A personal note on the length of a module

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1. Condition for finiteness of length

Unless specified otherwise, A is always a noetherian ring and M is a finitely generated A-module.

Lemma 1.1. *Let $M_1 \subset M_2$ be modules without any intermediate modules between them. Then, $M_2/M_1 \simeq A/m$ for a maximal ideal m.*

Proof. We may assume that $M_1 = 0$ and M_2 is a simple module, by the correspondence theorem. Let $0 \neq x \in M_2$. Since M_2 is a simple module, $Ax = M_2$ and $Ax \simeq A/\text{Ann}(x) \simeq M_2$. So the question is to find a condition for A/I, $I \subset A$ an ideal, not to have a nontrivial submodule. Hence, it is reduced to the following lemma: □

Lemma 1.2. *A-module A/I is simple iff I is a maximal ideal of A.*

Proof. For any ideal $J \supset I$, J/I is a submodule of A/I and any submodule of A/I is of this form by the correspondence theorem. Hence if A/I is simple, either $J = I$ or $J = A$ iff I is a maximal ideal. □

Corollary 1.3. *In particular, above maximal ideal $m = \text{Ann}(x)$ for some nonzero x, so that $m \in \text{Ass}(M)$.*

Recall the following trivial facts from Atiyah-MacDonalds:

Lemma 1.4. $\text{Supp}(A/I) = V(I) = \{p \in \text{Spec}(A) | I \subset P\}$.

Lemma 1.5. *For a short exact sequence $0 \to M' \to M \to M'' \to 0$, $\text{Supp}M = \text{Supp}M' \cup \text{Supp}M''$.*

So that we can prove one direction:

Proposition 1.6. *Suppose that $l_A(M) < \infty$. Then, $\text{Supp}M$ contains only maximal ideals. In particular, all of those maximal ideals are in $\text{Ass}M$, hence $\text{Ass}M = \text{Supp}M$.*

Proof. Let $M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{n+1} = 0$ be a composition series with M_i/M_{i+1} being simple. By Lemma 1.1, $M_i/M_{i+1} \simeq A/m_i$ for some maximal ideals m_i. Then, since $0 \to M_{i+1} \to M_i \to M_i/M_{i+1} \to 0$ is exact, by applying Lemma 1.5, we have $\text{Supp}M = \text{Supp}M_1 \cup \text{Supp}M_0/M_1 = \cdots = \cup \text{Supp}(A/m_i) = \{m_1, \cdots, m_n\}$.

□

Now we want to show the converse of above proposition:

Proposition 1.7. *Suppose that $\text{Supp}M$ contains only maximal ideals. Then, $l_A(M) < \infty$.*

Proof. By induction and by the correspondence theorem, it is enough to prove it when M is generated by a single element, i.e. $M = Ax$ for some nonzero $x \in M$. Since $Ax \simeq A/\text{Ann}(x)$ and $\text{Supp}M = \text{Supp}A/\text{Ann}(x) = V(\text{Ann}(x))$, $\text{Ann}(x)$ is contained only by maximal prime ideals.

Note that for any ring B, if we regard B as a B-module, $\text{Supp}B = \text{Spec}B$. Also, note that any A-submodule of A/I is of the form J/I for some ideal $J \supset I$ of A. Hence, that $\text{Supp}(A/\text{Ann}(x)) = V(\text{Ann}(x))$ contains only maximal ideals means, when $B = A/\text{Ann}(x)$, for ring B, all prime ideals are in fact maximal, i.e. $\dim B = 0$, i.e. B is an Artinian ring, because A was noetherian. Hence in fact, we have only finitely many maximal ideals and

Supp$M$ is in particular a finite set, and from the descending chain condition of submodules, i.e. ideals of B, we must have $l_B(B) < \infty$. Now, from the correspondence theorem,

$$l_A(A/\mathrm{Ann}(x)) = l_{A/\mathrm{Ann}(x)}(A/\mathrm{Ann}(x)) = l_B(B) < \infty,$$

which proves the claim.

□

We summarize above results as follows:

Theorem 1.8. *(condition for finite length) A is noetherian, M is a finitely generated A-module. Then, $l_A(M)$ iff* Supp$M$ *has only maximal ideals. If this is the case, then in fact,* Supp$M$ *must have only finitely many manximal ideals, and* Ass$M$ = Supp$M$.

Remark. Of course, not all finitely generated A-module M has a finite length, but, still we can have a finite chain of the following form:

$$M = M_0 \supset M_1 \supset \cdots \supset M_n \supset 0 = M_{n+1}$$

with $M_i/M_{i+1} \simeq A/p_i$ for some prime ideals p_i.

Proof. We use the fact that any nonzero finitely generated module M has $\mathrm{Ass}(M) \neq \phi$. Take any $p_1 \in \mathrm{Ass}M$, i.e. $p_1 = \mathrm{Ann}(x)$ for some nonzero $x \in M$, i.e. $A/p_1 = A/\mathrm{Ann}(x) \simeq Ax \subset M$ a submodule. Let $M^0 = 0$ and $M^1 = Ax$. If $M^1 = M$, then we are done. If not, $M/M^1 \neq 0$, so that again $\mathrm{Ass}(M/M^1) \neq \phi$ so that there is a module isomorphic to $A/p_2 \subset M/M^1$ for some prime ideal p_2. Hence there is $M^2 \subset M$ such that $M^2/M^1 \simeq A/p_2$. In this fashion, inductively we have $0 \subset M^1 \subset M^2 \subset M^3 \subset \cdots$ with $M^{i+1}/M^i \simeq A/p_{i+1}$. Since M is noetherian, this chain must terminate with a finite length.

□

2. Some applications

Corollary 2.1. *If $\dim A = 0$ and noetherian, then, for M, finitely generated A-module, $l_A(M) < \infty$ always.*

Proof. Supp$M$ contains only maximal ideals, because all prime ideals are maximal. □

Corollary 2.2. *Assume that $\dim A > 0$. Assume that A is an integral domain and $l_A(M) < \infty$. Then, M is a torsion module.*

Proof. $l_A(M) < \infty$ implies that Supp$M$ has only maximal ideals by the theorem. In particular, (0) is a prime ideal which is not maximal, hence, $\mathrm{Supp}M = V(\mathrm{Ann}M) \not\ni (0)$, i.e. $\mathrm{Ann}M \not\subset (0)$, i.e. Ann$M$ contains a nonzero $a \in A$ i.e. $aM = 0$. □

Remark. There are several useful results on Fulton's intersection theory about length in Appendix A. It is pointless to copy it down here, so please refer to the book.