

A note on Leray-Hirsch theorem

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Let X be a locally contractible topological space, and let $\phi : Y \rightarrow X$ be a fibration, i.e. for each $x \in X$, there is a neighborhood U_x of x such that

$$\phi^{-1}(U_x) \simeq Y_x \times U_x, \quad Y_x = \phi^{-1}(x)$$

i.e. Y is locally homeomorphic to a product.

Suppose that $H^*(Y_x, \mathbb{Z})$ are torsion free for each $x \in X$.

Theorem 0.1 (Leray-Hirsch). *Suppose that there are cohomology classes $\alpha_1, \dots, \alpha_N \in H^*(Y, \mathbb{Z})$ such that the subgroup $A^* \subset H^*(Y, \mathbb{Z})$ generated by $\alpha_1, \dots, \alpha_N$ is isomorphic to the group $H^*(Y_x, \mathbb{Z})$ via restriction $H^*(Y, \mathbb{Z}) \rightarrow H^*(Y_x, \mathbb{Z})$. Then, $H^*(Y, \mathbb{Z})$ is isomorphic to $A^* \otimes_{\mathbb{Z}} H^*(X, \mathbb{Z})$.*

Proof. Let $\mathbb{Z} \rightarrow \mathcal{F}^*$ be a flasque resolution on Y . We may assume that each $\deg \alpha_i$ has a fixed degree, by separating its homogeneous components. Let $k_i = \deg \alpha_i$. Let $\beta_i \in \Gamma(Y, \mathcal{F}^{k_i})$ be a closed element representing the class α_i . Since A^* is free, β_i 's also generate the group $\Gamma(Y, \mathcal{F}^{k_i}) = \Gamma(X, \phi_* \mathcal{F}^{k_i})$ and those sheaves are flasque, so that we have an inclusion of a complex $(A^*, 0) \hookrightarrow (\phi_* \mathcal{F}^*, \phi_* d)$. But, by the local triviality of ϕ , the assumption that A^* maps isomorphically to $H^*(Y_x, \mathbb{Z})$ is equivalent to the fact that the inclusion is a quasi-isomorphism. Hence

$$\mathbb{H}^*(X, A^*) \simeq \mathbb{H}^*(X, \phi_* \mathcal{F}^*) \simeq H^*(Y, \mathbb{Z})$$

. LHS is by the way, equal to $A^* \otimes_{\mathbb{Z}} H^*(X, \mathbb{Z})$, because A^* is a free \mathbb{Z} -module of finite rank. \square