}) = H^q_\sigma(X, \mathcal{F})$.

Given good filtrations X_q', X_q'' for $\mathcal{F}$, there exists a good filtration X_q''' for $\mathcal{F}$ containing both of the previous good filtrations. Call the corresponding complexes $C^\cdot(X, \mathcal{F})'$, $C^\cdot(X, \mathcal{F})''$ and $C^\cdot(X, \mathcal{F})'''$. Then, we have two quasi-isomorphisms from $C^\cdot(X, \mathcal{F})'''$ to $C^\cdot(X, \mathcal{F})'$ and $C^\cdot(X, \mathcal{F})''$.

□

13. November 11, 2004

14. November 16, 2004

We want to apply the theorem proven last time. We want to state some definitions first.

Definition 14.1. *Given a diagram* D *and a representation* $T : D \to (R - mod)$, *where* R *is a commutative noetherian ring. We say that* T *satisfies the stabilization hypothesis if for any finite subdiagram* $F \subset D$, *there is a finite subdiagram* G *containing* F *so that for all finite subdiagram* $F \subset G \subset H$ *(of course all contained in* D*), if we take the* $\mathrm{im}(\mathrm{End}T|H \to \mathrm{End}T|F)$, *then it is same as* $\mathrm{im}(\mathrm{End}T|G \to \mathrm{End}T|F)$.

(a) e.g. if for all $p \in O(D)$, if Tp is (in addition) an Artinian R-module, then for any finite $F \subset D$, $\prod_{p \in O(F)} \mathrm{End}_R Tp$ is an Artinian R-module, so the collection of

$$(\mathrm{End}T|H \to \prod_{p \in O(F)} \mathrm{End}Tp), \quad \text{for all } F \subset H, H \text{ finite subdiagram}$$

has a minimal element which equals the image of $\mathrm{End}T|G \to \prod p \in O(F)\mathrm{End}Tp$. So, the stabilization hypothesis is satisfies.

(b) In particular, if R itself is a Artinian ring, then (a) holds.

Remark 14.2. If stabilization holds, let F be a finite subdiagram. Put

$$E(F) = \cap_{H \supset F, H \text{finite}} \mathrm{im}(\mathrm{End}(T|H) \to \mathrm{End}(T|F)).$$

Clearly, this equals $\mathrm{im}(\mathrm{End}T|G \to \mathrm{End}T|F)$ for some finite G.

Then, for all $F \subset H \subset D$, H: finite subdiagram, whenever we have a situation

$$\begin{tikzcd} E(H) \arrow[d] & E(F) \arrow[d] \\ \mathrm{End}T|H \arrow[r, "p"] & \mathrm{End}T|F \end{tikzcd} ,$$

we have $p(E(H)) = E(F)$.

Note that $p(E(H)) \subset E(F)$ is true without any hypothesis on T.

Given stabilization hypothesis, we see that there is $H' \supset H$ with H' finite such that $E(F) = \mathrm{im}(\mathrm{End}T|H' \to \mathrm{End}T|F)$ so that $pE(F) = \mathrm{im}(\mathrm{End}T|H' \to \mathrm{End}T|F) \hookrightarrow E(F) = p(\mathrm{im}(\mathrm{End}T|H' \to \mathrm{End}T|H) \subset p(E(H))$ by the definition of $E(H)$.

(This is related to the generalized Tate conjecture.)

Thus, $F \mapsto E(F)$ is an inverse system of surjections.

Remark 14.3.

$$\mathrm{End}T = \{a \in \prod_{p \in O(D)} \mathrm{End}Tp | \forall p, q \in O(D), \forall m : p \to q \text{ in } M(D), \text{ satisfying the following diagram}\}$$

$$\begin{tikzcd} Tp \arrow[r, "Tm"] \arrow[d, "a(p)"'] & Tq \arrow[d, "a(q)"] \\ Tp \arrow[r, "Tm"'] & Tq \end{tikzcd}$$

It is evident that the natural map

$$\mathrm{End}T \overset{\simeq}{\to} \varprojlim_{F \subset D: \text{ for all finite subdiagram}} \mathrm{End}T|F$$

is an isomorphism.

Let

$$\mathrm{End}T - cmod = \varinjlim_{F \subset D, F \text{ finite}} \mathrm{im}(\mathrm{End}T \to \mathrm{End}T|F) - mod.$$

Here c in *cmod* stands for continuous. Then, there is a natural faithful exact R-linear functor $C(T) \to \mathrm{End}(T) - cmod$.

Proposition 14.4. *The following are equivalent:*

(1) *T satisfies the stabilization hypothesis.*
(2) *$C(T) \to \mathrm{End}(T) - cmod$ is an equivalence of categories.*
(3) *$C(T) \to \mathrm{End}(T) - cmod$ is absolutely fully faithful.*

Proof. (1) $\Rightarrow$ (2)

Since $F \mapsto E(F)$ is an inverse system of surjections, it follows that the natural map $\varprojlim_F E(F) \leftarrow \mathrm{End}T$ is an isomorphism.

(a) $C(T) \to \mathrm{End}(T) - cmod = \varinjlim_F E(F) - mod$ is fully faithful. Let $M, N \in \mathrm{Ob}\mathrm{End}T|F_0 - mod$ regarded as objects of $C(T)$. Recall that by definition

$$\mathrm{Hom}_{C(T)}(M,N) = \varinjlim_{D \supset F \supset F_0} \mathrm{Hom}_{\mathrm{End}T|F}(M,N) = \mathrm{Hom}_{\mathrm{im}(\mathrm{End}T|F \to \mathrm{End}T|F_0)}(M,N) = \mathrm{Hom}_{E(F_0)}(M,N)$$

which is precisely a morphism of the RHS.

(b) Essentially surjective. So, if $M \in \mathrm{ObEnd}T - cmod$, then, $M \in E(F_0) - mod$ for some F_0, but there is $F \supset F_0$ such that $\mathrm{End}T|F \twoheadrightarrow E(F_0) \hookrightarrow \mathrm{End}T|F_0$. Regard M as a module of $\mathrm{End}T|F$, thus as an object of $C(T)$.

□

Remark 14.5. Let B be a Borel subgroup of G then $G - rep \to B - rep$ is fully faithful. At the level of rings, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset A = M_2(R)$, $A - mod \to B - mod$ is fully faithful.

Definition 14.6. *Suppose that* $\mathcal{A}, \mathcal{B}$ *are abelian categories and* $F : \mathcal{A} \to \mathcal{B}$ *is an absolutely fully faithful if*

(a) F *is faithful and exact*

(b) $\forall A \in \mathrm{Ob}(\mathcal{A})$ *and all monomorphisms* $c : B \to FA$ *in* $\mathcal{B}$*, there is a monomorphism* $i : A' \to A$ *and*

$$\begin{tikzcd}
FA' \arrow[r, "Fi"] \arrow[d, "\simeq"] & FA \\
B \arrow[ur, "c"] &
\end{tikzcd}$$

Lemma 14.7. *Suppose that* $F : \mathcal{A} \to \mathcal{B}$ *is absolutely fully faithful. Then,*

(a) F *is fully faithful and*

(b) *for all* $A', A'' \in \mathrm{Ob}\mathcal{A}$*, the natural map* $\mathrm{Ext}^1_{\mathcal{A}}(A'', A') \to \mathrm{Ext}^1_{\mathcal{B}}(FA'', FA')$ *is one to one.*

Proof. (a) Suppose that $M, N \in \mathrm{Ob}\mathcal{A}$, $g : FM \to FN$ is a morphism in $\mathcal{B}$. Consider $S = graph(g) \subset FM \oplus FN = F(M \oplus N)$. This satisfies $S \hookrightarrow F(M \oplus N) \xrightarrow{Fp_1} FM$: isomorphism, where $p_1 : M \oplus N \to M$ is the projection. By the definition of absolutely faithfully flatness, there is $S' \hookrightarrow M \oplus N$ so that $FS' = S$ as a subobject of $FM \oplus FN$. Then, for $q : S' \hookrightarrow M \oplus N \xrightarrow{p_1} M$, by the exactness of F, $F \ker q = F \mathrm{coker} q = 0$ and by faithfulness of F, $\ker q = \mathrm{coker} q = 0$. Hence q is an isomorphism. Therefore $S = graph(g')$ where $g' : M \to N$.

(b) Let $\xi \in \mathrm{Ext}^1_{\mathcal{A}}(A'', A')$ be a representation, $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ is a short exact sequence in $\mathcal{A}$ that corresponds to it. Then, $F\xi$ is a representation of $0 \to FA' \xrightarrow{Fi} FA \xrightarrow{Fj} FA'' \to 0$. If $F\xi = 0$, there is $B \subset FA$ so that for the isomorphism $t : B \hookrightarrow FA \xrightarrow{Fj} FA''$ F is absolutely fully faithful, there is $C \hookrightarrow A$ so that $FC = B$ and $B \hookrightarrow A \to A''$ is an isomorphism because Ft is an isomorphism.

□

Lemma 14.8. *Let* $f : B \to A$ *is a ring homomorphism, and* A *is a finitely generated left* B*-module. Then,* $\mathcal{A} = A - mod \to \mathcal{B} = B - mod$ *is absolutely fully faithful if and only if* $f : B \to A$ *is onto.*

Proof. $(\Leftarrow)$ is obvious.

$(\Rightarrow)$ Let $A \in \mathrm{Ob}\mathcal{A}$. $F = ff : \mathcal{A} \to \mathcal{B}$, $A \in \mathrm{Ob}\mathcal{A}$ and $f(B) \hookrightarrow ff(A)$ is a subobject in $\mathcal{B}$. So, by the definition of absolutely fully faithfulness, there exists A-submodule $M \subset A$ so that $M = f(B)$ containing 1. Hence $M = A$ i.e. $f(B) = A$.

□

Suppose the following situation: we have two inverse systems $B_\lambda, \lambda \in \Lambda$ and $A_\lambda, \lambda\Lambda$ of R-algebras and a compatible sysmte of homomorphisms $B_\lambda \to A_\lambda$.

For example, Λ is the collection of all finite subdiagrams of D, $B(F) = \mathrm{im}(\mathrm{End}T \to \mathrm{End}T|F)$, $A(F) = \mathrm{End}(T|F)$. This induces a R-linear faithful exact functor

$$F : \mathcal{A} := \varinjlim_{\lambda \in \Lambda} A_\lambda - mod \to \mathcal{B} := \varinjlim_{\lambda \in \Lambda} B_\lambda - mod.$$

Lemma 14.9. *In the above, F is absolutely fully faithful if and only if for all pairs $\lambda \le \mu$, there exists ν with $\lambda \le \mu \le \nu$ so that the image of $B_\mu \to A_\lambda$ contains the image of $A_\nu \to A_\lambda$.*

Proof. ($\Rightarrow$) $A_\lambda \in \mathrm{Ob}\mathcal{A}$. $FA = A_\lambda$ considered as a $B_?$-module for any $? \ge \lambda$. So, for any given μ, λ, consider the $\mathrm{im}(B_\mu \to A_\lambda)$ is a $\mathcal{B}$-subobject of FA_λ. F is absolutely fully faithful, so, $\mathrm{im}(B_\mu \to A_\lambda)$ is an A_ν-module for some $\nu \ge \lambda$. Since $1 \in \mathrm{im}(B_\mu \to A_\lambda)$, we deduce that $\mathrm{im}(B_\mu \to A_\lambda) \supset \mathrm{im}(A_\nu \to A_\lambda)$.

($\Leftarrow$) is left as an exercise. □

Recall the following previous proposition: The following are equivalent:

(1) T satisfies the stabilization hypothesis
(2) $C(T) \to \mathrm{End}(T) - cmod$ is an exact functor of categories
(3) $C(T) \to \mathrm{End}(T) - cmod$ is absolutely fully faithful.

We proved that (1) implies (2). (2) $\Rightarrow$ (3) is clear. It remains to show that (3) implies (1).

After the previous lemma, let $\lambda = F, \mu = G, \nu = H$. Given $F \subset G$ there is H with $F \subset G \subset H$ and

$$\mathrm{im}(B(G) \to A(F)) = \mathrm{im}(\mathrm{End}(T) \to \mathrm{End}T|F) \supset \mathrm{im}(\mathrm{End}(T|H) \to \mathrm{End}(T|F))$$

(but, we have $\mathrm{End}(T) \to \mathrm{End}T|H \to \mathrm{End}T|F$, so therefore $\mathrm{im}(\mathrm{End}(T) \to \mathrm{End}(T|F) = \mathrm{im}(\mathrm{End}(T|H) \to \mathrm{End}(T|F))$. So, for any $H' \supset setH \supset F$, in $\mathrm{End}T \to \mathrm{End}T|H' \to \mathrm{End}T|H \to \mathrm{End}T|F$, the image of the first and the third equals in $\mathrm{End}(T|F)$, hence the second has the same image. In other words, the stabilization holds. This finishes the proof of the proposition.

We will now concentrate on Artinian rings.

Definition 14.10. *Let R be a commutative noetherian ring. Consider R-algebras A_λ where A_λ is finitely generated as R-module and A_λ is Artinian. A pro-R-algebra is $\varprojlim_\lambda A_\lambda$, with A_λ as above.*

Fix λ and concisder $\{\mathrm{im}(A_\mu \to A_\lambda : \forall \mu \ge \nu\}$ has a minimal element $A'_\lambda \subset A_\lambda$. And for all $\mu \ge \lambda$, $A'_\mu \to A'_\lambda$ is onto. And, $\varprojlim_\lambda A'_\lambda \to \varprojlim_\lambda A_\lambda$ is an isomorphism. For each λ,

$$I(\lambda) = \ker\left(A := \varprojlim_\mu A'_\mu \to A'_\lambda\right)$$

, $I(\lambda)$ is a system of two-sided ideals. Note that $A/I(\lambda) = A'_\lambda$. $A - cmod$ is the category of finitely generated A-modules annihilated by $I(\lambda)$ for some λ which is $\varinjlim_\lambda A_\lambda - mod = \varinjlim_\lambda A'_\lambda - mod$.

Lemma 14.11. *Let $f : B \to A$ is a continuous homomorphism of pro-R-algebras. Then, $A - cmod \to B - cmod$ is absolutely fully faithful if and only if $f : B \to A$ is onto.*

Proof. $\Leftarrow$ is clear. For $\Rightarrow$, it suffices to show that the composite $B \to A \to A/I(\lambda)$ is onto for all λ, $A = \varprojlim_{\lambda} A_\lambda$. Then, we can use the same argument as before. ☐

15. November 18th, 2004

Note missing.

16. November 23rd, 2004

17. November 30th, 2004